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INTERMEDIATE JACOBIANS AND ADE HITCHIN SYSTEMS

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Abstract. Let Σ be a smooth projective complex curve and $\mathfrak{g}$ a simple Lie algebra of type ADE with associated adjoint group G. For a fixed pair $(\Sigma, \mathfrak{g})$, we construct a family of quasi-projective Calabi-Yau threefolds parameterized by the base of the Hitchin integrable system associated to $(\Sigma, \mathfrak{g})$. Our main result establishes an isomorphism between the Calabi-Yau integrable system, whose fibers are the intermediate Jacobians of this family of Calabi-Yau threefolds, and the Hitchin system for G, whose fibers are Prym varieties of the corresponding spectral covers. This construction provides a geometric framework for Dijkgraaf-Vafa transitions of type ADE. In particular, it predicts an interesting connection between adjoint ADE Hitchin systems and quantization of holomorphic branes on Calabi-Yau manifolds.

1. Introduction

Large N duality has been a central element in many recent developments in topological string theory. **A**-model large N duality has led to exact results in the Gromov-Witten theory of quasi-projective Calabi-Yau threefolds equipped with a torus action [2, 5, 6, 1, 16, 15, 7]. **B**-model large N duality [8, 9] predicts a very interesting relation between matrix models, algebraic geometry and integrable systems, whose mathematical structure has not been understood so far. A first step in this direction has been taken in [4], where it was recognized that **B**-model large N duality is intimately connected to Hitchin integrable systems on projective curves. More precisely, the results of [4] relate the A_1 Hitching system on a smooth projective curve Σ of arbitrary genus to the large N limit of a holomorphic brane system. A key element in [4] is the construction of a family $\mathcal{X} \to \mathbf{L}$ of quasi-projective Calabi-Yau threefolds so that the associated intermediate Jacobian fibration $J^3(\mathcal{X}/\mathbf{L})$ is isogenous to the Prym fibration of the A_1 Hitchin system on Σ. Here $\mathbf{L} \simeq H^0(\Sigma, K_\Sigma^{\otimes 2})$ is the base of the A_1 Hitchin system.

The purpose of the present paper is to generalize this construction to ADE Hitchin systems on Σ. For a fixed simple Lie algebra $\mathfrak{g}$ of type ADE, we construct in section 2 a family $\mathcal{X} \to \mathbf{L}$ of quasi-projective Calabi-Yau threefolds parameterized by the base $\mathbf{L}$ of the corresponding ADE Hitchin system. In section 3 (Lemma 3.1) we show that the cohomology intermediate Jacobian $J^3(X)$ of a smooth generic threefold in the family $\mathcal{X} \to \mathbf{L}$ is an Abelian variety. Using the results of [12], it follows that the associated intermediate Jacobian fibration $J^3(\mathcal{X}/\mathbf{L})$ is an algebraic integrable system. For future reference, we will refer to such integrable systems as Calabi-Yau integrable systems.

Our main result (Theorem 3.2, section 3) states that the intermediate Jacobian fibration $J^3(\mathcal{X}/\mathbf{L})$ is isomorphic to the Prym fibration of the ADE Hitchin system for

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the adjoint group G associated to the Lie algebra $\mathfrak{g}$. Moreover, this is an isomorphism of integrable systems.

By analogy with the case of A_1 Hitchin systems considered in [4], this result suggests that an arbitrary ADE Hitchin system is related to the large N limit of a holomorphic brane system. This question will be investigated elsewhere.

There are several possible generalizations of our results inspired by situations often encountered in physics. As explained in [4, Section 3.1], the geometric set-up considered in this paper is related by linearization to the deformation theory of projective Calabi-Yau threefolds with curves of singularities. The case considered in this paper corresponds to Calabi-Yau threefolds with a curve Σ of split ADE singularities. There are two natural generalizations of this set-up. Namely, one can consider curves of ADE singularities with nontrivial monodromy and one can also allow the singularity type to jump at special points on Σ. In the first case, we expect a relation between Calabi-Yau integrable systems and Hitchin systems of type BCFG. In the second case, we expect a similar relation between Calabi-Yau integrable systems and meromorphic Hitchin systems on Σ. Both situations are of physical interest and will be studied in future work.

2. Moduli spaces

In this section we describe the relevant moduli spaces of non-compact Calabi-Yau manifolds, as well as the corresponding universal families. To set things up we fix the following data:

- a finite subgroup $\mathbf{\Gamma} \subset \mathrm{SL}(2, \mathbb{C})$, corresponding via the McKay correspondence to a simple Lie algebra $\mathfrak{g}$ of type ADE. Write $\mathfrak{t} \subset \mathfrak{g}$ for a Cartan subalgebra in $\mathfrak{g}$.
- a fixed integer $g \geq 2$.
- a smooth curve Σ of genus g.
- a $\mathbf{\Gamma}$-equivariant rank two holomorphic vector bundle V on Σ
- an isomorphism $\det V \cong K_\Sigma$.

Remark 2.1. The existence of a $\mathbf{\Gamma}$-equivariant structure can impose a constraint on V [20]:

Type A_1:: If $\mathbf{\Gamma} \leftrightarrow \mathsf{A}_1$, then V is unconstrained.

Type $\mathsf{A}_{n>1}$:: If $\mathbf{\Gamma} \leftrightarrow \mathsf{A}_{n>1}$, then we must have $V = L \oplus (K_\Sigma \otimes L^{-1})$ for some line bundle L on Σ. In fact, V will be polystable if and only if $\deg L = g - 1$.

Type D or E:: If $\mathbf{\Gamma} \leftrightarrow \mathsf{D}_{n>2}$ or $\mathbf{\Gamma} \leftrightarrow \mathsf{E}_{6,7,8}$, then we must have $V = \alpha \oplus \alpha$ for some line bundle α on Σ with $\alpha^{\otimes 2} = K_\Sigma$

These restrictions follow immediately by noticing that $\mathbf{\Gamma}$-equivariance reduces the structure group of V to the centralizer of $\mathbf{\Gamma}$ inside $\mathrm{GL}(2, \mathbb{C})$. In the three cases above this centralizer is $\mathrm{GL}(2, \mathbb{C})$, $\mathbb{C}^\times \times \mathbb{C}^\times$ and $\mathbb{C}^\times$.

To this data we can associate the Calabi-Yau threefold

$$X_0 := \mathrm{tot}(V)/\mathbf{\Gamma}$$

which fibers over Σ with fibers ALE spaces of type $\mathbf{\Gamma}$. This X_0 is the central fiber of a family $\mathcal{X} \to \boldsymbol{L}$ of non-compact Calabi-Yau threefolds that is constructed as follows.

Let W denote the Weyl group of $\mathfrak{g}$ acting on the Cartan algebra $\mathfrak{t}$. Consider the spaces

$$\begin{aligned} \widetilde{\boldsymbol{M}} &= H^0(\Sigma, K_\Sigma \otimes \mathfrak{t}) \\ \boldsymbol{M} &= H^0(\Sigma, K_\Sigma \otimes \mathfrak{t})/W \\ \boldsymbol{L} &= H^0(\Sigma, (K_\Sigma \otimes \mathfrak{t})/W). \end{aligned}$$

Remark 2.2. Notice that by construction $\boldsymbol{M}$ and $\boldsymbol{L}$ are affine algebraic varieties equipped with a natural $\mathbb{C}^\times$-action coming from the dilation action on the vector bundle $K_\Sigma \otimes \mathfrak{t}$. In fact $\boldsymbol{L}$ is isomorphic to a complex vector space of dimension $(g-1)\dim(\mathfrak{g})$ with an appropriately defined $\mathbb{C}^\times$-action. Indeed, the fiber bundle $(K_\Sigma \otimes \mathfrak{t})/W$ can be identified with the associated bundle $K_\Sigma^\times \times_{\mathbb{C}^\times} (\mathfrak{t}/W)$. By Chevalley's theorem [3] a choice of a basis of W-invariant homogeneous polynomials on $\mathfrak{t}$ identifies the cone $\mathfrak{t}/W$ with a vector space of dimension $\mathrm{rk}(\mathfrak{g})$ on which $\mathbb{C}^\times$ acts with weights given by the degrees of the polynomials in this basis. Thus $(K_\Sigma \otimes \mathfrak{t})/W$ can be viewed as a vector bundle of rank $\mathrm{rk}(\mathfrak{g})$ on Σ and so $\boldsymbol{L}$ is the vector space of global sections of this vector bundle. Finally from the definition it is clear that the natural map $\boldsymbol{M} \to \boldsymbol{L}$ is a closed immersion. This realizes $\boldsymbol{M}$ as a closed subcone in $\boldsymbol{L}$.

Consider the $\mathrm{rk}(\mathfrak{g}) + 1$ dimensional manifolds

$$\begin{aligned} \boldsymbol{U} &:= \operatorname{tot}((K_\Sigma \otimes \mathfrak{t})/W) \xrightarrow{\boldsymbol{u}} \Sigma, \\ \widetilde{\boldsymbol{U}} &:= \operatorname{tot}(K_\Sigma \otimes \mathfrak{t}) \xrightarrow{\tilde{\boldsymbol{u}}} \Sigma, \end{aligned}$$

and let $\boldsymbol{\pi} : \widetilde{\boldsymbol{U}} \to \boldsymbol{U}$ be the natural projection. We will make frequent use of the following important proposition, due to Balázs Szendröi [19, 20]:

Proposition 2.3. **(a):** *There exists a family of surfaces $\boldsymbol{q} : \mathcal{Q} \to \boldsymbol{U}$, uniquely characterized by the properties*

- $\mathcal{Q}_{|(\textit{zero section of } \boldsymbol{U} \to \Sigma)} \cong X_0$
- $\mathcal{Q}_{|(\textit{fiber of } \boldsymbol{U} \to \Sigma)} \cong \big(\textit{the miniversal unfolding of } \mathbb{C}^2/\boldsymbol{\Gamma}\big)$

(b): *There exists a family $\hat{\boldsymbol{q}} : \widehat{\mathcal{Q}} \to \widetilde{\boldsymbol{U}}$ of smooth surfaces, together with a map*

$$\begin{tikzcd} \widehat{\mathcal{Q}} \arrow[rr, "\varepsilon"] \arrow[dr] & & \boldsymbol{\pi}^*\mathcal{Q} \arrow[dl] \\ & \widetilde{\boldsymbol{U}} & \end{tikzcd}$$

which is a simultaneous resolution of all fibers of $\boldsymbol{\pi}^*\mathcal{Q} \to \widetilde{\boldsymbol{U}}$.

(c): *For every section $\boldsymbol{l} : \Sigma \to \boldsymbol{U}$ of $\boldsymbol{u}$, the fiber product*

$$X_{\boldsymbol{l}} := \mathcal{Q} \times_{\boldsymbol{q}, \boldsymbol{U}, \boldsymbol{l}} \Sigma$$

is a quasi-projective Gorenstein threefold with a trivial canonical class.

Proof. This is proven in [19, Propositions 2.5, 2.7 and 2.9] in a more general context by a cutting and regluing argument. The idea is to build the family $\widehat{\mathcal{Q}}$ from copies of the Brieskorn-Grothendieck versal deformation of $\mathbb{C}^2/\boldsymbol{\Gamma}$ via the cocycle defining the vector bundle V. In our simpler setup, one can also give a global argument by using the $\mathbb{C}^\times$ invariant Slodowy slice [18] through a subregular nilpotent element. We will not give the details here since this global construction is not essential for our considerations. $\square$

We can now construct our family $\mathcal{X} \to \boldsymbol{L}$. Define $\mathcal{X}$ as the pullback

$$\mathcal{X} := \mathsf{ev}_{\boldsymbol{L}}^* \mathcal{Q},$$

where $\mathsf{ev}_{\boldsymbol{L}} : \boldsymbol{L} \times \Sigma \to \boldsymbol{U}$ is the natural evaluation map. The map $\mathcal{X} \to \boldsymbol{L}$ is the composition $\mathcal{X} \to \boldsymbol{L} \times \Sigma \to \boldsymbol{L}$.

Similarly we can construct a family $\widetilde{\mathcal{X}} \to \widetilde{\boldsymbol{M}}$, where $\widetilde{\boldsymbol{M}} = H^0(\Sigma, \widetilde{\boldsymbol{U}})$ as the pullback

$$\widetilde{\mathcal{X}} := \mathsf{ev}_{\widetilde{\boldsymbol{M}}}^* \widehat{\mathcal{Q}},$$

where $\mathsf{ev}_{\widetilde{\boldsymbol{M}}} : \widetilde{\boldsymbol{M}} \times \Sigma \to \widetilde{\boldsymbol{U}}$ is the natural evaluation map. The projection $\widetilde{\mathcal{X}} \to \widetilde{\boldsymbol{M}}$ is the composition $\widetilde{\mathcal{X}} \to \widetilde{\boldsymbol{M}} \times \Sigma \to \widetilde{\boldsymbol{M}}$.

3. The main theorem

Note that the space $\boldsymbol{L}$ which was defined as moduli of noncompact Calabi-Yau manifolds has also an alternative description as moduli of $\mathfrak{g}$ cameral covers of Σ [10]. Indeed, the reader will recognize $\boldsymbol{L} = H^0(\Sigma, \boldsymbol{U})$ as the base of the Hitchin system

$$\boldsymbol{h} : \mathrm{Higgs}(\Sigma, G) \to \boldsymbol{L}$$

of topologically trivial G-Higgs bundles on Σ, where G is any complex Lie group with Lie algebra $\mathfrak{g}$. The case relevant to us is when $G = G_{\mathrm{ad}}$ is the adjoint form of $\mathfrak{g}$. In fact we can construct the universal $\mathfrak{g}$-cameral cover over $\boldsymbol{L}$ as the pullback

$$\widetilde{\boldsymbol{\Sigma}} := \mathsf{ev}_{\boldsymbol{L}}^* \widetilde{\boldsymbol{U}}.$$

The Hitchin fibration $\boldsymbol{h}$ is known [13, 11] to be a torsor over the relative Prym fibration $\mathrm{Prym}_G(\widetilde{\boldsymbol{\Sigma}}/\Sigma) \to \boldsymbol{L}$ for the cameral covers.

Consider the discriminant locus $\Delta \subset \boldsymbol{L}$. By this we mean the locus of all $\ell \in \boldsymbol{L} = H^0(\Sigma, \boldsymbol{U})$ which fail to be transversal to the branch divisor of the cover $\widetilde{\boldsymbol{U}} \to \boldsymbol{U}$. Outside of Δ both fibrations $\mathcal{X} \to \boldsymbol{L}$ and $\widetilde{\boldsymbol{\Sigma}} \to \boldsymbol{L}$ are smooth. Fix $\ell \in \boldsymbol{L}$ outside of the discriminant. We get a smooth Calabi-Yau $\pi : X \to \Sigma$ and a smooth cameral cover $p : \widetilde{\Sigma} \to \Sigma$ corresponding to ℓ. This geometry gives rise to two natural complex abelian Lie groups: the cohomology intermediate Jacobian $J^3(X)$ of X, and the Prym variety $\mathrm{Prym}_G(\widetilde{\Sigma}, \Sigma)$ of the cover $p : \widetilde{\Sigma} \to \Sigma$.

These groups have the following explicit description. Let $\boldsymbol{\Lambda}_G$ be the group of cocharacters of the maximal torus $\boldsymbol{T}_G \subset G$ of G. Since G is adjoint, $\boldsymbol{\Lambda}_G$ is naturally the weight lattice of the Langlands dual Lie algebra ${}^L\mathfrak{g}$, which is identified with $\mathfrak{g}$ since our Lie algebra is simply laced. The case of our theorem when $G = SO(3)$ was proven in [4]. From now on we will therefore assume that G is not of type A_1. In this case, the Prym is

$$\mathrm{Prym}_G(\widetilde{\Sigma}, \Sigma) = H^1\left(\Sigma, \left(p_*(\boldsymbol{\Lambda}_G \otimes \mathcal{O}_{\widetilde{\Sigma}}^\times)\right)^W\right). \tag{1}$$

More generally [11], the Prym associated with a G cameral cover, for any reductive G, is $\mathrm{Prym}(\widetilde{\Sigma},\Sigma) = H^1(\Sigma,\mathcal{T}_G)$, where $\mathcal{T}_G$ is a sheaf of commutative groups on Σ defined as

$$\mathcal{T}_G(U) := \left\{ t \in \Gamma(p^{-1}(U), \mathbf{\Lambda}_G \otimes \mathcal{O}^{\times}_{\widetilde{\Sigma}})^W \,\middle|\, \begin{array}{l}\text{for every root } \alpha \text{ of } \mathfrak{g} \text{ we}\\ \text{have } \alpha(t)_{|D^\alpha} = 1\end{array} \right\}.$$

In this formula we identify $\mathbf{\Lambda}_G \otimes \mathbb{C}^\times$ with $\boldsymbol{T}_G$ and we view a root α as a homomorphism $\alpha : \boldsymbol{T}_G \to \mathbb{C}^\times$. The divisor $D^\alpha \subset \widetilde{\Sigma}$ is the fixed divisor for the reflection $s_\alpha \in W$ corresponding to α. However, it was shown in [11, Theorem 6.5] that

$$\mathcal{T}_G = \left(p_*(\mathbf{\Lambda}_G \otimes \mathcal{O}^{\times}_{\widetilde{\Sigma}}) \right)^W$$

as long as the coroots of the group G are all primitive. As noted in the introduction of [11], this holds for all adjoint groups of type ADE *except* for our excluded case of $SO(3)$. Therefore we have the identity (1).

The intermediate Jacobians of X are Hodge theoretic invariants of the complex structure of X. For general non-compact threefolds, they are generalized tori (= quotients of a vector space by a discrete abelian subgroup) defined in terms of the mixed Hodge structure on the cohomology or the homology of X. However in our case, the third cohomology of X carries a pure Hodge structure as shown in the following lemma, so the intermediate Jacobians will be abelian varieties.

Lemma 3.1. *Suppose that X is a smooth non-compact Calabi-Yau threefold corresponding to $\ell \in \boldsymbol{L} - \Delta$. Then the mixed Hodge structure on $H^3(X,\mathbb{Z})$ is pure of weight 3 and of Hodge type $(1,2)+(2,1)$.*

Proof. To demonstrate the purity of the Hodge structure on $H^3(X,\mathbb{Z})$ we look at the map $\pi : X \to \Sigma$ onto the compact Riemann surface Σ. Let $\mathrm{crit}(\pi) \subset \Sigma$ be the finite set of critical values of π. Set $\Sigma^o := \Sigma - \mathrm{crit}(\pi)$, $X^o := \pi^{-1}(\Sigma^o)$, and let $j : \Sigma^o \hookrightarrow \Sigma$ and $\pi^o : X^o \to \Sigma^o$ denote the natural inclusion and projection maps. By the definition of X, explained in the previous section, the fibers of π^o are complex surfaces isomorphic to smooth fibers of the universal unfolding of the singularity $\mathbb{C}^2/\Gamma$. In particular, the second homology of every fiber of π^o is isomorphic to the root lattice of the Lie algebra $\mathfrak{g}$. By duality the second cohomology of every fiber of π^o is isomorphic to the weight lattice of $\mathfrak{g}$. Moreover, since all these fibers are deformation equivalent to the minimal resolution $\varepsilon : \widehat{\mathbb{C}^2/\boldsymbol{\Gamma}} \to \mathbb{C}^2/\boldsymbol{\Gamma}$ of $\mathbb{C}^2/\boldsymbol{\Gamma}$, it follows that every fiber of π^o is a deformation retract of the exceptional locus of ε and so is homotopy equivalent to a configuration of 2-spheres whose dual graph is the Dynkin diagram of $\mathfrak{g}$. This implies that for every $t \in \Sigma^o$ for the corresponding fiber $Q_t := (\pi^o)^{-1}(t)$ we have

$$H^0(Q_t,\mathbb{Z}) = \mathbb{Z}, \qquad H^2(Q_t,\mathbb{Z}) = \mathbf{\Lambda}_G,$$

and the rest of the cohomology of Q_t vanishes. A similar argument shows that the third cohomology group $H^3(Q_t,\mathbb{Z})$ also vanishes for singular fibers Q_t, with $t \in \mathrm{crit}(\pi)$. Indeed, since the section $\ell \in H^0(\Sigma,\mathbf{U})$ is transversal to the branch divisor of the cover $\widetilde{\mathbf{U}} \to \mathbf{U}$, the singular fibers of $\pi : X \to \Sigma$ have a single node. Therefore they are isomorphic to $\widehat{\mathbb{C}^2/\Gamma}$ with a (-2) curve contracted. In particular the singular fibers have the homotopy type of a tree of rational curves, and their third cohomology vanishes.

Therefore, by the Leray spectral sequence applied to the map $\pi : X \to \Sigma$ we get that

$$H^3(X, \mathbb{C}) = H^1(\Sigma, R^2\pi_*\mathbb{C}) = H^1(\Sigma, j_*R^2\pi^o_*\mathbb{C}).$$

Next observe that the (a priori mixed) Hodge structure on the second cohomology of each Q_t, $t \in \Sigma^o$ is pure and of type $(1, 1)$, and so $R^2\pi^o_*\mathbb{C}$ is a variation of pure Hodge structures of Tate type and weight two. Indeed, it is obvious that the Hodge structure on the second cohomology of $\widehat{\mathbb{C}^2/\mathbf{\Gamma}}$ is pure and of type $(1, 1)$, since the second homology of $\widehat{\mathbb{C}^2/\mathbf{\Gamma}}$ is spanned by the exceptional curves. The fact that Q_t are all deformations of the quasi-projective surface $\widehat{\mathbb{C}^2/\mathbf{\Gamma}}$ and the Gauss-Manin flatness of the weight filtration imply then that $H^2(Q_t, \mathbb{C})$ is pure and of type $(1, 1)$. More explicitly, by considering the versal deformation of the pair consisting of the minimal resolution of $\mathbb{P}^2/\Gamma$ and the divisor at infinity we can argue that each Q_t is a rational surface which admits a normal crossing compactification to a projective rational surface $\overline{Q}_t$ with a tree $D_t = \overline{Q}_t - Q_t$ of rational curves at infinity. Now writing the relative cohomology sequence for $(\overline{Q}_t, Q_t)$ of the pair and using the Gysin map we see that $H^2(Q_t, \mathbb{C})$ is of Tate type.

Finally, for any local system $\mathbb{L}$ of complex vector spaces on Σ^o with finite monodromy group W we know that $H^1(\Sigma, j_*\mathbb{L})$ is the W-invariant subspace of $H^1(\widehat{\Sigma}, j_*\widehat{\mathbb{L}})$, where $\widehat{\Sigma}$ is the W-cover of Σ determined by the monodromy, and $\widehat{\mathbb{L}}$ is the trivial local system on $\widehat{\Sigma}^o$ which is the pullback of $\mathbb{L}$. Now $H^1(\widehat{\Sigma}, j_*\widehat{\mathbb{L}})$ is the cohomology (with constant coefficients) of a smooth compact curve, so it carries a pure Hodge structure of weight 1 and of Hodge type $(0, 1) + (1, 0)$, so the same applies to its W-invariant subspace $H^1(\Sigma, j_*\mathbb{L})$. In our case, this is (up to a Tate twist, i.e. shifting of all types by (1,1)) the same as $H^3(X, \mathbb{C})$. The lemma is proven □

Since X is non-compact we will have to take extra care in distinguishing the intermediate Jacobians associated with the Hodge structures on $H^3(X, \mathbb{Z})$ and $H_3(X, \mathbb{Z})$. We will denote these tori by $J^3(X)$ and $J_3(X)$ respectively. Explicitly

$$J^3(X) = H^3(X, \mathbb{C})/(F^2H^3(X, \mathbb{C}) + H^3(X, \mathbb{Z})), \tag{2}$$

$$J_3(X) = H_3(X, \mathbb{C})/(F^{-1}H_3(X, \mathbb{C}) + H_3(X, \mathbb{Z})), \tag{3}$$

$$= H^3(X, \mathbb{C})/(F^2H^3(X, \mathbb{C}) + H_3(X, \mathbb{Z})), \tag{4}$$

where in the formula (4) the inclusion $H_3(X, \mathbb{Z})/(\text{torsion}) \hookrightarrow H^3(X, \mathbb{C})$ is given by the intersection pairing map on three dimensional cycles in X. More precisely, by the universal coefficients theorem we can identify $H^3(X, \mathbb{Z})/(\text{torsion})$ with the dual lattice $H_3(X, \mathbb{Z})^\vee := \mathrm{Hom}_{\mathbb{Z}}(H_3(X, \mathbb{Z}), \mathbb{Z})$. Combining this identification with the intersection pairing on the third homology of X we get a well defined map

$$\begin{array}{rccc} i: & H_3(X, \mathbb{Z}) & \longrightarrow & H^3(X, \mathbb{Z})/(\text{torsion}) \\ & a & \longmapsto & \langle a, \bullet \rangle \end{array}$$

which is injective on the free part of $H_3(X, \mathbb{Z})$. Combining i with the natural inclusion $H^3(X, \mathbb{Z})/(\text{torsion}) \subset H^3(X, \mathbb{C})$ we obtain the map appearing in (4). Furthermore since i is injective modulo torsion, it follows that the induced surjective map on intermediate Jacobians

$$J_3(X) \to J^3(X) \tag{5}$$

is a finite isogeny of tori. Note that had X been compact, the unimodularity of the Poincare pairing would have implied that (5) is an isomorphism and so we would not have had to worry about the distinction between $J_3(X)$ and $J^3(X)$.

By the previous lemma it follows that if we twist the Hodge structure on $H^3(X, \mathbb{C})$ by a Tate Hodge structure of weight (-2) we will get a pure effective Hodge structure of weight 1. In particular $J^3(X)$ and $J_3(X)$ are both abelian varieties which are dual to each other. The lemma also implies that

$$\begin{aligned} J_3(X) &= H_3(X, \mathbb{Z}) \otimes_{\mathbb{Z}} S^1 \\ J^3(X) &= H^3(X, \mathbb{Z}) \otimes_{\mathbb{Z}} S^1 \end{aligned}$$

as real tori. Furthermore the isogeny (5) can be identified explicitly as

$$\begin{tikzcd} J_3(X) \arrow[rr] \arrow[d, equal] & & J^3(X) \arrow[d, equal] \\ H_3(X,\mathbb{Z}) \otimes_{\mathbb{Z}} S^1 \arrow[rr, "i\otimes \mathrm{id}"'] & & H^3(X,\mathbb{Z}) \otimes_{\mathbb{Z}} S^1 \end{tikzcd}$$

Our main result is

Theorem 3.2. *Suppose G is the adjoint Lie group with Lie algebra $\mathfrak{g}$. Away from the discriminant, the relative Prym fibration* $\mathrm{Prym}_G(\widetilde{\mathbf{\Sigma}}/\Sigma) \to \boldsymbol{L}$ *is isomorphic to the cohomology intermediate Jacobian fibration $J^3(\mathcal{X}/\boldsymbol{L}) \to \boldsymbol{L}$ for the family $\mathcal{X} \to \boldsymbol{L}$. This isomorphism identifies the symplectic structure on Hitchin's space* $\mathrm{Prym}_G(\widetilde{\mathbf{\Sigma}}/\Sigma)$ *with the Poisson structure on $J^3(\mathcal{X}/\boldsymbol{L})$ coming from the Yukawa cubic on $\boldsymbol{L}$* [12].

Proof. First we show that the relative Prym fibration is isomorphic to the cohomology intermediate Jacobian fibration as families of polarized abelian varieties.

We divide the proof of this fact into three steps:

Step 1. $J^3(X) = H^3(X, S^1) \cong H^1(\Sigma, (p_*\mathbf{\Lambda}_G)^W \otimes S^1)$. Indeed, if $\pi : X \to \Sigma$ is the natural map, then we have $R^1\pi_*S^1 = 1$ and $R^3\pi_*S^1 = 1$. This follows from the explicit description of the homotopy type of a smooth fiber Q_t of π given in the proof of Lemma 3.1. By the Leray spectral sequence we get $H^3(X, S^1) = H^1(\Sigma, R^2\pi_*S^1)$. Furthermore

$$R^2\pi_*S^1 \cong (R^2\pi_*\mathbb{Z}) \otimes S^1 \cong (p_*\mathbf{\Lambda}_G)^W \otimes S^1.$$

The first isomorphism follows from the universal coefficients spectral sequence [14] and the divisibility of S^1. The identification $R^2\pi_*\mathbb{Z} \cong (p_*\mathbf{\Lambda}_G)^W$ over Σ follows from the corresponding identification over $\boldsymbol{U}$ which is classical.

Step 2. $H^1(\Sigma, (p_*\mathbf{\Lambda}_G)^W \otimes S^1) \cong H^1(\Sigma, (p_*(\mathbf{\Lambda}_G \otimes S^1))^W)$.

In fact we will show:

Lemma 3.3. *If $p : \widetilde{\Sigma} \to \Sigma$ has simple Galois ramification, then the natural map*

$$\nu : (p_* \mathbf{\Lambda}_G)^W \otimes S^1 \to (p_*(\mathbf{\Lambda}_G \otimes S^1))^W$$

is an isomorphism.

Proof. Since ν is tautologically an isomorphism away from the branch locus of $p : \widetilde{\Sigma} \to \Sigma$, we need only check that ν is an isomorphism of stalks at the branch points of p.

Suppose $b \in \Sigma - \Sigma^o$ is a branch point. We have

$$\begin{aligned} ((p_* \mathbf{\Lambda}_G)^W)_b &\cong \mathbf{\Lambda}_G^{\rho_\alpha} \\ ((p_*(\mathbf{\Lambda}_G \otimes S^1))^W)_b &\cong (\mathbf{\Lambda}_G \otimes S^1)^{\rho_\alpha}. \end{aligned} \tag{6}$$

where $\rho_\alpha : \mathbf{\Lambda}_G \to \mathbf{\Lambda}_G$ is the reflection corresponding to a root α. Indeed observe that

$$\begin{aligned} p_* \mathbf{\Lambda}_G &= i_*(p^o_* \mathbf{\Lambda}_G) \\ p_*(\mathbf{\Lambda}_G \otimes S^1) &= i_*(p^o_*(\mathbf{\Lambda}_G \otimes S^1)), \end{aligned}$$

where

$$\begin{array}{ccc} \widetilde{\Sigma} & \supset & \widetilde{\Sigma}^o \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle p^o} \\ \Sigma & \supset & \Sigma^o \end{array}$$

and p^o denotes the part of p away from ramification. In particular if $x \in \Sigma^o$ is a point near b, we have that

$$\begin{aligned} (p_* \mathbf{\Lambda}_G)_b &= (p_* \mathbf{\Lambda}_G)_x^{\mathrm{mon}_x} \\ &= (\mathrm{Fun}(p^{-1}(x), \mathbf{\Lambda}_G))^{\mathrm{mon}_x} \\ &= (\mathrm{Fun}(W, \mathbf{\Lambda}_G))^{\mathrm{mon}_x} \\ &= \mathrm{Fun}(W/s_\alpha, \mathbf{\Lambda}_G). \end{aligned}$$

Therefore

$$(p_* \mathbf{\Lambda}_G)_b^W = \mathrm{Fun}(W/s_\alpha, \mathbf{\Lambda}_G)^W = \mathbf{\Lambda}_G^{\rho_\alpha}.$$

An analogous argument gives the second identity in (6).

Thus our lemma is equivalent to showing that the natural map

$$\nu_b : \mathbf{\Lambda}_G^{\rho_\alpha} \otimes S^1 \to (\mathbf{\Lambda}_G \otimes S^1)^{\rho_\alpha}$$

is an isomorphism for one (hence all) roots α.

We will analyze the ADE types separately.

Suppose G is of type A_n. The short exact sequence

$$1 \to GL(1) \to GL(n+1) \to \mathbb{P}GL(n+1) \to 1$$

induces a short exact sequence of cocharacter lattices

$$0 \to \mathbf{\Lambda}_{GL(1)} \to \mathbf{\Lambda}_{GL(n+1)} \to \mathbf{\Lambda}_{\mathbb{P}GL(n+1)} \to 0.$$

Explicitly this is the sequence

$$0 \to \mathbb{Z} \to \mathbb{Z}^{n+1} \to \mathbf{\Lambda}_{\mathbb{P}GL(n+1)} \to 0,$$

where the map $\mathbb{Z} \to \mathbb{Z}^{n+1}$ is given by $1 \mapsto (1, 1, \ldots, 1)$.

Choose $\alpha = [(0,0,\dots,0,1,-1)]$. Then ρ_α is the transposition

$$(\lambda_1,\dots,\lambda_n,\lambda_{n+1}) \to (\lambda_1,\dots,\lambda_{n+1},\lambda_n)$$

and so $[(\lambda_1,\dots,\lambda_n,\lambda_{n+1})] \in \mathbf{\Lambda}_{\mathbb{P}GL(n+1)}$ is in $\mathbf{\Lambda}^{\rho_\alpha}_{\mathbb{P}GL(n+1)}$ if and only if $\lambda_n = \lambda_{n+1}$. Similarly $\boldsymbol{x} := [(x_1,\dots,x_{n+1})] \in \mathbf{\Lambda}_{\mathbb{P}GL(n+1)} \otimes S^1$ satisfies $\rho_\alpha(\boldsymbol{x}) = \boldsymbol{x}$ if and only if

$$(1,\dots,1,x_n x_{n+1}^{-1}, x_n^{-1}x_{n+1}) = (x,\dots,x,x,x)$$

for some $x \in S^1$. Since by assumption $n > 1$ we see that ν_b is surjective, hence an isomorphism. Note that in the excluded case $n = 1$ the map ν_b has a non-trivial cokernel $\mathbb{Z}/2$.

If G is of type D_n, $n > 2$, we use the basic sequence

$$0 \to \mathbf{\Lambda}_{SO(2n)} \to \mathbf{\Lambda}_{\mathbb{P}SO(2n)} \to \mathbb{Z}/2 \to 0$$

and the fact that $\mathbf{\Lambda}_{SO(2n)}$ can be identified with the square lattice $\mathbb{Z}^n$. After tensoring with S^1 we get

$$1 \to \{\pm 1\} \to \mathbf{\Lambda}_{SO(2n)} \otimes S^1 \to \mathbf{\Lambda}_{\mathbb{P}SO(2n)} \otimes S^1 \to 1$$

where the map $\{\pm1\} \to \mathbf{\Lambda}_{SO(2n)} \otimes S^1 = (S^1)^n$ sends (-1) to $(-1,-1,\dots,-1)$. Since we are assuming that $n > 2$, this yields the surjectivity of ν_b. Again in the excluded case of $n = 2$ we have $\operatorname{coker}(\nu_b) = \mathbb{Z}/2$.

Finally if G is of type E_n, for $n = 6,7,8$, then $\mathbf{\Lambda}_G$ can be identified with the quotient:

$$0 \to \mathbb{Z} \to H^2(dP_n,\mathbb{Z}) \to \mathbf{\Lambda}_G \to 0,$$

where dP_n is a general del Pezzo surface of degree $9-n$ and under the map $\mathbb{Z} \to H^2(dP_n,\mathbb{Z})$, the generator $1 \in \mathbb{Z}$ goes to the anti-canonical class $3\ell - \sum_{i=1}^n e_i$. Here we think of dP_n as the blow-up of $\mathbb{P}^2$ at n general points. Its cohomology has an orthogonal basis $\{\ell, e_1,\dots,e_n\}$ satisfying $\ell^2 = 1$ and $e_i^2 = -1$. Geometrically, ℓ is the pullback of the hyperplane class on $\mathbb{P}^2$ and the e_i's are the exceptional curves.

For our root α we take $\alpha = e_1 - e_2$. Then ρ_α acts as the Picard-Lefschetz reflection $\rho_\alpha(x) = x + \langle x, e_1 - e_2\rangle \cdot (e_1 - e_2)$. In particular ρ_α interchanges e_1 with e_2 and fixes the rest of the basis. Now the same reasoning as above shows that ν_b is surjective and hence an isomorphism.

Step 3. The inclusion $S^1 \subset \mathbb{C}^\times$ of groups induces a natural inclusion of sheaves

(7) $$\iota : \mathbf{\Lambda}_G \otimes S^1 \hookrightarrow \mathbf{\Lambda}_G \otimes \mathcal{O}^\times_{\widetilde{\Sigma}}.$$

We claim that ι induces an isomorphism of tori

$$h^1(\iota) : H^1(\Sigma, (p_*(\mathbf{\Lambda}_G \otimes S^1))^W) \xrightarrow{\sim} H^1(\Sigma, (p_*(\mathbf{\Lambda}_G \otimes \mathcal{O}^\times_{\widetilde{\Sigma}}))^W).$$

Indeed, observe that $H^1(\Sigma, (p_*(\mathbf{\Lambda}_G \otimes S^1))^W)$ is isogenous to $H^1(\widetilde{\Sigma}, \mathbf{\Lambda}_G \otimes S^1)^W$ and similarly $H^1(\Sigma, (p_*(\mathbf{\Lambda}_G \otimes \mathcal{O}^\times_{\widetilde{\Sigma}}))^W)$ is isogenous to $H^1(\widetilde{\Sigma}, \mathbf{\Lambda}_G \otimes \mathcal{O}^\times_{\widetilde{\Sigma}})^W$. Under these isogenies the map $h^1(\iota)$ is compatible with the map

$$H^1(\widetilde{\Sigma}, \mathbf{\Lambda}_G \otimes S^1)^W \to H^1(\widetilde{\Sigma}, \mathbf{\Lambda}_G \otimes \mathcal{O}^\times_{\widetilde{\Sigma}})^W$$

and so $h^1(\iota)$ is surjective with at most a finite kernel.

Let C be the cone of the map of sheaves (7). Since the constant sheaf $\mathbb{C}^{\times}_{\widetilde{\Sigma}}$ has a resolution

$$\mathbb{C}^{\times}_{\widetilde{\Sigma}} \to \mathcal{O}^{\times}_{\widetilde{\Sigma}} \to \Omega^{1}_{\widetilde{\Sigma}},$$

and since $\mathbb{C}^{\times} = S^1 \times \mathbb{R}$, it follows that C is quasi-isomorphic to a complex of $\mathbb{R}$-vector spaces on $\widetilde{\Sigma}$ with cohomology sheaves $\mathcal{H}^0 C \cong \mathbf{\Lambda}_G \otimes \Omega^1_{\widetilde{\Sigma}}$ (considered as a sheaf of $\mathbb{R}$-vector spaces, and $\mathcal{H}^1 C \cong \mathbf{\Lambda}_G \otimes \mathbb{R}$. This implies that

$$\operatorname{cone}\left[(p_*(\mathbf{\Lambda}_G \otimes S^1))^W \to (p_*(\mathbf{\Lambda}_G \otimes \mathcal{O}^{\times}_{\widetilde{\Sigma}}))^W \right]$$

is a complex of sheaves of $\mathbb{R}$-vector spaces on Σ and so its hypercohomology can not be a torsion group. This implies that $h^1(\iota)$ is injective and finishes the proof of the identification of the cameral Pryms with the intermediate Jacobians.

To finish the proof of the theorem, it remains to show that the family $\mathcal{X} \to \boldsymbol{L}$ of non-compact Calabi-Yau manifolds gives rise to a Yukawa cubic field on $\boldsymbol{L}$, which coincides with the cubic defining the symplectic structure [12] on the Higgs moduli space. This is equivalent to showing that for a smooth Calabi-Yau X in $\boldsymbol{L}$, there is a unique up to scale non-vanishing holomorphic three form Ω on X, which is compatible with the Seiberg-Witten $\mathfrak{t}$-valued one form η on the corresponding cameral cover $\widetilde{\Sigma}$.

First we recall that the cameral cover $\widetilde{\Sigma}$ was defined as the pullback of the cover $\widetilde{\boldsymbol{U}} \to \boldsymbol{U}$ via a a map $\Sigma \to \boldsymbol{U}$. In this picture, the Seiberg-Witten $\mathfrak{t}$-valued holomorphic one form η on $\widetilde{\Sigma}$ becomes simply the pullback of the tautological section of the pullback of $\mathfrak{t} \otimes K_{\Sigma}$ to $\widetilde{\boldsymbol{U}} = \operatorname{tot}(\mathfrak{t} \otimes K_{\Sigma})$.

Next observe that as long as V is semistable, the singular Gorenstein Calabi-Yau $X_0 = \operatorname{tot}(V)/\Gamma$ has a unique up to scale non-vanishing holomorphic three form Ω_0. Indeed, the ratio of any two such forms will be a global holomorphic function on X_0, and so will pullback to a global holomorphic function on $\operatorname{tot}(V)$. But every such function can be written as a convergent series of functions which are polynomial along the fibers of $\operatorname{tot}(V) \to \Sigma$. However these polynomial functions can be thought of as sections in $S^{\bullet}V^{\vee}$, and since V is semistable of positive degree, it follows that all such sections are constants. By semicontinuity this implies that for $\boldsymbol{l} \in \boldsymbol{L}$ in a small neighborhood of zero, the Calabi-Yau manifold $X_{\boldsymbol{l}}$ has a unique up to scale holomorphic three form $\Omega_{\boldsymbol{l}}$. Since our universal family $\mathcal{X} \to \boldsymbol{L}$ is preserved by the natural action of $\mathbb{C}^{\times}$ on $\boldsymbol{L}$, this shows that $\Gamma(X_{\boldsymbol{l}}, \Omega^3_{X_{\boldsymbol{l}}}) = \mathbb{C}$ for all $\boldsymbol{l}$ in $\boldsymbol{L}$.

Let now $\boldsymbol{l} \in \boldsymbol{L} - \Delta$ and let $\pi : X \to \Sigma$ and $p : \widetilde{\Sigma} \to \Sigma$ be the corresponding Calabi-Yau threefold and cameral cover. We have a commutative diagram of spaces

$$\begin{tikzcd} \widehat{X} \arrow[d, "\varepsilon"'] \arrow[dr, "\hat{\pi}"] & \\ X \times_{\Sigma} \widetilde{\Sigma} \arrow[r, "\tilde{\pi}"] \arrow[d, "f"'] & \widetilde{\Sigma} \arrow[d, "p"] \\ X \arrow[r, "\pi"'] & \Sigma \end{tikzcd}$$

where $\widehat{X} := \widehat{\mathcal{Q}} \times_{\widetilde{\boldsymbol{U}}} \widetilde{\Sigma}$ is the small resolution of $X \times_{\Sigma} \widetilde{\Sigma}$, induced from $\widehat{\mathcal{Q}} \to \mathcal{Q}$. Note that in this diagram $X \times_{\Sigma} \widetilde{\Sigma}$ is Gorenstein and all the other spaces are smooth.

Let K_π, $K_{\tilde{\pi}}$, $K_{\hat{\pi}}$ denote the relative canonical classes of the morphisms π, $\tilde{\pi}$, $\hat{\pi}$. Since the square in the above diagram is a fiber square we have $K_{\tilde{\pi}} = f^*K_\pi$. Since $X \times_\Sigma \widetilde{\Sigma}$ is Gorenstein and $\varepsilon : \widehat{X} \to X \times_\Sigma \widetilde{\Sigma}$ is small, we have that $K_{\widehat{X}} = \varepsilon^* K_{X\times_\Sigma\widetilde{\Sigma}}$ and therefore $K_{\hat{\pi}} = \varepsilon^* K_{\tilde{\pi}}$. This gives an identification

$$(f \circ \varepsilon)^* K_X = K_{\hat{\pi}} \otimes (p \circ \hat{\pi})^* K_\Sigma.$$

Let Ω denote the unique up to scale non-vanishing holomorphic three form on X. Then Ω is a global nowhere vanishing section of K_X and so $(f \circ \varepsilon)^*\Omega$ is a non-vanishing section of $K_{\hat{\pi}} \otimes (p \circ \hat{\pi})^* K_\Sigma = \Omega^2_{\hat{\pi}} \otimes (p \circ \hat{\pi})^* \Omega^1_\Sigma$. Using the section $\widehat{\Omega} := (f \circ \varepsilon)^*\Omega$ we can construct a period map from $\widetilde{\Sigma}$ to the total space of $\mathfrak{t} \otimes p^*K_\Sigma$. Indeed, fix a base point $o \in \widetilde{\Sigma}$ and an identification $H^2(\widehat{X}_o, \mathbb{C}) \cong \mathfrak{t}$. Let $s \in \widetilde{\Sigma}$ and let $v \in (p^*K_\Sigma^{-1})_s$. The contraction of $\widehat{\Omega}_{|\widehat{X}_s}$ with $\hat{\pi}^*(v)$ is a closed two form on $\widehat{X}_s$ which can be transported by the Gauss-Manin connection along a path from s to o to give an element in $\mathfrak{t} = H^2(\widehat{X}_o, \mathbb{C})$. Since by construction $R^2\hat{\pi}_*\mathbb{C}$ is a trivial local system on $\widetilde{\Sigma}$, this construction is independent of the choice of a path and gives a well defined map $p^*K_\Sigma^{-1} \to \mathfrak{t} \otimes \mathcal{O}_{\widetilde{\Sigma}}$ of holomorphic vector bundles, or equivalently a holomorphic section $\hat{\eta}$ in $\mathfrak{t} \otimes p^*K_\Sigma$ on $\widetilde{\Sigma}$. Finally, to show that $\hat{\eta}$ coincides with the Seiberg-Witten form, note that the period map $\hat{\eta}$ is the composition of the inclusion $\widetilde{\Sigma} \hookrightarrow \widetilde{U}$ and the universal period map $\underline{\hat{\eta}} : \widetilde{U} \to \mathfrak{t} \otimes \tilde{u}^*K_\Sigma$ corresponding to the canonical section $\underline{\widehat{\Omega}} \in H^0(\widehat{\mathcal{Q}}, \Omega^2_{\hat{\boldsymbol{q}}} \otimes (\tilde{u} \circ \boldsymbol{q})^* K_\Sigma)$. Using the cut-and-paste construction of $\widehat{\mathcal{Q}}$ from [19] one can check that the universal period map $\underline{\hat{\eta}}$ is given by the tautological section. Indeed, if we choose a local frame of V on an open $D \subset \Sigma$ and if we write ζ for the corresponding local frame of $K_\Sigma \cong \wedge^2 V$, then over the local patch $D \subset \Sigma$ we have $\widetilde{\boldsymbol{U}}_{|D} \cong D \times \mathfrak{t}$, $\widehat{\mathcal{Q}}_{|D} \cong D \times \widehat{Y}$, where $\widehat{Y} \to \mathfrak{t}$ is the Brieskorn-Grothendieck simultaneous resolution of the versal deformation family of $\mathbb{C}/\Gamma$. In these terms we have $\underline{\widehat{\Omega}} = p_D^*\zeta \otimes p_{\widehat{Y}}^*\omega$, where $\omega \in \Omega^2_{\widehat{Y}/\mathfrak{t}}$ is the canonical fiberwise symplectic form on $\widehat{Y}$. Now the statement follows by the well known fact [17, Section 4] that the period map $\mathfrak{t} \to \mathfrak{t}$ given by ω is proportional to the identity, and by the invariance-under-gluing statement of [19]. $\square$

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References

[1] M. Aganagic, A. Klemm, M. Mariño, and C. Vafa, *The topological vertex*, Comm. Math. Phys. **254** (2005), no. 2, 425–478.

[2] M. Aganagic, M. Mariño, and C. Vafa, *All loop topological string amplitudes from Chern-Simons theory*, Comm. Math. Phys. **247** (2004), no. 2, 467–512.

[3] C. Chevalley, *Invariants of finite groups generated by reflections*, Amer. J. Math. **77** (1955) 778–782.
[4] D.-E. Diaconescu, R. Dijkgraaf, R. Donagi, C. Hofman, and T. Pantev, *Geometric transitions and integrable systems*, Nuclear Phys. B **752** (2006), no. 3, 329–390.
[5] D.-E. Diaconescu, B. Florea, and A. Grassi, *Geometric transitions and open string instantons*, Adv. Theor. Math. Phys. **6** (2002), no. 4, 619–642.
[6] ———, *Geometric transitions, del Pezzo surfaces and open string instantons*, Adv. Theor. Math. Phys. **6** (2002), no. 4, 643–702.
[7] D.-E. Diaconescu, B. Florea, and N. Saulina, *A vertex formalism for local ruled surfaces*, Comm. Math. Phys. **265** (2006), no. 1, 201–226.
[8] R. Dijkgraaf and C. Vafa, *Matrix Models, Topological Strings, and Supersymmetric Gauge Theories*, Nucl. Phys. **B 644** (2002), no. 3,. Hep-th/0206255.
[9] ———, *On geometry and matrix models*, Nuclear Phys. B **644** (2002), no. 1-2, 21–39.
[10] R. Donagi, *Spectral covers*, in Current topics in complex algebraic geometry (Berkeley, CA, 1992/93), Vol. 28 of *Math. Sci. Res. Inst. Publ.*, 65–86, Cambridge Univ. Press, Cambridge (1995).
[11] R. Donagi and D. Gaitsgory, *The gerbe of Higgs bundles*, Transform. Groups **7** (2002), no. 2, 109–153.
[12] R. Donagi and E. Markman, *Cubics, integrable systems, and Calabi-Yau threefolds*, in Proceedings of the Hirzebruch 65 Conference on Algebraic Geometry (Ramat Gan, 1993), Vol. 9 of *Israel Math. Conf. Proc.*, 199–221, Bar-Ilan Univ., Ramat Gan (1996).
[13] G. Faltings, *Stable G-bundles and projective connections*, J. Algebraic Geom. **2** (1993), no. 3, 507–568.
[14] B. Iversen, Cohomology of sheaves, Universitext, Springer-Verlag, Berlin (1986), ISBN 3-540-16389-1.
[15] J. Li, C.-C. Liu, K. Liu, and J. Zhou, *A Mathematical Theory of the Topological Vertex* (2004). Math.AG/0408426.
[16] C.-C. Liu, K. Liu, and J. Zhou, *A proof of a conjecture of Mariño-Vafa on Hodge integrals*, J. Differential Geom. **65** (2003), no. 2, 289–340.
[17] N. Shepherd-Barron, *On simple groups and simple singularities*, Israel J. Math. **123** (2001) 179–188.
[18] P. Slodowy, Simple singularities and simple algebraic groups, Vol. 815 of *Lecture Notes in Mathematics*, Springer, Berlin (1980), ISBN 3-540-10026-1.
[19] B. Szendrői, *Artin group actions on derived categories of threefolds*, J. Reine Angew. Math. **572** (2004) 139–166.
[20] B. Szendröi, *Sheaves on fibered threefolds and quiver sheaves* (2005). Math.AG/0506301.

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