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ON $[A,A]/[A,[A,A]]$ AND ON A W_n-ACTION ON THE CONSECUTIVE COMMUTATORS OF FREE ASSOCIATIVE ALGEBRAS

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Abstract. We consider the lower central series of the free associative algebra A_n with n generators as a Lie algebra. We consider the associated graded Lie algebra. It is shown that this Lie algebra has a huge center which belongs to the cyclic words, and on the quotient Lie algebra by the center there acts the Lie algebra W_n of polynomial vector fields on $\mathbb{C}^n$. We compute the space $[A_n, A_n]/[A_n, [A_n, A_n]]$ and show that it is isomorphic to the space $\Omega^2_{closed}(\mathbb{C}^n) \oplus \Omega^4_{closed}(\mathbb{C}^n) \oplus \Omega^6_{closed}(\mathbb{C}^n) \oplus \dots$.

Introduction

Let A be an associative algebra. A free resolution $\mathcal{R}^\bullet$ of A is a free graded differential algebra $\mathcal{R}^\bullet = \bigoplus R^i$, $i \in \mathbb{Z}_{\leq 0}$, with differential Q of degree $+1$ such that the cohomology of Q is only in degree zero, and is canonically isomorphic to A as algebra. Such a resolution can be used for computation of "higher derived functors" for A. For example, higher cyclic homology of A is the higher derived functor for the functor

$$A \to A/[A,A].$$

It means that for the computation of cyclic homology of A we have to take an arbitrary free resolution $\mathcal{R}^\bullet$ of A and consider the quotient $\mathcal{R}^\bullet/[\mathcal{R}^\bullet, \mathcal{R}^\bullet]$. The differential Q acts in $\mathcal{R}^\bullet/[\mathcal{R}^\bullet, \mathcal{R}^\bullet]$, and cohomology of Q is the higher derived functor of $A \to A/[A,A]$.

It is natural to try to compute "higher derived" for other functors. Surely there is a lot of interesting and important functors, but unfortunately in the most cases it is rather hard to compute the higher derived functors. The cyclic homology is an exception, because it can be expressed in the terms of usual Hochschild homology. Another relatively simple case is the functor of abelianization

$$A \to A^{ab} \simeq A/J,$$

where J is the two-sided ideal generated by the brackets $[a,b]$, $a, b \in A$. Let A be the polynomial algebra $\mathbb{C}[x_1, \dots, x_n]$. A resolution $\mathcal{R}^\bullet$ of A can be constructed in terms of the dual Grassmannian algebra $\Lambda^\bullet(\xi_1, \dots, \xi_n)$. Let $A_+^\vee$ be the kernel of the augmentation map

$$\Lambda^\bullet(\xi_1, \dots, \xi_n) \to \mathbb{C}.$$

Then $A_+^\vee$ is a graded algebra and the resolution $\mathcal{R}^\bullet$ of A is a free graded algebra generated by the dual shifted space $(A_+^\vee)^*[-1]$. The differential in $\mathcal{R}^\bullet$ is given by the coproduct

$$(A_+^\vee)^* \to (A_+^\vee)^* \otimes (A_+^\vee)^*.$$

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On the quotient

$$\mathcal{R}^\bullet/\mathcal{R}^\bullet[\mathcal{R}^\bullet,\mathcal{R}^\bullet]\mathcal{R}^\bullet$$

the differential acts by zero. Therefore the "higher abelianization" of $\mathbb{C}[x_1,\dots,x_n]$ is an algebra of functions on the super vector space $A_+^\vee[-1]$ (for example, the "higher abelianization" of $\mathbb{C}[x_1,x_2]$ is the algebra $\mathbb{C}[u_1,u_2,u_{1,2}]$, $\deg u_i=0$, $\deg u_{1,2}=-1$).

In this paper we study the functor

$$A\to A/[A,[A,A]].$$

In order to say something about the higher functors we need to know what is the quotient $A/[A,[A,A]]$ for a free algebra A. A related question is to determine the higher functors for

$$A\to A/A[A,[A,A]]A$$

Our first result is an explicit computation of the last quotient for the free algebra $T(V)$ of an n-dimensional vector space V; we also denote this algebra by A_n. Let

$$\Omega^\bullet=S^\bullet(V)\otimes\Lambda^\bullet(V)$$

be the de Rham complex of the polynomial differential forms on the dual space V^*. The differential d on $\Omega^\bullet$ determines the bivector field $d\wedge d=\nu$. Note that

$$d^2=\frac{1}{2}[d,d]=0,$$

therefore $[\nu,\nu]=0$ and $\Omega^\bullet$ with the bracket

$$[\omega_1,\omega_2]=(-1)^{\deg\omega_1}\cdot 2d\omega_1\wedge d\omega_2$$

is a $\mathbb{Z}_2$-graded Poisson algebra. A quantization of (Ω,ν) may be given by the very simple formula:

$$\omega_1*\omega_2=\omega_1\wedge\omega_2+(-1)^{\deg\omega_1}d\omega_1\wedge d\omega_2.$$

Denote the result of quantization by $\Omega(V)_*$. Then $\Omega(V)_*$ is a $\mathbb{Z}_2$-graded associative algebra and the algebra of even forms with the quantized product $\Omega^{even}(V)_*$ is a subalgebra of $\Omega(V)_*$ generated by the space $V\hookrightarrow\Omega^0(V)$.

Proposition. *The quotient algebra*

$$B=A_n/A_n[A_n,[A_n,A_n]]A_n$$

is isomorphic to $\Omega^{even}(V)_*$. *The map* $B\to\Omega^{even}(V)_*$ *restricted to the space of generators* V *is the identity map.*

The second commutator

$$[\Omega^{even}(V)_*,[\Omega^{even}(V)_*,\Omega^{even}(V)_*]]$$

vanishes, so the natural homomorphism $A_n\to\Omega^{even}(V)_*$ induces a map

$$\theta:[A_n,A_n]/[A_n,[A_n,A_n]]\to[\Omega^{even}(V)_*,\Omega^{even}(V)_*].$$

It is easy to see that $[\Omega^{even}(V)_*,\Omega^{even}(V)_*]$ coincides with the space of exact (=closed of degree >0) even forms. We prove the map θ is an isomorphism

$$\theta:[A_n,A_n]/[A_n,[A_n,A_n]]\to\Omega^{even>0}_{closed}(V)$$

This result is equivalent to the fact that

$$[A_n,[A_n,A_n]]=[A_n,A_n]\cap(A_n\cdot[A_n,[A_n,A_n]]\cdot A_n)$$

Let us summarize. We consider the following functors on the category of associative algebras:

$$F_1 : B \to B/[B, B],$$
$$F_2 : B \to B/[B, [B, B]],$$
$$G_1 : B \to B/B[B, B]B,$$
$$G_2 : B \to B/B[B, [B, B]]B,$$
$$F_{1,2} : B \to [B, B]/[[B, B], B],$$
$$G_{1,2} : B \to B/(B[B, [B, B]]B + [B, B])$$

Higher derived functors for $F_1, F_2, G_1, G_2, F_{1,2}, G_{1,2}$ can be found without a lot of problems when we know the higher abelianization and cyclic homology.

For example, let $B = \mathbb{C}[x_1, x_2]$. We know that a free resolution of $\mathbb{C}[x_1, x_2]$ is the algebra generated by u_1, u_2 and $u_{1,2}$. The differential Q is defined by

$$Q(u_i) = 0,\ Q(u_{1,2}) = u_1u_2 - u_2u_1.$$

The functor G_2 applied to the resolution gives us the even part of the algebra of forms as a superspace:

$$\mathbb{C}[u_1, u_2, u_{1,2}; du_1, du_2, du_{1,2}].$$

The differential Q acts nontrivially only on $u_{1,2}$: $Qu_{1,2} = du_1 \wedge du_2$. So the higher derived functor for the functor G_2 is just the cohomology of this differential.

Now consider the exact sequence of functors

$$0 \to F_{1,2} \to F_2 \to F_1 \to 0.$$

Using it we can find "higher derived" for F_2.

In the beginning of this work we tried to analyze the numerical results on the dimensions $A^{\ell}_{n,k}$ of graded components for the algebra A_n for $n = 2, 3$ and small k, ℓ, obtained by Eric Rains on MAGMA (see (17) below). Here $A_{n,k}$ is the quotient space of k-commutators of A_n modulo $k+1$-commutators, and $A^{\ell}_{n,k}$ is the component of $A_{n,k}$ of monomials of length ℓ. For example, $A_{n,1} = A_n/[A_n, A_n]$ is the space of cyclic words on n variables, and the dimensions of $A^{\ell}_{n,1}$ grow exponentially as ℓ tends to ∞. Our first observation was the unexpected phenomenon that the spaces $A^{\ell}_{n,k}$ for $k \geq 2$ grow *polynomially* on ℓ. We saw this from (17) for small k, ℓ and $n = 2, 3$. In general, it is still a conjecture.

Then, if they grow polynomially, we tried to think about them as about some tensor fields, more precisely, (maybe not irreducible) W_n-modules. The present paper is the result of our attempt to understand these two phenomena–the polynomial growth of $A^{\ell}_{n.k}$, and a structure of W_n-module on it.

Another strange thing which appeared from our results is that the space $[A_n, A_n]/[A_n, [A_n, A_n]]$ is an *associative algebra*. The algebra structure is very unclear from this definition, but it follows from the description of the last space as $\Omega^{even>0}_{closed*}(V)$. It is a commutative algebra with the usual wedge product of differential forms.

At the moment we can not find such a theory for higher $A_{n,k}$, $k > 2$. We have some conjectures which hopefully will be published somewhere.

Now let us outline the contents of the paper:

In Section 1 we prove "by hand" that $[A_n, A_n]/[A_n, [A_n, A_n]]$ is isomorphic to the space of closed 2-forms on $\mathbb{C}^n$ for $n = 2, 3$.

In Section 2 we develop our main technique, aimed to find $A_n/(A_n[A_n, [A_n, A_n]]A_n)$ and $[A_n, A_n]/[A_n, [A_n, A_n]]$ for general n. We prove our results here modulo Lemma 2.2.2.2 which is proven in Section 3. To prove this Lemma in Section 3 we use a theorem describing all irreducible W_n-modules of a reasonable class.

In Section 4 we define a W_n-action on the quotient (by the center) $\widetilde{gr}(A_n)$ of the associated graded Lie algebra of A_n with respect to the lower central filtration. We also consider many examples there.

1. The isomorphism $[A_n, A_n]/[A_n, [A_n, A_n]] \simeq \Omega^2_{closed}(\mathbb{C}^n)$ for $n = 2, 3$

In this Section we compute "by hand" the quotient $[A_n, A_n]/[A_n, [A_n, A_n]]$ for $n = 2$ and $n = 3$. To make the exposition more clear, we first define the concept of a non-commutative 1-form.

1.1. Non-commutative 1-forms. Let A be an associative algebra. A 1-form on A is a finite sum of expressions $a \cdot db \cdot c$, where $a, b, c \in A$ modulo the following two relations:

(i) the cyclicity: $t \cdot a \cdot db \cdot c = a \cdot db \cdot c \cdot t$ for any $a, b, c, t \in A$,
(2) the Leibniz rule: $d(a \cdot b) = (da) \cdot b + a \cdot db$.

We say that a 1-form on A is exact if it has the form $\omega = da$, $a \in A$.

Consider the space $\Omega^1_A/d\Omega^0_A$ of 1-forms on A modulo exact 1-forms. We can reduce any 1-form to an expression $a \cdot db$ using the cyclicity. Next, modulo the exact forms $a \cdot db + b \cdot da = 0$. Thus, there is a map of the space $\Lambda^2(A) \to \Omega^1_A/d\Omega^0_A$ which is clearly surjective. What is its kernel?

We have the relation: $a_1 d(a_2 a_3) = a_1 d(a_2) a_3 + a_1 a_2 d(a_3) = a_3 a_1 d(a_2) + a_1 a_2 d(a_3)$ which is

$$a_1 \wedge (a_2 a_3) = (a_3 a_1) \wedge a_2 + (a_1 a_2) \wedge a_3 \tag{1}$$

or, in more symmetric form,

$$(a_1 a_2) \wedge a_3 + (a_2 a_3) \wedge a_1 + (a_3 a_1) \wedge a_2 = 0 \tag{2}$$

It is clear that there are no other relations. We proved the following result:

Lemma. *For any associative algebra A, the space $\Omega^1_A/d\Omega^0_A$ is isomorphic to $\Lambda^2(A)/(relations(2))$.*

□

1.2. A Lemma. Consider now the case $A = A_n$, the free associative algebra with n generators over $\mathbb{C}$. We have the following Lemma:

Lemma. (i) $\Lambda^2(A_n)/(relations(2)) = [A_n, A_n]$,
(ii) $\left(\Lambda^2(A_n/[A_n, A_n])\right)/(relations\ (2)) = [A_n, A_n]/[A_n, [A_n, A_n]]$

Proof. For any associative algebra A, we have the short exact sequence:

(3) $$0 \longrightarrow HC_1(A) \longrightarrow \left(\Lambda^2 A\right)/(\textit{relations }(2)) \longrightarrow [A,A] \longrightarrow 0$$

where the last map is the commutator map: $a \wedge b \mapsto [a,b]$. This map is well-defined because of the following relation which holds for any associative algebra A:

$$[a,bc]+[b,ca]+[c,ab]=0$$

The kernel of this map is the first cyclic homology $HC_1(A)$. Now the statement (i) of lemma follows from the fact that $HC_1(A_n)=0$.

Consider A as a Lie algebra with the bracket $[a,b]=a\cdot b-b\cdot a$. Then the short exact sequence (3) is a sequence of A-modules, and the action of A on $HC_1(A)$ is trivial. Consider the corresponding long exact sequence in Lie algebra homology:

$$\cdots \to HC_1(A) \to \left(\Lambda^2(A/[A,A])\right)/(a\wedge(b\cdot c)+b\wedge(c\cdot a)+c\wedge(a\cdot b)=0) \to$$
$$[A,A]/[A,[A,A]] \to 0$$

One should explain the middle term. It is not true that $\Lambda^2(A)/[A,\Lambda^2(A)]$ is $\Lambda^2(A/[A,A])$, and we should use the identity $a\wedge(b\cdot c)+b\wedge(c\cdot a)+c\wedge(a\cdot b)=0$. We have by this identity:

$$[a,b\wedge c]=[a,b]\wedge c+b\wedge[a,c]=a\wedge[b,c]$$

which explains the middle term. Again, the statement (ii) of lemma follows from the fact that $HC_1(A_n)=0$. $\square$

Remark. The definition of a k-form on an associative algebra A, generalizing the definition of 1-form here, is given in [K2]. It is proven there that the de Rham complex of the algebra A_n obeys the Poincare lemma. It is also proven that the space of closed two-forms on A_n is $[A_n,A_n]$. From this point of view, the statement (i) of the Lemma above is the Poincaré lemma for 1-forms.

1.3. The case $n=2$. We denote $A_{n,2}=[A_n,A_n]/[A_n,[A_n,A_n]]$, and we denote by $A_{n,2}^{\ell}$ the graded component in $A_{n,2}$ consisting of the monomials of the length ℓ. We prove here the following theorem:

Theorem. $\dim A_{2,2}^{\ell}=\ell-1$

Proof. By Lemma 1.2 we know that $[A_n,A_n]/[A_n,[A_n,A_n]] \simeq \left(\Lambda^2(A_n/[A_n,A_n])\right)/(\textit{relations }(2))$. The idea is to show that any element in $\left(\Lambda^2(A_2/[A_2,A_2])\right)/(\textit{relations }(2))$ is equivalent to an element of the form $x_1^k\wedge x_2^m$, where $k,m\geq 1$.

On the other hand, the space $\Omega^2_{closed}(\mathbb{C}^2)$ of closed 2-forms of "length" ℓ (the length is the eigenvalue of the operator Lie_e where $e=x_1\frac{\partial}{\partial x_1}+x_2\frac{\partial}{\partial x_2}$ is the Euler vector field) on the 2-dimensional vector space has also dimension $\ell-1$. (We denote by the same letters x_1,x_2 the generators of the algebra A_2, and the coordinates in $\mathbb{C}^2$, which should not lead to a confusion). Denote by $\widetilde{m}$ the image of a monomial $m\in A_2$ in the commutative algebra $\mathbb{C}[x_1,x_2]=A_2/A_2[A_2,A_2]A_2$. Then we have the map $[m_1,m_2]\mapsto d(\widetilde{m_1})\wedge d(\widetilde{m_2})$, which clearly defines a map $[A_2,A_2]/[A_2,[A_2,A_2]]\to \Omega^2_{closed}(\mathbb{C}^2)$. We want to prove that this map is an isomorphism.

We start with the isomorphism $\Omega^1_{A_2}/d\Omega^0_{A_2}\simeq[A_2,A_2]$. Now we consider the space of non-commutative 1-forms adb modulo exact forms and modulo forms of the types

$a \cdot d([x,y])$ and $[a,b]dx$. This space is isomorphic to the quotient $[A_2,A_2]/[A_2,[A_2,A_2]]$. Let us compute this space.

Any 1-form can be reduced to the sum of forms of the type $(dx_1)a(x_1,x_2)$ and $d(x_2)b(x_1,x_2)$ where a and b are non-commutative monomials. Consider the form $(dx_1)a(x_1,x_2)$. Suppose $a = a_1 \cdot a_2$, then in our quotient space

$$(dx_1) \cdot a_1 \cdot a_2 = (dx_1) \cdot a_2 \cdot a_1 \tag{4}$$

Using this operation, we can suppose that $a(x_1,x_2)$ has form $a = x_1^k \cdot a_1$ where a_1 starts with x_2 (or is equal to 1). We have in our quotient space:

$$(dx_1) \cdot x_1^k \cdot a_1 = \frac{1}{k+1}(dx_1^{k+1}) \cdot a_1 \tag{5}$$

Indeed, $(dx_1^{k+1}) \cdot a_1 = \sum_{i=0}^k x_1^i \cdot dx_1 \cdot x_1^{k-i} \cdot a_1$. Consider a summand $x_1^i \cdot dx_1 \cdot x_1^{k-i} \cdot a_1$. Using the cyclic symmetry of 1-forms, the latter is the same that $dx_1 \cdot x_1^{k-i} \cdot a_1 \cdot x_1^i$. Now using the equation (4), we see that it is the same that $dx_1 \cdot x_1^k \cdot a_1$. Equation (5) is proven.

Now we proceed in the same way: we represent a_1 as $a_1 = x_2^m \cdot x_1^n \cdot a_2$. Using the property (4) we see that $dx_1 \cdot x_2^m \cdot x_1^n \cdot a_2 = dx_1 \cdot x_1^n \cdot a_2 \cdot x_2^m = \frac{1}{n} d(x_1^{k+1+n}) \cdot (a_2 \cdot x_2^m)$, and so on. Finally, we obtain that $(dx_1) \cdot a(x_1,x_2)$ is equivalent to a form of the type $d(x_1^N) \cdot x_2^M$, which proves the theorem.

□

1.4. The case $n = 3$.

Theorem. *The dimension of the space $A_{3,2}^{\ell}$ is $\ell^2 - 1$.*

Proof. Again, it is the dimension of the polynomial closed two-forms on $\mathbb{C}^3$ of length ℓ. Our proof is analogous to the proof in the case $n = 2$.

First we reduce a non-commutative 1-form to the form $d(x_1^k) \cdot a(x_2,x_3)$. Then, modulo exact forms, it is $-da(x_2,x_3) \cdot x_1^k$. We can proceed as above to reduce any form to the type $d(x_i^{k_1}) \cdot x_j^{k_2} \cdot x_m^{k_3}$ where $x_j^{k_2}$ and $x_m^{k_3}$ commute ($\{i,j,m\} = \{1,2,3\}$). □

We cannot apply this proof for $n > 3$. On the other hand, computations showed that, starting from $n = 4$, the dimension of the space $A_{n,2}^{\ell}$ is *greater* than the dimension of the corresponding closed 2-forms on $\mathbb{C}^n$ (see (17) in Section 4.2). We give the answer in the next Section.

2. The quotient $[A_n,A_n]/[A_n,[A_n,A_n]]$ for general n

Consider the following algebra structure on the (commutative) even forms on an n-dimensional vector space $\Omega^{even}(\mathbb{C}^n)$:

$$\omega_1 \circ \omega_2 = \omega_1 \wedge \omega_2 + d\omega_1 \wedge d\omega_2 \tag{1}$$

where d is the de Rham differential. Notice that this product is neither commutative nor skew-commutative. Later on, we consider only this algebra structure on Ω^{even}.

Remark. For any (not necessarily commutative) differential graded associative algebra $A^\bullet$ we can define a new algebra $A^\bullet_\star$ with the product

$$a \star b = a \cdot b + (-1)^{\deg a}(da) \cdot (db) \tag{2}$$

(a,b are homogeneous) which is also associative.

We have a map $\varphi_n\colon A_n \to \Omega^{even}(\mathbb{C}^n)_*$ which maps $x_i \in A_n$ to $x_i \in \Omega^0(\mathbb{C}^n)$, and we extend it to A_n in the unique way to get a map of algebras.

2.1.

2.1.1.

Lemma. *The map $\varphi_n\colon A_n \to \Omega^{even}(\mathbb{C}^n)_*$ is surjective.*

Proof. Prove first that any monomial on the coordinates $\{x_i\}$'s belongs to the image. Let M_1 and M_2 be two such monomials belonging to the image of φ_n; we prove that $M_1 \cdot M_2$ also does. If $M_1 = \varphi_n(R_1)$, and $M_2 = \varphi_n(R_2)$, then $M_1 \cdot M_2 = \frac{1}{2}\varphi_n(R_1 \cdot R_2 + R_2 \cdot R_1)$. This proves that any monomial on $\{x_i\}$'s belongs to the image because linear monomials $x_1, \ldots, x_n$ belong to the image by definition. Analogously we prove that any even monomial on $\{dx_i\}$'s belongs to the image, and the general statement. □

2.1.2.

Lemma. (i) *The map φ_n maps the commutator $[A_n, A_n]$ to the closed (=exact) forms of degree > 0 $\Omega^{even+}_{closed}(\mathbb{C}^n)$, and the map $\varphi_n\colon [A_n, A_n] \to \Omega^{even+}_{closed}(\mathbb{C}^n)$ is surjective,*

(ii) *the triple commutator $[A_n, [A_n, A_n]]$ is mapped by φ_n to 0,*

(iii) *the kernel of the map φ_n is $K_n = A_n \cdot [A_n, [A_n, A_n]] \cdot A_n$.*

Proof. (i): it is clear that $[\omega_1, \omega_2] = 2d\omega_1 \wedge d\omega_2$. Therefore, the image $[A_n, A_n]$ belongs to closed forms. Surjectivity can be proved analogously with the lemma above,

(ii) it is clear from (i),

(iii) it follows from (ii) that K_n belongs to the kernel of φ_n, because φ_n is a map of associative algebras, and its kernel is a two-sided ideal. Now it is sufficient to prove that the algebra A_n/K_n is isomorphic under φ_n to $\Omega^{even}(\mathbb{C}^n)_*$. This can be deduced from the following presentation of the *commutative* algebra $\Omega^{even}(\mathbb{C}^n)$ by generators and relations: it is generated by $\{x_i\}$ and $\{dx_i \wedge dx_j\}$ with the usual commutativity relations and the relation

$$(dx_i \wedge dx_j) \cdot (dx_k \wedge dx_l) = -(dx_i \wedge dx_k) \cdot (dx_j \wedge dx_l)$$

Therefore, $\Omega^{even}(\mathbb{C}^n)$ is a commutative algebra generated by $\{x_i\}$ and $\{\eta_{i,j}\}$, where $\eta_{i,j} = -\eta_{j,i}$ and with the relations

$$\eta_{i,j} \cdot \eta_{k,l} + \eta_{i,k} \cdot \eta_{j,l} = 0 \tag{3}$$

Now let us consider the algebra A_n/K_n. It is generated by $\{x_i\}$ and $\{[x_i, x_j]\}$. We should check the relations (3), that is

$$[x_i, x_j] \cdot [x_k, x_l] + [x_i, x_k] \cdot [x_j, x_l] \in K_n \tag{4}$$

This follows from the identity in the free algebra:

$$\begin{aligned} &[x_i, x_j] \cdot [x_k, x_l] + [x_i, x_k] \cdot [x_j, x_l] = \\ &[[x_j, x_k], x_i x_l] + x_i[x_k, [x_j, x_l]] + [[x_i, x_j], x_k]x_l - [[x_i x_l, x_k], x_j] \end{aligned} \tag{5}$$

The Lemma is proven. □

2.2. The main theorem. We prove here the following theorem:

Theorem. *The map φ_n induces an isomorphism* $\varphi_n\colon [A_n, A_n]/[A_n, [A_n, A_n]] \xrightarrow{\sim} \Omega^{even+}_{closed}$.

By Lemma 2.1.2 above, the Theorem follows from the Lemma:

Key-Lemma. *The intersection* $[A_n, A_n] \bigcap (A_n \cdot [A_n, [A_n, A_n]] \cdot A_n) = [A_n, [A_n, A_n]]$.

We prove this Lemma and the Theorem in the rest of this Section and in Section 3.

2.2.1. Consider the space $[A_n, A_n \cdot [A_n, [A_n, A_n]]]$. It is clear that this space belongs to the intersection $[A_n, A_n] \bigcap (A_n \cdot [A_n, [A_n, A_n]] \cdot A_n)$. We first prove the Key-Lemma in this particular case.

Lemma. *The space* $[A_n, A_n \cdot [A_n, [A_n, A_n]]]$ *belongs to* $[A_n, [A_n, A_n]]$.

Proof. Let t_1, t_2, t_3, t_4, t_5 be arbitrary monomials in A_n. We need to prove that

$$[t_1, t_2 \cdot [t_3, [t_4, t_5]]] \in [A_n, [A_n, A_n]] \tag{6}$$

We have:

$$\begin{aligned} [t_1, t_2[t_3, [t_4, t_5]]] &= [t_1, t_2t_3[t_4, t_5]] - [t_1, t_2[t_4, t_5]t_3] \\ &= [t_1, t_2t_3[t_4, t_5]] - [t_1, t_3t_2[t_4, t_5]] + [t_1, [t_3, t_2[t_4, t_5]]] \end{aligned} \tag{7}$$

Notice that the third summand in the last line belongs to $[A_n, [A_n, A_n]]$. Now we apply the identity:

$$[a, bc] + [b, ca] + [c, ab] = 0 \tag{8}$$

We set $a = t_1$, $b = t_2t_3$ or t_3t_2, and $c = [t_4, t_5]$. By (8) we have:

$$\begin{aligned} [t_1, t_2[t_3, [t_4, t_5]]] &= [t_1, t_2t_3[t_4, t_5]] - [t_1, t_3t_2[t_4, t_5]] + [t_1, [t_3, t_2[t_4, t_5]]] \\ &= -[t_2t_3, [t_4, t_5]t_1] - [[t_4, t_5], t_1t_2t_3] \\ &\quad + [t_3t_2, [t_4, t_5]t_1] + [[t_4, t_5], t_1t_3t_2] \\ &\quad + [t_1, [t_3, t_2[t_4, t_5]]] \\ &= [[t_3, t_2], [t_4, t_5]t_1] - [[t_4, t_5], t_1t_2t_3] \\ &\quad + [[t_4, t_5], t_1t_3t_2] + [t_1, [t_3, t_2[t_4, t_5]]] \end{aligned} \tag{9}$$

□

2.2.2. *The proof of the Theorem.* By Lemma 1.2, we have an isomorphism $\theta\colon \left(\Lambda^2(A_n/[A_n, A_n])\right)/(\textit{relations}\ (2)) \xrightarrow{\sim} [A_n, A_n]/[A_n, [A_n, A_n]]$. The map θ is induced by the map $\theta\colon a \wedge b \mapsto [a, b]$.

Recall that we denote by K_n the kernel of the map of algebras $\varphi_n\colon A_n \to \Omega^{even}(\mathbb{C}^n)_*$, that is, the space $K_n = A_n \cdot [A_n, [A_n, A_n]] \cdot A_n$ by Lemma 2.1.2(iii). By Lemma 2.2.1, the bracket $[K_n, A_n] \in [A_n, [A_n, A_n]]$, and, therefore, the map θ defines a map

$$\theta\colon \left(\Lambda^2(A_n/(K_n + [A_n, A_n]))\right)/(\textit{relations}\ (2)) \xrightarrow{\sim} [A_n, A_n]/[A_n, [A_n, A_n]] \tag{10}$$

Here in the last formula we reduce the relation $a \wedge bc + b \wedge ca + c \wedge ab = 0$ modulo $K_n + [A_n, A_n]$.

2.2.2.1.

Lemma. *The space $A_n/(K_n+[A_n,A_n])$ is isomorphic to $\Omega^{even}(\mathbb{C}^n)/\mathrm{Im}d$, and the isomorphism is given by the map φ_n.*

Proof. It follows from Lemma 2.1.1 and 2.1.2 □

2.2.2.2. Now we deduce our Theorem from the following result:

Lemma. *The space $\Lambda^2(\Omega^{even}/\mathrm{Im}d)/(relations\ \omega_1\bigwedge(\omega_2\wedge\omega_3)+\omega_2\bigwedge(\omega_3\wedge\omega_1)+\omega_3\bigwedge(\omega_1\wedge\omega_2)=0)$ is isomorphic to Ω^{even+}_{closed}, and the isomorphism is given by the formula $\alpha\bigwedge\beta\mapsto d\alpha\wedge d\beta$.*

We prove Lemma 2.2.2.2 in the next Section.

3. A proof of Lemma 2.2.2.2

3.1. The irreducible W_n-modules of the class $\mathcal{C}$. We need to prove that the space $\Lambda^2(\Omega^{even}/\mathrm{Im}d)/(relations\ \omega_1\bigwedge(\omega_2\wedge\omega_3)+\omega_2\bigwedge(\omega_3\wedge\omega_1)+\omega_3\bigwedge(\omega_1\wedge\omega_2)=0)$ is isomorphic to Ω^{even+}_{closed}. Both sides are modules over the Lie algebra W_n of polynomial vector fields on an n-dimensional vector space $\mathbb{C}^n$. We are going to define some (very general) class $\mathcal{C}$ of W_n-modules; all W_n-modules we meet here belong to this class.

Consider the vector field $e=\sum_{i=1}^n x_i\frac{\partial}{\partial x_i}$. We characterize the class $\mathcal{C}$ of W_n-modules L by the two conditions:

(i) the operator e is semisimple on L with finite-dimensional eigenspaces,
(ii) the eigenvalues of e are bounded from below on L.

We are going to describe all irreducible modules over W_n of the class $\mathcal{C}$. Let W_n^0 be the Lie subalgebra of W_n of vector fields vanishing at the origin. Then W_n^0 has the subalgebra W_n^{00} of vector fields vanishing at the origin with zero of at least second order. Actually W_n^{00} is an ideal in W_n^0, and $W_n^0/W_n^{00}\xrightarrow{\sim}\mathfrak{gl}_n$.

Let D be a Young diagram, and let F_D be the corresponding $\mathfrak{gl}_n$-module (see [Ful]). Denote by $\mathcal{F}_D$ the coinduced module $\mathcal{F}_D=Hom_{U(W_n^0)}(U(W_n),F_D)$.

Theorem. (i) *All representations $\mathcal{F}_D$ are irreducible except in the case when D is just a column, that is $F_D=\Lambda^i(\mathbb{C}^n)^*$, $\mathcal{F}_D=\Omega^i(\mathbb{C}^n)$; in this case $\mathcal{F}_D$ contains the image of the de Rham differential $d\Omega^{i-1}(\mathbb{C}^n)$, which is irreducible,*
(ii) *the modules $\mathcal{F}_D$ for D not a column, $\Omega^i(\mathbb{C}^n)/d\Omega^{i-1}(\mathbb{C}^n)$, and the trivial representation exhaust all irreducible W_n-modules of the class $\mathcal{C}$.*

3.2. Let V be an n-dimensional vector space, we prefer to work in coordinate-free way. Then $\Omega^i(V)$ is coinduced from $\Lambda^i(V^*)$.

Lemma. *The map $i\colon V^*\otimes(\Omega^{even}(V)/\mathrm{Im}d)\to\Lambda^2(\Omega^{even}/\mathrm{Im}d)/(relations\ \omega_1\bigwedge(\omega_2\wedge\omega_3)+\omega_2\bigwedge(\omega_3\wedge\omega_1)+\omega_3\bigwedge(\omega_1\wedge\omega_2)=0)$ is surjective. Here we consider V^* as linear 0-forms, and the map i is the composition of the inclusion with the subsequent factorization.*

Proof. Denote by $\overline{\Lambda^2(\Omega^{even}/\mathrm{Im}d)}$ the quotient space $\Lambda^2(\Omega^{even}/\mathrm{Im}d)/(relations\ \omega_1\bigwedge(\omega_2\wedge\omega_3)+\omega_2\bigwedge(\omega_3\wedge\omega_1)+\omega_3\bigwedge(\omega_1\wedge\omega_2)=0)$. Write $\omega_1=x_i\wedge\omega_1^{(1)}$. We have: $x_i\omega_1^{(1)}\bigwedge\omega_2+(\omega_1^{(1)}\wedge\omega_2)\bigwedge x_i+(\omega_2 x_i)\bigwedge\omega_1^{(1)}=0$ in the quotient space. The second summand $(\omega_1^{(1)}\wedge\omega_2)\bigwedge x_i$ belongs to $\overline{V^*\otimes(\Omega^{even}(V)/\mathrm{Im}d)}$.

Thus, modulo this image, we can freely move all x_i's from ω_1 to ω_2. We can do it many times. Finally, instead of ω_1 we will have a form without x_i's, that is, a form $dx_{i_1} \wedge \dots \wedge dx_{i_k}$. This form is exact, and therefore is zero in $\Lambda^2(\Omega^{even}/\mathrm{Im}d)$. We are done. $\square$

3.3. Thus, we have a surjective map i of $V^* \otimes \Omega^{2k}(V)/\mathrm{Im}d$ to $\Lambda^2(\Omega^{even}/\mathrm{Im}d)/(relations\ \omega_1 \bigwedge(\omega_2 \wedge \omega_3) + \omega_2 \bigwedge(\omega_3 \wedge \omega_1) + \omega_3 \bigwedge(\omega_1 \wedge \omega_2) = 0)$, and the latter is mapped to $\Omega^{2k+2}_{closed}(V)$ by the formula $\omega_1 \wedge \omega_2 \mapsto d\omega_1 \wedge d\omega_2$. This map j is clearly surjective. We need to prove that the induced map $(V^* \otimes \Omega^{2k}(V)/\mathrm{Im}d)/relations \to \Omega^{2k+2}_{closed}(V)$ is an isomorphism. Our tool is Theorem 4.1.

The main point is that $(V^* \otimes \Omega^{2k}(V)/\mathrm{Im}d)$ is *not* a W_n-module, only $(V \otimes \Omega^{2k}(V)/\mathrm{Im}d)/relations$ is. But it is still a $\mathfrak{gl}_n$-module, and in the proof we will use the representation theory of $\mathfrak{gl}_n$-modules.

First of all, we describe the $\Omega^k(V)$ as a $\mathfrak{gl}_n$-module. The answer is given in Figure 1. It follows from the Littlewood-Richardson rule applied to $\Lambda^k(V^*) \otimes S^N(V^*)$.

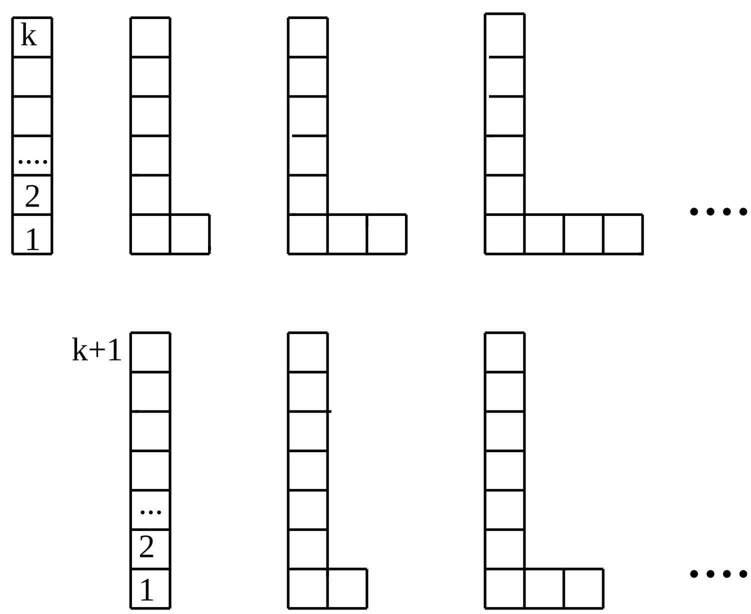

FIGURE 1. The space $\Omega^k(V)$ as a $\mathfrak{gl}_n$-module

As a W_n-module, $\Omega^k(V)$ has a submodule $Im_k = \Omega^k(V)_{closed} = d\Omega^{k-1}(V)$, and the quotient module $\Omega^k(V)/Im_k \simeq Im_{k+1}$ by the Poincaré lemma. It is clear that as a $\mathfrak{gl}_n$-module, the submodule Im_k is the first line in Figure 1, while the quotient-module $\Omega^k(V)/Im_k \simeq Im_{k+1}$ is the second line.

Now we should pass from $\Omega^k(V)/d\Omega^{k-1}(V)$ to $V\otimes(\Omega^k(V)/d\Omega^{k-1}(V))$. The space $\Omega^k(V)/d\Omega^{k-1}(V)$ is the second line in Figure 1, and now we add a new box to each Young diagram by the Littlewood-Richardson rule. We obtain a number of diagrams, but only one among them will be a column (of the height $k+2$).

Now consider the situation of Lemma 2.2.2.2 (k is even, $k=2m$). The only column of the height $2m+2$ maps isomorphically to the corresponding component in $\Omega^{2m+2}(V)_{closed}$ under the map of Lemma 2.2.2.2. Therefore, there are no columns in the kernel of the map $i\colon V^*\otimes(\Omega^{even}(V)/\mathrm{Im}d)\to\Lambda^2\left(\Omega^{even}/\mathrm{Im}d\right)/(relations\ \omega_1\bigwedge(\omega_2\wedge\omega_3)+\omega_2\bigwedge(\omega_3\wedge\omega_1)+\omega_3\bigwedge(\omega_1\wedge\omega_2)=0)$. But when we consider the map $\bar{i}\colon(V^*\otimes(\Omega^{even}(V)/\mathrm{Im}d))/relations\to\Lambda^2\left(\Omega^{even}/\mathrm{Im}d\right)/(relations\ \omega_1\bigwedge(\omega_2\wedge\omega_3)+\omega_2\bigwedge(\omega_3\wedge\omega_1)+\omega_3\bigwedge(\omega_1\wedge\omega_2)=0)$, then its kernel is smaller then the kernel of the map i, and is a W_n-module. It does not contain any column in the $\mathfrak{gl}_n$-decomposition. We know from Theorem 3.1 the description of all W_n-modules of the class $\mathcal{C}$. It could not be $d\Omega^l(V)$ because it does not contain any column. Therefore, it is the coinduced module with some Young diagram D which is not a column. It is clear by the Littlewood-Richardson rule that $D\otimes S^\bullet(V^*)$ is *bigger* than what we have from the second line of Figure 1 multiplied by V^*.

Lemma 2.2.2.2 is proven □

Theorem 2.2 and the Key-Lemma 2.2 are proven. □

4. A W_n-action on $\tilde{gr}A_n$

4.1. The Theorem on W_n-action. Consider the Lie algebra grA_n. Consider its first component $A_n/[A_n,A_n]$. Consider the image under the canonical projection $p\colon A_n\to A_n/[A_n,A_n]$ of the kernel $K_n=A_n[A_n,[A_n,A_n]]A_n$. Denote this image by $\mathcal{Z}$.

Lemma. *$\mathcal{Z}$ belongs to the center of the Lie algebra grA_n.*

Proof. It follows from Lemma 2.2.1. □

Denote the quotient Lie algebra $grA_n/\mathcal{Z}$ by $\widetilde{gr}A_n$. It is $\mathbb{Z}_{\geq 1}$-graded Lie algebra.

Theorem. *On each graded component of $\widetilde{gr}A_n$ there acts the Lie algebra W_n of polynomial vector fields on an n-dimensional vector space. The bracket is W_n-equivariant.*

Proof. Consider the Lie algebra $Der(A_n)$ of the derivations of A_n *as an associative algebra*. This Lie algebra can be easily described: a derivation of a free associative algebra is uniquely defined by its values on the generators $\{x_i\}_{i=1,\dots,n}$, and these values may be arbitrary. This Lie algebra acts on A_n, and this induces an action on any quotient like $A_n/[A_n,A_n]$, $A_n/A_n[A_n,A_n]A_n$, etc. In particular, it acts on $A_n/A_n[A_n,[A_n,A_n]]A_n$. This space is isomorphic to $\Omega^{even}(\mathbb{C}^n)_*$ as an algebra (considered as an algebra in degree 0) by Lemmas 2.1.1 and 2.1.2(iii). Then we have a map $\varphi\colon Der(A_n)\to Der(\Omega^{even}(\mathbb{C}^n)_*)$. Denote by $\Im$ the image of this map.

Lemma. *The Lie algebra $\Im$ acts on $\widetilde{gr}A_n$, and the bracket in $\widetilde{gr}A_n$ is $\Im$-equivariant.*

Proof. We need to prove that $\Im$ acts on each consecutive quotient. For this we need to prove that if we apply a derivation such that all generators x_i are mapped to $A_n[A_n,[A_n,A_n]A_n$ to a k-commutator in A_n, the image will belong to $(k+1)$-commutator. This follows (for any k) immediately from Lemma 2.2.1. □

Now we should just construct a Lie subalgebra isomorphic to W_n in $\Im$. This is the Lie algebra W_n which acts in the natural way on all even forms in $\Omega^{even}(\mathbb{C}^n)_*$. It is clear that this subalgebra belongs to the image $\Im$. Indeed, each derivation of $A_n/A_n[A_n,[A_n,A_n]]A_n$ is defined by its values on the generators $\{x_i\}$. These values are defined up to $A_n[A_n,[A_n,A_n]]A_n$. Take an arbitrary lift of each value in A_n, we get a derivation of A_n. This speculation proves also that the map $Der(A_n) \to Der(A_n/A_n[A_n,[A_n,A_n]]A_n)$ is surjective, and $\Im = Der(\Omega^{even}(\mathbb{C}^n)_*)$. Consider the Lie algebra W_n acting on *commutative* forms $\Omega^{even}(\mathbb{C}^n)$ in the natural way. Then it acts on the quantized algebra $\Omega^{even}(\mathbb{C}^n)_*$ as well. In particular, the canonical W_n acts on $\tilde{gr}A_n$. $\square$

4.2. Examples.

Example. The first graded component is the Lie algebra $\widetilde{gr}^1(A_n) = \Omega^{even}/\mathrm{Im}d$. The bracket $\Lambda^2(\widetilde{gr}^1(A_n)) \to \widetilde{gr}^2(A_n) = [A_n,A_n]/[A_n,[A_n,A_n]] = \Omega^{even+}_{closed}$ is the map $\omega_1 \bigwedge \omega_2 \to (d\omega_1) \wedge (d\omega_2)$ which is clearly W_n-equivariant.

Example. The following computation was made using MAGMA by Eric Rains.

Let $F_1 = A_n$, and $F_k = [A_n, F_{k-1}]$ for $k > 1$. Denote $A_{n,k} = F_k/F_{k+1}$, and denote by $A^\ell_{n,k}$ the graded component of $A_{n,k}$ consisting from the monomials of degree ℓ. Consider the bigraded Hilbert series for A_n:

$$H_n = \sum_{\ell\geq 0,\ k\geq 1} \dim A^\ell_{n,k} u^k t^\ell \tag{1}$$

For $n=2$ and for $n=3$ the bigraded Hilbert series are:

$$
\begin{aligned}
&H_2(u,t) = \\
&\quad (u) \\
&+ (2u)t \\
&+ (3u+u^2)t^2 \\
&+ (4u+2u^2+2u^3)t^3 \\
&+ (6u+3u^2+4u^3+3u^4)t^4 \\
&+ (8u+4u^2+6u^3+8u^4+6u^5)t^5 \\
&+ (14u+5u^2+8u^3+13u^4+15u^5+9u^6)t^6 \\
&+ (20u+6u^2+10u^3+18u^4+26u^5+30u^6+18u^7)t^7 \\
&+ (36u+7u^2+12u^3+23u^4+37u^5+54u^6+57u^7+30u^8)t^8 \\
&+ (60u+8u^2+14u^3+28u^4+48u^5+80u^6+108u^7+110u^8+56u^9)t^9 \\
&+ \mathcal{O}(t^{10})
\end{aligned}
\tag{2}
$$

$$
\begin{aligned}
&H_3(u,t) = \\
&\quad (u) \\
&+ (3u)t \\
&+ (6u+3u^2)t^2 \\
&+ (11u+8u^2+8u^3)t^3 \\
&+ (24u+15u^2+24u^3+18u^4)t^4 \\
&+ (51u+24u^2+48u^3+72u^4+48u^5)t^5 \\
&+ (130u+35u^2+80u^3+162u^4+206u^5+116u^6)t^6 \\
&+ \mathcal{O}(t^7)
\end{aligned}
$$

We will find the consecutive quotients as W_n-modules of the class $\mathcal{C}$, for small k, and $n=2,3$. We have already proved that $A_{n,2}$ is a W_n-module. Consider the space $A_{n,3}$ for $n=2$. We are going to show that the dimensions $\dim A^{\ell}_{2,3}$ are exactly like the character of a W_2-module. Indeed, we know from (2) that $\dim A^{\ell}_{2,3} = 2(\ell-2)$ for $3 \le \ell \le 9$. Consider a W_2-module coinduced from a Young diagram D_1 showed in Figure 2 with a 2-dimensional $\mathfrak{gl}_2$-module on the level $\ell=3$. Then on a level ℓ this W_2-module should have dimension $2S_{\ell-3,2}$ where $S_{k,2}$ is dimension of symmetric polynomials in 2 variables of degree k. We have: $S_{k,2}=k+1$. This shows that the spaces $A^{\ell}_{2,3}$ have the character of a W_2-module for $\ell \le 9$.

Consider now $A_{2,4}$. Here the polynomial for $A^{\ell}_{2,4}$ is $3t^4+8t^5+13t^6+18t^7+23t^8+28t^9+\mathcal{O}(t^{10})$. First take the difference with $3t^4+6t^5+\dots+3(k+1)t^{k+4}+\dots$ which is the character of a W_2-module. The difference is $\sum_{k\ge0} 2(k+1)t^{k+5}$ which is a character of W_2-module.

The case $A_{2,5}$ is analogous.

It is more interesting to consider the case $n = 3$. Consider $A_{3,3}$.

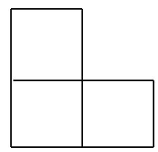

FIGURE 2. The Young diagram D_1

The polynomial for $A^{\ell}_{3,3}$ is $p_{3,3} = 8t^3 + 24t^4 + 48t^5 + 80t^6 + \mathcal{O}(t^7)$. The first coefficient 8 is the dimension of the space generated by Lie words $[x_i, [x_j, x_k]]$ on 3 letters x_1, x_2, x_3 (some of i, j, k may coincide). This space has dimension 8, and as $\mathfrak{gl}_3$-module it corresponds to the Young diagram D_1 (see Figure 2). The irreducible $\mathfrak{gl}_3$-module corresponding to this Young diagram has dimension 8. Consider the W_3-module coinduced from this $\mathfrak{gl}_3$-module. This coinduced character is $8\sum_{k\geq 0} \frac{(k+1)(k+2)}{2} t^{k+3}$. It is exactly our $p_{3,3}$ up to t^6.

Finally, consider $A_{3,4}$.

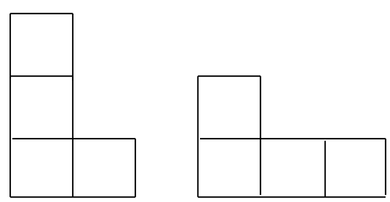

FIGURE 3. The Young diagrams D_2 (Left) and D_3 (right)

Here we have: $p_{3,4} = 18t^4 + 72t^5 + 162t^6 + \mathcal{O}(t^7)$. The first space with dimension 18 is the space of Lie brackets $[x_i, [x_j, [x_k, x_l]]]$ on 3 letters x_1, x_2, x_3. As $\mathfrak{gl}_3$-module, it is the direct sum of two representations with Young diagrams D_2 and D_3 (see Figure 3). The $\mathfrak{gl}_3$-module with the left diagram, D_2, has dimension 3, and the representation with the right diagram, D_3, has dimension 15. The coinduced W_3-module from the direct sum of these two representations on the level 4 has the character $18\sum_{k\geq 0} \frac{(k+1)(k+2)}{2} t^{k+4}$. Subtract this character from $p_{3,4}$. The difference is $18t^5 + 54t^6 + \mathcal{O}(t^7)$ which again coincides with the character of the coinduced module from $D_4 \oplus D_5$ up to level 5 (see Figure 4). The irreducible $\mathfrak{gl}_3$-module corresponding to the Young diagram D_4 has dimension 3, and the irreducible $\mathfrak{gl}_3$-module corresponding to D_5 has dimension 15. The number of boxes in the Young diagram should be equal to the length of the words in the corresponding representation of $\mathfrak{gl}_3$, that is, to the level ℓ.

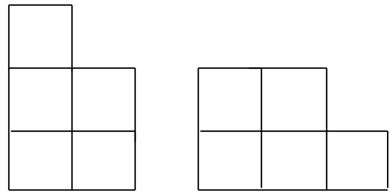

FIGURE 4. The Young diagrams D_4 (Left) and D_5 (right)

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