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SURFACES WITH $K^2 < 3\chi$ AND FINITE FUNDAMENTAL GROUP

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Abstract. In this paper we continue the study of $\pi_1^{\text{alg}}(S)$ for minimal surfaces of general type S satisfying $K_S^2 < 3\chi(S)$. We show that, if $K_S^2 = 3\chi(S) - 1$ and $|\pi_1^{\text{alg}}(S)| = 8$, then S is a Campedelli surface.

In view of the results of [MP1] and [MP2], this implies that the fundamental group of a surface with $K^2 < 3\chi$ that has no irregular étale cover has order at most 9, and if it has order 8 or 9, then S is a Campedelli surface.

To obtain this we establish some classification results for minimal surfaces of general type such that $K^2 = 3p_g - 5$ and such that the canonical map is a birational morphism. We also study rational surfaces with a $\mathbb{Z}_2^3$-action.

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1. Introduction

The algebraic fundamental group of a surface of general type is an important invariant of the surface. If a surface is irregular then of course π_1^{alg} is infinite, but regular surfaces may also have infinite fundamental group.

For minimal surfaces S of general type with $K^2 < 3\chi$ through the work of several authors ([Bo], [Ho2], [Re1], [Re2], [Xi1], [Xi2], [Mu1], [Mu2], [CM], [MP1], [MP2]) one has a quite good picture of what can occur, and in particular a very good understanding of the geometry of S when S has infinite fundamental group (cf. the introduction of [MP1]). It turns out that S has an infinite fundamental group if and only if it has an irregular étale cover. Also, if S has finite fundamental group the groups that can occur form a very limited family. In particular if $K^2 \leq 3\chi - 2$, then the order of π_1^{alg} is at most 5 and if $|\pi_1^{\text{alg}}(S)| = 5$ then S is a Godeaux surface (i. e. $K^2 = 1, p_g(S) = 0, \chi(S) = 1$) (see [MP2]). If $K^2 = 3\chi - 1$, then the order of π_1^{alg} is at most 9 and if $|\pi_1^{\text{alg}}(S)| = 9$ then S is a Campedelli surface (i. e. $K^2 = 2, p_g(S) = 0, \chi(S) = 1$).

In this paper we sharpen this last result and we prove the following:

Theorem 1.1. *Let S be a minimal complex surface of general type such that $K_S^2 = 3\chi(S) - 1$. If the group $\pi_1^{\text{alg}}(S)$ has order 8, then $\chi(S) = 1$.*

By the results in [MP1, MP2], we have then the following:

Theorem 1.2. *Let S be a minimal complex surface of general type with $K_S^2 \leq 3\chi(S) - 1$ which has no irregular étale cover. Then the group $\pi_1^{\text{alg}}(S)$ is finite of order ≤ 9.*

If $|\pi_1^{\text{alg}}(S)| = 8$ or 9, then $\chi(S) = 1$, $K_S^2 = 2$, namely S is a numerical Campedelli surface.

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Note that there exist Campedelli surfaces with algebraic fundamental group of orders 8 and 9 (cf. [Re2] and [MP3]).

The idea for the proof of Theorem 1.1 is the following. Let $V \to S$ be the étale cover of degree 8 given by $\pi_1^{\mathrm{alg}}(S)$. Then V is regular and $K_V^2 = 3p_g(V) - 5$, i.e. these invariants are *close* to the *Castelnuovo line* $K^2 = 3p_g - 7$.

Canonical surfaces with $3p_g - 7 \leq K^2 \leq 3p_g - 6$ have been studied and classified in [Ha1], [AK] and [Ko]. The case $K^2 = 3p_g - 5$, of interest here, is however still widely open. In §3 we start the analysis of canonical surfaces with these invariants and obtain the information we need for the proof of Theorem 1.1. Namely, we consider a surface V with $K_V^2 = 3p_g(V) - 5$, for which the canonical map is a birational morphism of V onto its image Σ. One proves that V is regular (see Lemma 3.1) and that, if $p_g \geq 14$, then Σ lies on a rational threefold scroll X of minimal degree, whose ruling pulls back to a base point free pencil of curves of genus 3 on V (see Theorem 3.5 for a more precise statement). This result opens up the possibility for a detailed classification of these canonical surfaces. However we do not dwell on this here, since Theorem 3.5 is what we need for our proof of Theorem 1.1. The proof of the results of §3 use Castelnuovo's theory, in particular the geometric information on a set of points in uniform position which impose *few* conditions on quadrics (see [Ha2], [EH], [Ci2]).

Next, having in mind the application to the proof of Theorem 1.1, we consider, in §4, the following general situation: S is a minimal surface of general type, $V \to S$ is an étale Galois cover with Galois group G of order 8, the canonical map of V has degree 2 onto a rational surface W. The canonical involution lies in the centre of $\mathrm{Aut}(V)$, hence the action of G on V descends to an action of G on W. Moreover, a result by Beauville [Be1] implies that $G = \mathbb{Z}_2^3$. We analyse this situation by considering the *relative minimal model* à la Mori of the pair (W, G). We show that it is a conic bundle and that the subgroup H of G that sends each fibre to itself has order 2. We describe in detail the whole situation in Proposition 4.4.

Finally, in §5 we go back to the problem we started with.

If S has $K_S^2 = 3\chi(S) - 1$ with $G = \pi_1^{\mathrm{alg}}(S)$ of order 8, and $V \to S$ is the étale cover given by G, we prove that either the canonical map of V is a birational morphism to its image, or it has degree 2 onto a rational surface (see Proposition 5.2). Then, by using the aforementioned results of §§3, 4, we separately discuss the two cases, excluding the latter, and proving that the former is compatible only with $\chi(S) = 1$, i.e. S has to be a Campedelli surface.

Notation and conventions. We work over the complex numbers. All varieties are projective algebraic. All the notation we use is standard in algebraic geometry. We just recall the definition of the numerical invariants of a smooth surface S: the self-intersection number K_S^2 of the canonical divisor K_S, the *geometric genus* $p_g(S) := h^0(K_S) = h^2(\mathcal{O}_S)$, the *irregularity* $q(S) := h^0(\Omega_S^1) = h^1(\mathcal{O}_S)$ and the *holomorphic Euler characteristic* $\chi(S) := 1 + p_g(S) - q(S)$.

2. Preliminaries

We recall some general facts that will be used in the following sections.

2.1. Fixed point formulae. Let S be a smooth complex compact surface and let ι be an involution of S, namely an automorphism of order 2 of S. The fixed locus of ι is the union of a smooth (possibly empty) divisor R and of ν isolated points. We recall the following two well-known formulae:

Holomorphic Fixed Point Formula (see [AS], pg. 566):

$$\sum_{i=0}^{2}(-1)^i \mathrm{Trace}(\iota|H^i(S,\mathcal{O}_S)) = \frac{\nu - RK_S}{4} \tag{2.1}$$

Topological Fixed Point Formula (see [Gr], (30.9)):

$$\sum_{i=0}^{4}(-1)^i \mathrm{Trace}(\iota|H^i(S,\mathbb{C})) = \nu + e(R), \tag{2.2}$$

where $e(R) = -R^2 - RK_S$ is the topological Euler characteristic of R.

If S is a rational surface, as it is often the case in this paper, then (2.1) takes a particularly simple form:

$$\nu = 4 + K_S R \tag{2.3}$$

If, in addition, the divisor R is empty, then (2.3) gives $\nu = 4$.

Remark 2.1. Formulae (2.1) and (2.3) are stated here for a smooth surface, but clearly they hold also for a surface with rational singularities, provided that the involution acts freely on the singular locus.

2.2. Free group actions on pencils of low genus. We will use the following result from [MP2].

Lemma 2.2. *Let Y be a regular surface of general type, let $G \neq \{1\}$ be a finite group that acts freely on Y and let $|F|$ be a G-invariant free pencil of curves of genus $g(F) \leq 4$. Then only the following possibilities can occur:*

(i) *$G = \mathbb{Z}_2^2$, $g(F) = 3$ and G acts faithfully on $|F|$;*
(ii) *$G = \mathbb{Z}_3$, $g(F) = 4$;*
(iii) *$G = \mathbb{Z}_2$, $g(F) = 3$.*

2.3. Threefolds with special curve sections. The results of this section are related to the classification of varieties with small *Delta–genus* (cf. [CR], [EH], [Fu1], [Fu2], [Fu3], [Fu4], [Fu5]).

Recall that a variety in a projective space is a *scroll* over a smooth curve Δ if it is swept out by a 1–dimensional family $\mathcal{R}$ of $n-1$-dimensional linear subspaces parametrized by a curve birational to Δ. The elements of the family $\mathcal{R}$ are called *rulings*. A linearly normal variety which is a scroll over $\mathbb{P}^1$ is called a *rational normal scroll*. These have been classically described by del Pezzo (see [EH]).

Throughout this subsection we denote by $X \subset \mathbb{P}^{r+1}$ a non–degenerate, irreducible threefold, by Y its general hyperplane section and by γ the arithmetic genus of a general curve section of X. In addition, we assume that X is *linearly normal*, namely that $h^1(\mathbb{P}^{r+1}, \mathcal{I}_{X,\mathbb{P}^{r+1}}(1)) = 0$.

We need the following:

Lemma 2.3. *If $\gamma > 0$ and Y is a scroll over a curve Δ, then also X is a scroll over Δ.*

Proof. Let $f: W \to X$ be a desingularization of X and set $f^*(Y) = S$. Note that S is smooth. The divisor S is also nef and big, hence for $i < 3$ one has $H^i(W, \mathcal{O}_W(-S)) = 0$. This implies that $h^1(W, \mathcal{O}_W) = h^1(S, \mathcal{O}_S)$ and the map $\mathrm{Alb}(S) \to \mathrm{Alb}(W)$ is an isogeny.

Let R be the proper transform on S of a general line of the ruling of Y. The Albanese map $a_S: S \to \mathrm{Alb}(S)$ maps onto Δ with general fibre R. Since the system $|S|$ on W is birational and S is general, it follows that the Albanese image of W is also a curve Δ'.

Let Π be the general fibre of the Albanese map of W, which is irreducible. Then the intersection of S with Π is also irreducible and, by the above argument, it coincides with R. This yields that f maps Π to a linear subspace of $\mathbb{P}^r$ of dimension $n-1$, proving the assertion. □

If $\gamma \leq 2$, then the possibilities for X are quite restricted:

Proposition 2.4. (i) *If $\gamma = 0$, then X is either a rational normal scroll or $X \subset \mathbb{P}^6$ is the cone over the Veronese surface in $\mathbb{P}^5$;*
(ii) *if $\gamma = 1$ and X is not a scroll then $r \leq 9$;*
(iii) *if $\gamma = 2$ and X is not a scroll then $r \leq 11$.*

Proof. Assume $\gamma = 0$. Then the general curve section of X is smooth irreducible, hence a rational normal curve, and it is a classical result (see [EH, Theorem 1]) that X is as in statement (i).

Assume $\gamma > 0$ and X is not a scroll. By Lemma 2.3 Y is not a scroll, either. Let $Y_0 \to Y$ be the minimal desingularization of Y and let D be the pull-back on Y_0 of the general hyperplane section of Y. The divisor D is smooth and nef, hence statements (ii) and (iii) follow immediately by applying [CR, Theorem 7.3] to the pair (Y_0, D). □

2.4. Castelnuovo theory. We recall some facts from Castelnuovo theory. For any variety $T \subset \mathbb{P}^m$ we denote by h_T the Hilbert function of T, namely $h_T(n)$ is the dimension of the image of the restriction map $H^0(\mathcal{O}_{\mathbb{P}^m}(n)) \to H^0(\mathcal{O}_T(n))$.

Given a variety $X \subset \mathbb{P}^m$, we denote its general hyperplane section by Y. Then

$$h_X(n) - h_X(n-1) \geq h_Y(n) \tag{2.4}$$

for all $n \geq 1$. Equality holds in (2.4) for all $n \geq 1$ if and only if $h^1(\mathbb{P}^m, \mathcal{I}_{X,\mathbb{P}^m}(n)) = 0$ for all $n \geq 0$, i.e. if and only if X is *projectively normal* (see [Ha2, Lemma 3.1 and Remark 3.1.1]; see [Ci2, Remark 1.8, (ii)], for a more precise result).

Of special interest to us here is the Hilbert function of a finite set $\Gamma \subset \mathbb{P}^{r-1}$. If the d points of Γ are in general position and $d \geq 2r-1$, then it is easy to check that $h_\Gamma(2) \geq 2r-1$. The cases in which $h_\Gamma(2)$ is close to this lower bound have been studied by several authors:

Proposition 2.5. (Castelnuovo, [Ha2, Lemma 3.9]) *Let $r \geq 3$ and let $\Gamma \subset \mathbb{P}^{r-1}$ be a finite set of $d \geq 2r+1$ points in general position. Then $h_\Gamma(2) = 2r-1$ if and only if Γ is contained in a rational normal curve of degree $r-1$ in $\mathbb{P}^{r-1}$, cut out by all the quadrics containing Γ.*

In order to state the remaining results, we need to recall the following definition (cf. [Ha2, p.85-86]). A finite set $\Gamma \subset \mathbb{P}^{r-1}$ of d points is said to be in *uniform position* if, for any subset $\Gamma' \subset \Gamma$ of d' points and for any $n \geq 0$, one has $h_{\Gamma'}(n) = min\{d', h_\Gamma(n)\}$. This definition is motivated by the fact that the general hyperplane section of an irreducible projective curve consists of points in uniform position (cf. [Ha2, Lemma 3.4]).

Proposition 2.6. ([Fa, p.54]; [Ha2, p.106]) *Let $r \geq 4$ and let $\Gamma \subset \mathbb{P}^{r-1}$ be a finite set of $d \geq 2r+3$ points in uniform position. If $h_\Gamma(2) = 2r$, then Γ is contained in an elliptic normal curve of degree r in $\mathbb{P}^{r-1}$, cut out by all the quadrics containing Γ.*

Proposition 2.7. ([Ci2, Theorem 3.8], cf. also [Pe, Proposition 4.3]) *Let $r \geq 6$, $d > \frac{8}{3}r$ and let $\Gamma \subset \mathbb{P}^{r-1}$ be a finite set of $d \geq 2r+1$ points in uniform position. If $h_\Gamma(2) = 2r+1$, then Γ is contained in an irreducible curve of degree $\leq r+1$ in $\mathbb{P}^{r-1}$, cut out by all the quadrics containing Γ.*

3. Canonical surfaces with $K^2 = 3p_g - 5$.

Throughout this section we assume that V is a smooth minimal projective surface of general type such that:

- $p_g := p_g(V) \geq 4$, $K^2 := K_V^2 = 3p_g(V) - 5$;
- the canonical system of $|K_V|$ is base point free;
- the canonical map of V is birational.

Our analysis is similar to those of [Ha1], [AK] and [Ko] for canonical surfaces with $K^2 = 3p_g - 7$ and $K^2 = 3p_g - 6$.

Our first remark is that V must be regular:

Lemma 3.1. *One has $q(V) = 0$.*

Proof. By [De, Theorem 3.2], a minimal surface of general type with birational canonical map satisfies:

$$K^2 \geq 3p_g + q(V) - 7.$$

Therefore, for V satisfying the hypothesis of the lemma, one has $q(V) \leq 2$. It follows in particular that $\chi(V) \geq p_g - 1$, and therefore $K^2 = 3p_g - 5 \leq 3\chi(V) - 2 < 3\chi(V)$.

Suppose that V is irregular. By [Ho2], the condition $K^2 < 3\chi$ implies that the Albanese image of V is a curve and that the general fibre F of the Albanese map is an hyperelliptic curve. Since the restriction of $|K_V|$ to F is contained in $|K_F|$, the canonical map of V is not birational, contradicting the assumptions. □

Set $r = p_g - 2$, $d = K^2 = 3r + 1$ and $g = K^2 + 1 = 3r + 2$.

Let Σ denote the canonical image of V. Our assumptions imply that Σ is a non degenerate surface of degree d in $\mathbb{P}^{r+1}$. We denote by Δ the general hyperplane section of Σ, which is a non–degenerate curve of degree d in $\mathbb{P}^r$. The curve Δ is linearly normal, since $q(V) = 0$ by Lemma 3.1. The curve $C = \phi^*(\Delta)$ is a general curve in $|K_V|$, hence it is smooth of genus g.

We let Γ be the general hyperplane section of Δ. By [Ha2, Lemma 3.4], the set Γ consists of d distinct points in uniform position spanning a $\mathbb{P}^{r-1}$. As in §2.3, we denote by h_Γ [resp. by h_Δ, h_Σ] the Hilbert function of Γ [resp. of Δ, Σ]. From formula (2.4) one has:

$$h_\Delta(n) - h_\Delta(n-1) \geq h_\Gamma(n) \tag{3.1}$$

$$h_\Sigma(n) - h_\Sigma(n-1) \geq h_\Delta(n) \tag{3.2}$$

for all $n \geq 1$.

For any non–negative integers n, m, one also has:

$$h_\Gamma(n+m) \geq \min\{3r+1, h_\Gamma(n) + h_\Gamma(m) - 1\} \tag{3.3}$$

(see [Ha2, Corollary 3.5]).

Lemma 3.2. *One has:*

- (i) $h_\Delta(2) \leq g = 3r+2$ *and, if equality holds, then* $h^1(\mathbb{P}^r, \mathcal{I}_{\Delta,\mathbb{P}^r}(2)) = 0$ *and* Δ *is smooth;*
- (ii) $2r - 1 \leq h_\Gamma(2) \leq 2r+1$.

Proof. Note that $h_\Delta(2)$ is the dimension of the image of the restriction map

$$H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(2)) \to H^0(\Delta, \mathcal{O}_\Delta(2)).$$

By pulling back to V via the canonical map, this is bounded above by the dimension of the image of the map

$$H^0(V, \mathcal{O}_V(2K_V)) \to H^0(C, \mathcal{O}_C(2K_V)) = H^0(C, \omega_C).$$

Since $h^0(C, \omega_C) = g$, assertion (i) follows immediately. Note now that $h_\Gamma(1) = r$. Hence (ii) follows by (i) and by formulae (3.1) and (3.3). □

We first examine the cases $2r - 1 \leq h_\Gamma(2) \leq 2r$.

Proposition 3.3. *Assume that* $r \geq 4$ *(i.e.* $p_g \geq 6$*). If* $h_\Gamma(2) = 2r - 1 + i$*, where* $i = 0$ *or* 1*, then the intersection of all quadrics containing* Σ *is a linearly normal threefold* X *of degree* $r - 1 + i$ *with curve sections of geometric genus* i*.*

Proof. Assume first $i = 0$. Then Proposition 2.5 (Castelnuovo's Lemma) says that the intersection of all quadrics containing Γ is a rational normal curve Z of degree $r - 1$.

Since $q(V) = 0$ by Lemma 3.1, Δ is linearly normal and so the restriction map

$$H^0(\mathbb{P}^r, \mathcal{I}_{\Delta,\mathbb{P}^r}(2)) \to H^0(\mathbb{P}^{r-1}, \mathcal{I}_{\Gamma,\mathbb{P}^{r-1}}(2))$$

is surjective. As a consequence, we have that the intersection of all quadrics containing Δ is a surface Y of degree $r-1$, whose general hyperplane section is Z.

Since Σ is linearly normal, also the map

$$H^0(\mathbb{P}^{r+1}, \mathcal{I}_{\Sigma,\mathbb{P}^{r+1}}(2)) \to H^0(\mathbb{P}^r, \mathcal{I}_{\Delta,\mathbb{P}^r}(2))$$

is surjective, and, as in the preceding paragraph, we conclude that the intersection of all quadrics containing Σ is a threefold $X \subset \mathbb{P}^{r+1}$ of degree $r - 1$. The threefold X is linearly normal because Σ is also linearly normal.

If $i = 1$, then, by Proposition 2.6, the intersection of all quadrics containing Γ is a linearly normal curve E of degree r and genus 1 in $\mathbb{P}^{r-1}$. Then the same argument as above proves the assertion. □

Next we consider the cases $h_\Gamma(2) = 2r + 1$ and we prove a similar result.

Proposition 3.4. *Assume that $r \geq 6$ (i. e. $p_g \geq 8$). If $h_\Gamma(2) = 2r + 1$, then the intersection of all quadrics containing Σ is a linearly normal threefold X of degree $r+1$ with curve sections of arithmetic genus either 1 or 2. Moreover, Σ is projectively normal and smooth in codimension 1.*

Proof. If $h_\Gamma(2) = 2r + 1$, then we have $h_\Delta(2) = 3r + 2$ by (3.1) and by Lemma 3.2. So, again by Lemma 3.2, Δ is smooth and $h^1(\mathbb{P}^r, \mathcal{I}_{\Delta,\mathbb{P}^r}(2)) = 0$. In particular Σ is smooth in codimension 1.

Moreover, one has $h^1(\mathbb{P}^r, \mathcal{I}_{\Delta,\mathbb{P}^r}(n)) = 0$, for all $n \geq 3$ by a well known result of Castelnuovo's (see [Ci1, Teorema 1.1]). Hence Δ is projectively normal. Then also Σ is projectively normal, since we have exact sequences:

$$H^1(\mathbb{P}^{r+1}, \mathcal{I}_{\Sigma,\mathbb{P}^{r+1}}(n-1)) \to H^1(\mathbb{P}^{r+1}, \mathcal{I}_{\Sigma,\mathbb{P}^{r+1}}(n)) \to H^1(\mathbb{P}^r, \mathcal{I}_{\Delta,\mathbb{P}^r}(n)) = 0$$

for all $n \geq 1$.

Now by Lemma 2.7 the set Γ is contained in a curve Z of degree $\delta \leq r+1$ which is the intersection of all quadrics containing Γ. Since $\delta \leq r + 1$ the arithmetic genus of Z is ≤ 2, by Castelnuovo's theorem (cf. [Ha2, Theorem 3.7]). The same argument as in the proof of Proposition 3.3 shows that the intersection of all quadrics containing Σ is a threefold X whose general curve section is Z.

Now we want to show that $\delta = r + 1$. Suppose otherwise. If $\delta = r - 1$, then Z is a rational normal curve and $h_\Gamma(2) = h_Z(2) = 2r - 1$, contradicting $h_\Gamma(2) = 2r + 1$. So $\delta \geq r$. If Z is rational, then X is not linearly normal (see [EH]) and Σ is not linearly normal either, a contradiction. If $\delta = r$, then Z is a linearly normal smooth elliptic curve, and $h_\Gamma(2) = h_Z(2) = 2r$, which again contradicts $h_\Gamma(2) = 2r + 1$. □

In conclusion we have:

Theorem 3.5.
(i) *If $p_g \geq 8$, then the intersection of all the quadrics containing the canonical image Σ of V is a linearly normal threefold X of degree $\delta \leq p_g - 1$ whose curve sections have arithmetic genus ≤ 2;*
(ii) *if $p_g \geq 14$, then X is a rational normal scroll and the moving part $|D|$ of the pull back $|F|$ on V of the ruling of X is a base point free pencil of curves of genus 3.*
(iii) *if $p_g \geq 17$, then the pull back $|F|$ on V of the ruling of X is a base point free pencil of curves of genus 3.*

Proof. Assertion (i) follows immediately from Propositions 3.3 and 3.4.

By Corollary 2.4 and assertion (i), the hypothesis $p_g \geq 14$ ensures that the threefold X is a scroll. Since V is regular by Lemma 3.1, X must be a rational normal scroll. We denote by $|F|$ the pull back on V of the ruling of X and we write $|F| = Z + |D|$, where Z is the fixed part and $|D|$ is the moving part of $|F|$. The pencil $|D|$ has genus ≥ 3 because the canonical map of V is birational.

One has $h_\Sigma(4) \le P_4(V) = 19p_g - 29$. By using relations of the type (2.4), (3.3) etc. for X, its surface and curve sections (cf. [Ha2], Lemma (3.1) and Remark (3.1.1)), one finds $h_X(4) \ge 20p_g - 45$. Hence for $p_g \ge 17$, one has $h_\Sigma(4) < h_X(4)$ and so there is a quartic hypersurface containing Σ but not X. It follows that F has genus 3, $Z = 0$ and the pencil $|F|$ is free.

Assume now $14 \le p_g \le 16$. Arguing as above, one shows $h_\Sigma(5) \le 31p_g - 49$ and $h_X(5) \ge 35p_g - 84$. Thus there is a quintic hypersurface containing Σ and not containing X. It follows that $K_V D \le 5$ and therefore $D^2 = 0$ by the index theorem. Now $K_V D$ is even by the adjunction formula, hence $K_V D = 4$ and $g(D) = 3$. □

4. Canonical double planes with a free $\mathbb{Z}_2^3$-action

4.1. The set-up. In this section we study the following situation:

- S is a minimal surface of general type;
- $\pi\colon V \to S$ is an étale Galois cover with Galois group G isomorphic to $\mathbb{Z}_2^3$;
- the canonical map of V is 2-to-1 onto a rational surface.

Remark 4.1. By [Be1, Corollary 5.8] (cf. [MP1, Proposition 4.1]) a group G that acts freely on a surface whose canonical map is of degree 2 on a rational surface is isomorphic to $\mathbb{Z}_2^k$ for some k.

Denote by ι the involution associated with the canonical map of V. Then ι is in the center of $\mathrm{Aut}(V)$, and it induces an involution $\bar\iota$ of S. Let $\alpha\colon V \to W := V/\iota$ and $\bar\alpha\colon S \to \bar W := S/\bar\iota$ be the quotient maps. Notice that G acts also on W and we have the following commutative diagram:

$$\begin{array}{ccc} V & \xrightarrow{\alpha} & W \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\bar\pi} \\ S & \xrightarrow{\bar\alpha} & \bar W \end{array} \tag{4.1}$$

where the vertical maps are G-covers.

Let $\tilde V$ be the blow up of V at the isolated fixed points of ι. Since the set of isolated fixed points of ι is G-invariant, the action of G on V lifts to an action on $\tilde V$, which is again free. The involution ι lifts to an involution of $\tilde V$ that we denote by the same letter. The quotient surface $\tilde W := \tilde V/\iota$ is smooth and the natural map $\tilde W \to W$ is the minimal resolution of the singularities of W.

4.2. The minimal model of $(\tilde W, G)$. Clearly, the group G acts also on $\tilde W$. Then (cf. [BB], [Zh]) $\tilde W$ dominates a *minimal pair* (X, G), namely there is a smooth rational surface X with the following properties:

- the group G acts on X and there is a birational G-equivariant morphism $\tilde W \to X$;
- every birational G-equivariant morphism $X \to X'$, where X' is a smooth rational surface with a G-action, is an isomorphism.

By [Zh, Theorem 4] there are the following possibilities for X:

(a) The group $\mathrm{Pic}(X)^G$ has rank ≥ 2. In this case, there is a G-invariant Mori fibration $f\colon X \to \mathbb{P}^1$. In particular, the smooth fibres of f are isomorphic to $\mathbb{P}^1$ and the singular fibres of f are the union of two irreducible -1-curves meeting in a point.

(b) The group $\mathrm{Pic}(X)^G$ has rank 1. In this case, X is a Del Pezzo surface.

The following elementary observation is the key of our analysis of the G-action on $\tilde{W}$.

Lemma 4.2. *The stabilizer $G_P < G$ of any point $P \in \tilde{W}$ is cyclic.*

Proof. Let $P' \in \tilde{V}$ be a point that maps to P. Assume by contradiction that there exist nonzero elements $g_1 \neq g_2 \in G_P$. Since G acts freely on $\tilde{V}$, we have $g_1(P') = \iota(P') = g_2(P')$. It follows that P' is a fixed point of $g_1 g_2$ on $\tilde{V}$, a contradiction. □

The statement of Lemma 4.2 holds also for the minimal model (X, G):

Lemma 4.3. *For every $P \in X$ the stabilizer $G_P < G$ is cyclic.*

Proof. Let Y be a smooth surface with a faithful G-action, and a point $P \in Y$ with nontrivial stabilizer G_P. By Cartan's Lemma (see [Car]) there exist local analytic coordinates in a neighbourhood of P with respect to which the action of G_P is linear. Hence G_P is isomorphic either to $\mathbb{Z}_2$ or to $\mathbb{Z}_2^2$. Denote by $B_2(Y)$ the subset of points of Y with stabilizer isomorphic to $\mathbb{Z}_2^2$.

By Lemma 4.2, it is enough to show that if $B_2(Y)$ is nonempty, then $B_2(Y')$ is also nonempty for any pair (Y', G) that dominates (Y, G). We will prove this by induction on the number k of blow-ups of which the map $\varepsilon\colon Y' \to Y$ is composed.

Let $P \in Y$ be a fundamental point of ε^{-1} and consider the blow up $Y'' \to Y$ of all the points in the G-orbit of P. Then the action of G on Y induces an action on Y'' and ε is the composition of the G-equivariant birational maps $Y' \to Y''$ and $Y'' \to Y$. If $P \notin B_2(Y)$, then $B_2(Y'')$ contains the inverse image of $B_2(Y)$, hence it is nonempty. The claim now follows by induction, since the number of blowups in $Y' \to Y''$ is strictly smaller than k. So assume that $P \in B_2(Y)$. Then by Cartan's Lemma, there are generators g_1, g_2 of G_P and local analytic coordinates (x, y) near P such that G_P acts by:

$$(x, y) \overset{g_1}{\mapsto} (x, -y); \quad (x, y) \overset{g_2}{\mapsto} (-x, y). \tag{4.2}$$

In particular, the fixed loci of g_1 and g_2 are divisorial near P. Let $E_P \subset Y''$ be the exceptional divisor over P. Then (4.2) shows that $g_1 g_2$ acts trivially on E_P. Hence the points of intersection of E_P with the strict transforms of the divisorial part of the fixed loci of g_1 and g_2 are in $B_2(Y'')$. The claim now follows by induction as in the previous case. □

Proposition 4.4. (i) *The surface X is a G-invariant conic bundle;*

(ii) *the subgroup $H < G$ of the elements that map every fibre of f to iself is cyclic generated by an element $h \neq 0$;*

(iii) *the fixed locus of h is a smooth irreducible bisection D_0 of f of genus $g(D_0) \equiv 1 \mod 4$. There are $2g(D_0) + 2$ singular fibres of f. The divisor D_0 intersects every singular fibre of f at the singular point and h exchanges the two components of each singular fibre;*

(iv) *if $g \in G \setminus H$, then the fixed locus of g consists of two pairs of points, lying on two smooth fibres of f.*

Remark 4.5. An involution h as in Proposition 4.3 is called a *De Jonquières involution* of degree $g(D_0)+2$ (cf. [BB, §2]). Property (iii) implies that X is a minimal model for h.

Proof. We divide the proof into several steps.

Step 1: *If X is a Del Pezzo surface and* $\mathrm{Pic}(X)^G$ *has rank 1, then $K_X^2 = 8$ does not occur.*
Assume by contradiction that this is the case. Then X is either $\mathbb{P}^1 \times \mathbb{P}^1$ or X is the blow up of $\mathbb{P}^2$ in a point. If X is equal to $\mathbb{P}^1 \times \mathbb{P}^1$ then the condition that $\mathrm{Pic}(X)^G$ has rank 1 implies that G contains at least one (and hence four) elements that exchange the two copies of $\mathbb{P}^1$. An element g of this type has the form $(x,y) \mapsto (a^{-1}y, ax)$, where $a \in \mathrm{Aut}(\mathbb{P}^1)$. The fixed locus of g is the graph of the automorphism a of $\mathbb{P}^1$, and therefore it is a curve of type $(1,1)$. Since two such curves have nonempty intersection, we have a contradiction to Lemma 4.3. If X is the blow up of the plane in a point, then every automorphism of X maps the unique -1-curve of X to itself. Since the canonical class is also preserved by every automorphism, G acts trivially on $\mathrm{Pic}(X)$, which has rank 2, contradicting the assumptions.

Step 2: *If X is a Del Pezzo surface and* $\mathrm{Pic}(X)^G$ *has rank 1, then there is precisely one element $h \in G \setminus \{0\}$ such that the fixed locus of h contains a divisor D_h.*
Assume first that the fixed locus of every nonzero element of G has dimension 0. The quotient surface $Y := X/G$ has only nodes as singularities and we have $K_X^2 = 8K_Y^2$, hence we have $K_X^2 = 8$, contradicting Step 1. So there is at least an element h such that the divisorial part D_h of the fixed locus of h is nonempty. Since D_h is invariant under the group G, there exists a positive rational number r such that D_h is numerically equivalent to $-rK_X$. Hence D_h is ample. Since this argument applies to the divisorial part of the fixed locus of any nonzero $g \in G$, by Lemma 4.3 the fixed locus of every nonzero $g \neq h$ is a finite set.

Step 3: *The case X Del Pezzo and* $\mathrm{Pic}(X)^G$ *of rank 1 does not occur, hence there is a G-invariant Mori fibration $f\colon X \to \mathbb{P}^1$.*

By Step 2, there is a subgroup $G_1 < G$ of order 4 such that the fixed locus of every element of G_1 is a finite set. (Recall that by (2.3) the fixed locus of a nonzero element of G_1 consists of 4 points). By Cartan's Lemma the quotient surface $Y := X/G_1$ has only nodes as singularities, and we have $K_X^2 = 4K_Y^2$, hence K_X^2 is divisible by 4. Then we have $K_X^2 = 4$ by Step 1. Notice that in this case K_X is not divisible in $\mathrm{Pic}(X)$, since X is not minimal. It follows that the curve D_h is linearly equivalent to $-rK_X$, where $r > 0$ is an integer. Let ν be the number of isolated fixed points of h. Formula 2.3 gives:

$$4 = \nu - K_X D_h = \nu + rK_X^2 = \nu + 4r,$$

i.e. $\nu = 0$, $r = 1$. Hence the quotient surface $T := X/h$ is smooth. If we denote by $p\colon X \to T$ the quotient map, then the adjunction formula gives $p^*K_T = 2K_X$, hence T is a Del Pezzo surface with $K_T^2 = 8$. The Picard number of T is 2, hence the group $\mathrm{Pic}(X)^h$ has rank 2. The group $G/{<h>}$ acts on T. Let $\gamma \in G/{<h>}$ be a nonzero element and denote by t_γ the trace of γ in $H^2(T,\mathbb{C})$. Then, since the fixed locus of

γ consists of 4 points, the topological trace formula (2.2) gives $2 + t_\gamma = 4$, namely γ acts on $H^2(T, \mathbb{C})$ as the identity. It follows that $\mathrm{Pic}(X)^G$ has rank 2, contradicting the assumptions.

Hence we have proven that $\mathrm{Pic}(X)^G$ has rank greater than 1, and therefore we conclude that X has the structure of a G-invariant conic bundle, proving (i).

Step 4: $X = \mathbb{P}^1 \times \mathbb{P}^1$ *does not occur.*
Assume that this is the case. By Step 3, the group G acts trivially on $\mathrm{Pic}(X)$, hence in particular it preserves the projections $p_i \colon \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$, $i = 1, 2$. Thus every element of G is of the form $(x, y) \mapsto (ax, by)$, where $a, b \in \mathrm{Aut}(\mathbb{P}^1)$ have order 2. Consider the homomorphism $\psi_1 \colon G \to \mathrm{Aut}(\mathbb{P}^1)$ induced by the projection p_1. Since $\mathrm{Aut}(\mathbb{P}^1)$ does not contain a subgroup isomorphic to $\mathbb{Z}_2^3$, the kernel of ψ_1 contains a nontrivial element g_1 of the form $(x, y) \mapsto (x, by)$. By the same argument, G contains also a nontrivial element g_2 of the form $(x, y) \mapsto (ax, y)$. Then the fixed loci of g_1 and g_2 intersect, contradicting Lemma 4.3.

Step 5: *The subgroup $H < G$ consisting of the elements that map the general fibre of f to itself is isomorphic to $\mathbb{Z}_2$. The quotient group G/H acts freely on the set of singular fibres of f.*
By definition, the group H acts faithfully on the general fibre of f, which is a $\mathbb{P}^1$, and G/H acts faithfully on the base of the fibration f, which is also a $\mathbb{P}^1$. Since $\mathbb{P}^1$ has no faithful $\mathbb{Z}_2^3$-action, both H and G/H have order at most 4. In particular, H has either order 2 or 4. Let $h \in H$ be a nonzero element. The fixed locus of h on a general fibre of f consists of two points. Hence the fixed locus of h contains a bisection D_0 of the fibration f. If H has order 4, then f has no reducible fibre since otherwise the singular point of the fibre would be fixed by every element of H, contradicting Lemma 4.3. Hence $f \colon X \to \mathbb{P}^1$ is a $\mathbb{P}^1$-bundle. Since by Lemma 4.3 the fixed loci of the three nonzero elements of H are disjoint, X contains three disjoint bisections. This is possible only if $X = \mathbb{P}^1 \times \mathbb{P}^1$, contradicting Step 4. So H has order 2.

Assume by contradiction that the action of G/H on the set of singular fibres of f is not free. Then there is a singular fibre of f that is mapped to itself by a subgroup $H_0 \subset G$ that contains properly H. It follows that the singular point of the fibre is fixed by H_0, contradicting Lemma 4.3.

Step 6: *Let h be the generator of H. Then the fixed locus of h is a smooth bisection D_0 of f, which intersects the reducible fibres of f at the singular point and the smooth fibres at two distinct points. The two irreducible components of a singular fibre are exchanged by h.*
As we remarked in Step 5, the fixed locus of h contains a bisection D_0 of the fibration f. The divisorial part D_h of the fixed locus of h is smooth, hence D_h does not contain any fibre of f. We claim that if $A_1 + A_2$ is a reducible fibre of f, then h exchanges A_1 and A_2 and, as a consequence, D_h intersects $A_1 + A_2$ at the common point of A_1 and A_2. Assume by contradiction that $h(A_1) = A_1$. By Step 5 the group G/H acts freely on the set of reducible fibres of f, hence the G-orbit of A_1 is a disjoint union of four -1-curves, contradicting the minimality of (X, G).

Now, in order to show that the fixed locus of h coincides with D_0, it is enough to prove that D_0 meets every smooth fibre of f at two distinct points. Assume by contradiction that there exists a point $P \in X$ such that D_0 is tangent at P to the fibre C of f containing P. Then h acts as the identity on the space $T_P D_0 = T_P C$. By Cartan's Lemma this contradicts the fact that h maps C to itself but it does not fix C pointwise.

Step 7: *The curve D_0 is irreducible and $g(D_0) - 1$ is divisible by 4. The fibration f has precisely $2g(D_0) + 2$ reducible fibres.*
By Lemma 4.3, the quotient group G/H acts freely on D_0. Assume that D_0 is reducible. In this case f is a $\mathbb{P}^1$-bundle by Step 6, and D_0 is the disjoint union of two sections of f. Since G/H has order 4 there exists a nonzero $\gamma \in G/H$ that maps each component of D_0 to itself. Since these components are rational curves, this contradicts the fact that G/H acts freely on D_0. Hence D_0 is irreducible and $g(D_0) - 1$ is divisible by 4 by the Hurwitz formula. By Step 6 the singular fibres of f are in 1-to-1 correspondence with the ramification points of the double cover $D_0 \to \mathbb{P}^1$ induced by f, hence there are $2g(D_0) + 2$ of them by the Hurwitz formula.

Step 8: *If $g \notin H$, then the fixed locus of g consists of 4 points.* Consider now an element $g \in G \setminus H$. No reducible fibre of f can be mapped to itself by g, since otherwise the singular point of the fibre would be fixed by both g and h, contradicting Lemma 4.3. Hence the fixed locus of g is contained in two smooth fibres of f and, again by Lemma 4.3, it consists of two pairs of points, one on each fibre. □

Remark 4.6. The minimal pair (X, G) can be constructed as follows. Let $\psi \colon Z \to \mathbb{P}^1$ be a geometrically ruled rational surface and let $C \subset Z$ be a smooth bisection of genus $g(C) > 0$ such that C is divisible by 2 in $\mathrm{Pic}(Z)$. Let P_1, P_2, P_3 be points of C such that for $i = 1, 2, 3$ the curve C is tangent to the fibre of ψ in P_i (there are $2g(C) + 2 \geq 4$ such points by the Hurwitz formula). Let $\varepsilon \colon Z' \to Z$ be the surface obtained by blowing up first the points P_1, P_2, P_3 and then for $i = 1, 2, 3$ the intersection point of the exceptional curve over P_i with the strict transform of the fibre of ψ through P_i. Let $\psi' \colon Z' \to \mathbb{P}^1$ be the fibration induced by ψ. The map ψ' has 3 reducible fibres F_1, F_2, F_3, which are the pull back of the fibres of ψ containing P_1, P_2, P_3, respectively. For every $i = 1, 2, 3$ we write $F_i = 2E_i + N_{2i-1} + N_{2i}$, where E_i is a -1-curve, N_{2i-1} is the strict transform of the fibre of ψ through P_i and N_{2i} is the strict transform of the exceptional curve over P_i. The curves $N_1, \dots N_6$ are disjoint -2-curves. One has:

$$\varepsilon^* C = C' + \sum_{i=1}^{3} (2E_i + N_{2i}),$$

where C' is the strict transform.

Let g_1, g_2, h be generators of G and let $\chi_1, \chi_2, \chi_3 \in G^*$ be the dual basis of characters. Write $g_3 = g_1 g_2$ and set:

$$D_{g_1} = N_1,\ D_{g_2} = N_3,\ D_{g_3} = N_5,\ D_h = C'$$
$$D_{g_1 h_1} = N_2,\ D_{g_2 h_2} = N_4,\ D_{g_3 h_3} = N_6.$$

The reduced fundamental relations (cf. [Pa, Proposition 2.1]) for a G-cover with branch divisors D_g as above are the following:

$$2L_{\chi_1} \equiv D_{g_1} + D_{g_1 h} + D_{g_3} + D_{g_3 h} = N_1 + N_2 + N_5 + N_6,$$
$$2L_{\chi_2} \equiv D_{g_2} + D_{g_2 h} + D_{g_3} + D_{g_3 h} = N_3 + N_4 + N_5 + N_6,$$

$$2L_{\chi_3} \equiv D_h + D_{g_1 h} + D_{g_2 h} + D_{g_3 h} = C' + N_2 + N_4 + N_6.$$

It is not difficult to check that the above relations admit a solution, which is unique since Z' is regular. Hence by [Pa, Proposition 2.1] there exists a G-cover $X' \to Z'$ with reduced building data L_{χ_i}, D_g as above. The surface X' is smooth by [Pa, Proposition 3.1] and it is rational, since the pull back of a general fibre of ψ' is the union of 4 disjoint smooth rational curves. For $i = 1, \dots 6$ the inverse image of N_i is the disjoint union of four -1-curves. Let X be the surface obtained from X' by contracting all these -1-curves. Then the G-action on X' descends to X and (X, G) is a minimal model as in the statement of Proposition 4.4.

4.3. Numerical invariants. Consider now diagram (4.1). The surfaces W and $\bar{W}$ are rational surfaces. The singularities of W and $\bar{W}$ are nodes, which are the images of the isolated fixed points of ι, respectively $\bar{\iota}$. We let ρ, respectively $\bar{\rho}$, denote the number of isolated fixed points of ι, respectively $\bar{\iota}$ and we let B, respectively $\bar{B}$, denote the branch divisor of α, respectively $\bar{\alpha}$. Recall that B and $\bar{B}$ are smooth curves.

Set $G_0 := G \times < \iota > \subseteq \mathrm{Aut}(V)$. For every $\gamma \in G_0$, $\gamma \neq \iota$, denote by n_γ the number of isolated fixed points of γ on V and denote by D_γ the divisorial part of the fixed locus of γ on V. Notice that for $g \in G$ we have $n_g = 0$, $D_g = 0$. Similarly, we denote by Δ_g the divisorial part of the fixed locus of g on W and by ν_g the number of isolated fixed points of g on W. By the commutativity of diagram 4.1, we have:

$$\bar{B} = \bar{\pi}(B) + \sum_{g \neq 1} \bar{\pi}(\Delta_g) \tag{4.3}$$

Analogously, we have the following formula for the number of isolated fixed points:

$$\bar{\rho} = \frac{1}{8}\rho + \frac{1}{8} \sum_{g \neq 1} n_{g\iota} \tag{4.4}$$

Recall (see §4.1) that W has canonical singularities. In particular, it is Gorenstein and 2-factorial, hence the intersection number $K_W D$ is defined for any divisor D of W.

Using the analysis of the previous subsection we can prove:

Proposition 4.7. *Let $1 \neq g \in G$. Then:*

(i) *if $C \subset W$ is a component of Δ_g, then $K_W C \geq 0$, and equality holds if and only if C is a -2-curve;*
(ii) $n_{g\iota} = 2\nu_g = 8 + 2K_W \Delta_g$*;*
(iii) *for $g = h$* (cf. Proposition 4.4)*, the divisor Δ_h contains an hyperelliptic curve D with $g(D) \equiv 1 \mod 4$.*

Proof. Recall that, since the G-action on W is induced by a free action on V, G acts freely on the branch locus of ι, and in particular on the nodes of W. Hence, if we denote by Δ the set of points P of W such that the stabilizer $G_P < G$ is non trivial, then α is étale above Δ. In particular, for every curve $C \subset \Delta$ one has $2K_W C = K_V \alpha^* C$, and statement (i) follows since V is minimal of general type.

The equality $n_{g\iota} = 2\nu_g$ is a consequence of the previous remark and of $n_g = 0$. In addition, we have $\nu_g = 4 + K_W \Delta_g$ by (2.3) and Remark 2.1.

Finally statement (iii) follows directly by Proposition 4.4. □

Corollary 4.8. *One has* $\bar\rho \geq 8$*;*

Proof. The result follows from formula (4.4) and Proposition 4.7. □

Let $\eta\colon S' \to S$ be the blow up of the isolated fixed points of $\bar\iota$ and let $Y \to \bar W$ be the minimal desingularization. Note that $\bar B$, which is a curve on $\bar W$, can be seen as a curve on Y. Let $p\colon S' \to Y$ be the map induced by $S \to \bar W$. The map p is a flat double cover of smooth surfaces, given by an equivalence relation $2L \equiv \bar B + N$, where N is the exceptional divisor of $Y \to \bar W$.

By standard formulae for double covers one has:

$$H^0(K_{S'}) = H^0(K_Y) \oplus H^0(K_Y + L),$$
$$H^0(2K_{S'}) = H^0(2(K_Y + L)) \oplus H^0(2K_Y + L).$$

In the above decompositions the first summand corresponds to "even" sections and the second one to "odd" sections.

The dimension of $H^0(2K_Y + L)$ can be computed as follows (cf. [CCM, §3] for similar computations):

Proposition 4.9. *One has:*
(i) $K_S^2 \equiv \bar\rho \mod 2$*;*
(ii) $2h^0(2K_Y + L) = 6 + K_S^2 - \bar\rho - 2\chi(S)$.

Proof. Since $K_{S'} = p^*(K_Y + L)$, we have:

$$K_S^2 - \bar\rho = K_{S'}^2 = 2(K_Y + L)^2, \tag{4.5}$$

proving statement (i).

Computing $\chi(S) = \chi(S')$ by means of the standard formulae for double covers, we get:

$$L(K_Y + L) = 2\chi(S) - 4. \tag{4.6}$$

By the Kawamata-Viehweg vanishing theorem, we have $h^0(2K_Y + L) = \chi(2K_Y + L)$, hence

$$h^0(2K_Y + L) = 1 + \frac{1}{2}(2K_Y + L)(K_Y + L) = 1 + (K + L)^2 - \frac{1}{2}L(K_Y + L) \tag{4.7}$$

By (4.5) and (4.6), this is equivalent to statement (ii). □

5. The fundamental group of surfaces with $K^2 = 3\chi - 1$

The aim of this section is to prove the following:

Theorem 5.1. *Let S be a minimal complex surface of general type such that $K_S^2 = 3\chi(S) - 1$. If the group $\pi_1^{\mathrm{alg}}(S)$ has order 8, then $\chi(S) = 1$.*

We fix the notation and assumptions that will hold throughout all the section.

We consider a smooth minimal complex surface of general type S such that:

- $K_S^2 = 3\chi(S) - 1$;
- $G := \pi_1^{\mathrm{alg}}(S)$ has order 8.

We denote by $\pi\colon V \to S$ the (algebraic) universal cover of S. The surface V has invariants $K_V^2 = 8K_S^2$, $\chi(V) = 8\chi(S)$, $q(V) = 0$, $p_g(V) = \chi(V) - 1$. In particular one has $K_V^2 = 3\chi(V) - 8 = 3p_g(V) - 5$. We start by analyzing the canonical map of V.

Proposition 5.2. *Set $r := \chi(V) - 2$, let Σ be the canonical image of V and let $\varphi\colon V \to \Sigma \subset \mathbb{P}^r$ be the canonical map. Then there are the following possibilities:*

(i) *φ is a degree 2 map and Σ is a rational surface;*

(ii) *φ is birational and the system $|K_V|$ is free.*

Proof. We start by showing that φ is not composed with a pencil. This also follows from [Ho1] but here we give a direct proof. Assume by contradiction that this is the case and write $|K_V| = Z + |rF|$, where $|F|$ is a pencil. Then:

$$3r - 2 = K_V^2 \geq rK_VF$$

which implies $K_VF \leq 2$. Now the index theorem gives $F^2 = 0$, hence by the adjunction formula one has $K_VF = 2$ and $|F|$ is a G-invariant genus 2 pencil, contradicting Lemma 2.2. So Σ is a surface.

Let d be the degree of φ. Write $|K_V| = Z + |M|$, where Z is the fixed part and $|M|$ is the moving part. If $Z \neq 0$, then $MZ \geq 2$ by the 2-connectedness of canonical divisors. We have the following chain of inequalities:

$$\begin{aligned} 3r - 2 = K_V^2 &= K_VM + K_VZ \geq K_VM = \\ &= M^2 + MZ \geq M^2 \geq d \deg \Sigma. \end{aligned} \tag{5.1}$$

Since a nondegenerate surface of $\mathbb{P}^r$ has degree $\geq r - 1$, (5.1) gives $d \leq 3$.

Assume $d = 3$. In this case (5.1) gives $\deg \Sigma = r - 1$ and $MZ \leq 1$. So we have $MZ = 0$ and $Z = 0$. The relation $K_V^2 = 3 \deg \Sigma + 1$ implies that $|K_V|$ has precisely one base point, which must be fixed by every element of G, contradicting the fact that G acts freely on V. So $d = 3$ does not occur.

Assume $d = 2$. Then (5.1) gives $\deg \Sigma \leq \frac{3}{2}r - 1 < 2r - 2$ (recall that $r \geq 6$) and by [Be1, Lemme 1.4] the surface Σ is ruled. In addition, Σ is regular, since it is dominated by V, and therefore it is a rational surface.

Assume $d = 1$. Then by the Castelnuovo inequality ([Be1, Remarques 5.6, 1)]) we have $\deg \Sigma \geq 3r - 4$. Hence (5.1) gives $MZ \leq 2$. In addition, if $MZ = 2$ then $K_VZ = 0$ and $Z^2 = -2$, hence Z is a connected configuration of -2-curves and the group G maps Z to itself. Considering the action of G on the Dynkin diagram associated to Z, it is easy to check that every element of G either maps a component of Z to itself or it exchanges two components of Z that intersect in a point. In either case, this contradicts the assumption that G act freely on V. This proves $MZ = 0$, hence $Z = 0$ and $|K_V|$ has no fixed part.

Assume that the base locus $\mathcal{B}$ of $|K_V|$ is not empty. Since G maps $\mathcal{B}$ to itself, the cardinality of $\mathcal{B}$ is equal to $8m$ for some positive integer m. Then we have $K_V^2 - 8m = 3r - 2 - 8m \geq \deg \Sigma$, contradicting $\deg \Sigma \geq 3r - 4$. □

The two possibilities for the canonical map of V given in Proposition 5.2 correspond to the situations studied in sections 3 and 4. First of all we rule out case (i), namely we prove:

Proposition 5.3. *The canonical map of V is birational.*

Proof. Assume by contradiction that the canonical map of V is not birational. Then by Proposition 5.2 the canonical map of V is of degree 2 onto a rational surface. By [Be1, Cor. 5.8] (cf. [MP1, Prop. 4.1]) the group G is isomorphic to $\mathbb{Z}_2^3$ and we can apply the results of section 4. Throughout the proof we use freely all the notation of section 4.

Recall that we have $\bar{\rho} \geq 8$ by Corollary 4.8. On the other hand, by Proposition 4.9, (ii), we have $\bar{\rho} \leq K_S^2 - 2\chi(S) + 6 = \chi(S) + 5$. Hence $\chi(S) \geq 3$, and $p_g(S) \geq 2$.

Let D be the smooth curve of W defined in Proposition 4.7, (iii). The intersection number $K_W D$ is strictly positive by Proposition 4.7. In addition, as we observed in the proof of Proposition 4.7, one has $2K_W D = K_V \alpha^* D$. Since the divisor $\alpha^* D$ is clearly G-invariant and G acts freely on V, $K_V \alpha^* D$ is divisible by 8 and we may write $K_W D = 4\epsilon$, for an integer $\epsilon > 0$. Set:

$$4\gamma := \sum_{g \in G \setminus <h>} K_W \Delta_g + K_W(\Delta_h - D) + \frac{1}{2}\rho.$$

Note that since $D \leq \Delta_h$, $\Delta_h - D$ is effective. By Proposition 4.7 (i), K_W is nef on the components of Δ_g and thus $\gamma \geq 0$.

By formulae (4.4) and Proposition 4.7, one has:

$$\bar{\rho} = \epsilon + 7 + \gamma \tag{5.2}$$

So the statement of Proposition 4.9 can be rewritten as follows:

$$2h^0(2K_Y + L) = \chi(S) - 2 - \epsilon - \gamma = p_g(S) - 1 - \epsilon - \gamma \tag{5.3}$$

In particular we conclude that $\epsilon \leq p_g(S) - 1$.

Let $\bar{D} := \bar{\pi}(D)$ and denote again by $\bar{D}$ the corresponding curve in Y. Recalling that $K_Y + L$ pulls back on S' to the canonical bundle and chasing through diagram (4.1), one checks that $(K_Y + L)\bar{D} = \epsilon$. Since $g(\bar{D}) \geq 1$, we obtain $h^0(\bar{D}, \mathcal{O}_{\bar{D}}(K_Y + L)) \leq \epsilon$.

Consider the restriction map:

$$H^0(Y, K_Y + L) \to H^0(\bar{D}, \mathcal{O}_{\bar{D}}(K_Y + L)).$$

Taking into account that $h^0(K_Y + L) = p_g(S)$, we see that we can write $K_Y + L = \bar{D} + A$, where $h^0(Y, A) \geq p_g(S) - \epsilon > 0$.

Since $g(\bar{D}) \geq 1$ and the surface Y is regular, one has $h^0(Y, K_Y + \bar{D}) \geq 1$. Since $2K_Y + L = K_Y + \bar{D} + A$, we conclude that $h^0(Y, 2K_Y + L) \geq p_g(S) - \epsilon$, which contradicts (5.3). □

We notice the following corollary of Proposition 5.2 and Proposition 5.3:

Corollary 5.4. *Let S be a numerical Campedelli surface, namely a smooth minimal surface of general type with $K_S^2 = 2$ and $p_g(S) = 0$. Assume that $\pi_1^{\text{alg}}(S)$ has order 8 and let V be the (algebraic) universal cover of S. Then the system $|K_V|$ is free and the canonical map of V is birational.*

Finally we can give the:

Proof of Theorem 5.1. The system $|K_V|$ is free and the canonical map of V is birational by Proposition 5.2 and Proposition 5.3. The invariants of V satisfy $K_V^2 = 3p_g(V) - 5$. Assume that $\chi(S) \geq 2$. In this case $\chi(V) \geq 16$, hence $p_g(V) \geq 15$ and, by Theorem 3.5 (ii), the intersection of all quadrics containing the canonical image Σ

of V is a rational normal scroll of dimension 3 and the pull back on V of the ruling of X is a free pencil $|F|$ of curves of genus 3. Since, excepting the cone over the smooth quadric of $\mathbb{P}^3$, a rational normal scroll has only one ruling, the pencil $|F|$ is intrinsically attached to V. Therefore $|F|$ is G-invariant, contradicting Lemma 2.2. This proves that $\chi(S) = 1$. □

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