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AN INITIAL VALUE PROBLEM FOR TWO-DIMENSIONAL IDEAL INCOMPRESSIBLE FLUIDS WITH CONTINUOUS VORTICITY

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Abstract. We study an initial value problem for the two-dimensional Euler equation. In particular, we consider the case where initial data belongs to a critical or subcritical Besov space, and initial vorticity is continuous with compact support. Under these assumptions, we conclude that the solution to the Euler equation loses an arbitrarily small amount of regularity as time evolves.

1. Introduction

In this paper, we study the Euler Equation modeling incompressible fluid flow on $\mathbb{R}^2$, given by

$$\begin{aligned} &\partial_t v(t,x) = -v\cdot\nabla v - \nabla p,\\ &\nabla\cdot v = 0, \qquad\qquad (1)\\ &v|_{t=0} = v_0, \end{aligned}$$

where the vorticity of the fluid is given by

$$\omega = \omega(v) = \partial_1 v^2 - \partial_2 v^1.$$

It is known that when initial data for (1) is in the supercritical Sobolev space $W^{s+1,p}(\mathbb{R}^2)$, where $sp > 2$, the solution does not lose any regularity as time evolves (see, for example, [4], [5]). Much less is known when initial data belongs to the critical Sobolev spaces, where $sp = 2$, or to the subcritical Sobolev spaces, where $sp < 2$. We therefore restrict our attention to these two cases. In addition, we assume that the initial vorticity is continuous with compact support.

The motivation for this paper is a result of Bahouri and Chemin found in [1], where the authors show that a lower bound for the Sobolev exponent of $\omega(t)$ is determined by the log-Lipschitz norm of $v(t)$. They define

$$V(t) = \sup_{|x-y|\le 1} \frac{|v(t,x)-v(t,y)|}{|x-y|(1-\log|x-y|)},$$

and they prove the following theorem:

Theorem 1. *Let v be a solution to* (1) *such that $\omega(v_0) = \omega_0 \in L^\infty(\mathbb{R}^2)\cap W^{s,p}(\mathbb{R}^2)$, for $sp \le 2$ and $s \in (0,1]$. Fix $s' < s$, and define $\sigma(s',t) = s'\exp(-\int_0^t V(\tau)d\tau)$. Then $\omega(t) \in W^{\sigma(s',t),p}(\mathbb{R}^2)$ for all $t \in \mathbb{R}$.*

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In this paper, we show that, if we also assume ω_0 is continuous with compact support, then we can improve the lower bound for loss of regularity to an arbitrarily small amount. More precisely, we show that, given $\epsilon > 0$ arbitrarily small, and $T > 0$ fixed, if $\omega_0 \in W^{s,p}(\mathbb{R}^2) \cap C_c(\mathbb{R}^2)$ with $sp \leq 2$, $p \in (1, \infty)$, and $s \in (0, 2)$, then $\omega(t)$ belongs to $W^{s-\epsilon,p}(\mathbb{R}^2)$ for all $t \in [0, T]$. As in [1], we study the vorticity equation corresponding to the Euler equation. When $n = 2$, the vorticity equation is given by

$$\begin{aligned} &\partial_t \omega + v \cdot \nabla \omega = 0, \\ &\omega|_{t=0} = \omega_0. \end{aligned} \tag{2}$$

To prove our result, we show that $\omega(t)$ belongs to the Besov space $B^{s-\epsilon}_{p,\infty}$ (see Definition 3) for all t in a finite time interval $[0, T]$. Our general approach is to localize the frequency of the terms of (2), which results in a new equation with a commutator term on the right-hand side:

$$\begin{aligned} &\partial_t \Delta_q \omega + v \cdot \nabla \Delta_q \omega = [v \cdot \nabla, \Delta_q]\omega, \\ &\Delta_q \omega|_{t=0} = \Delta_q \omega_0. \end{aligned} \tag{3}$$

We then prove the necessary estimate on the L^p norm of the commutator on the right-hand side of (3), and apply a Gronwall argument to show that $\omega(t)$ is in $B^{s-\epsilon}_{p,\infty}$.

The main novelty of this paper is that our methods allow us to draw conclusions for $\omega_0 \in B^s_{p,\infty}(\mathbb{R}^2) \cap C_c(\mathbb{R}^2)$ for all $s \in (0, 2)$, whereas, if we restrict our attention to the case $s \in (0, 1]$, we can prove the result by combining methods used in [1] with the following theorem, which we prove in Section 3:

Theorem 2. *Let v be a solution to* (1) *such that $\omega(v_0) = \omega_0 \in C_c(\mathbb{R}^2)$. Let $g(t)$ be the measure-preserving homeomorphism in $\mathbb{R}^2$ satisfying $\partial_t g(t,x) = v(t, g(t,x))$. Given $\delta > 0$ and $T > 0$, it follows that $||g(t)^{-1} - Id||_{C^{1-\delta}} \in L^\infty([0,T])$.*

2. Littlewood-Paley Decomposition and Function Spaces

In this section, we set notation and recall the definitions of the function spaces which we use throughout the proof of our main theorem.

Proposition 1. *There exists two radial functions $\chi \in S$ and $\varphi \in S$ satisfying the following properties:*

(i) supp $\chi \subset \{\xi \in \mathbb{R}^2 : 0 \leq |\xi| \leq \frac{4}{3}\}$,
(ii) supp $\varphi \subset \{\xi \in \mathbb{R}^2 : \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$,
(iii) $\chi(\xi) + \sum_{j=0}^{\infty} \varphi_j(\xi) = 1$,

where $\varphi_j(\xi) = \varphi(2^{-j}\xi)$ (so $\check{\varphi}_j(x) = 2^{jn}\check{\varphi}(2^j x)$).

Proof. See [6]. □

Observe that, if $|j - j'| \geq 2$, then supp $\varphi_j \cap$ supp $\varphi_{j'} = \emptyset$, and, if $j \geq 1$, then supp $\varphi_j \cap$ supp $\chi = \emptyset$.

Definition 1. Let $f \in S'$. We define

$$\Delta_{-1} f = \chi(D) f = \check{\chi} * f.$$

For $j \geq 0$, $\Delta_j f = \check{\varphi}_j * f$.
For $j \leq -2$, $\Delta_j f = 0$.
For $k \in \mathbb{Z}$, $S_k f = \sum_{j=-1}^{k-1} \Delta_j f = \chi(2^{-k}D)f$.

Definition 2. Let $s \in \mathbb{R}$. We define the Zygmund space C_*^s to be the space of tempered distributions f such that

$$||f||_{C_*^s} := \sup_{q \geq -1} 2^{qs}||\Delta_q f||_{L^\infty} < \infty.$$

It is well known that the norm on C_*^s is equivalent to the classical C^s norm when s is not an integer and $s > 0$. For a proof of this, see [3], Proposition 2.3.1.

Definition 3. Let $s \in \mathbb{R}$, $p, q \in [1, \infty]$. We define the inhomogeneous Besov space $B_{p,q}^s$ to be the space of tempered distributions f such that

$$||f||_{B_{p,q}^s} := (\sum_{j=-1}^{\infty} 2^{jqs}||\Delta_j f||_{L^p}^q)^{\frac{1}{q}} < \infty.$$

When $q = \infty$, write

$$||f||_{B_{p,\infty}^s} := \sup_{j \geq -1} 2^{js}||\Delta_j f||_{L^p}.$$

Remark. Note that $B_{\infty,\infty}^s = C_*^s$.

Definition 4. The space of log-Lipschitz functions, denoted by LL, is the space of bounded functions f on $\mathbb{R}^n$ such that

$$||f||_{LL} := ||f||_{L^\infty} + \sup_{|x-y| \leq 1} \frac{|f(x) - f(y)|}{|x-y|(1 - \log|x-y|)} < \infty.$$

An important tool throughout the proof of the main theorem will be a decomposition introduced by J.-M. Bony in [2]. We recall the definition of the paraproduct and remainder used in this decomposition.

Definition 5. Define the paraproduct of two functions f and g by

$$T_f g = \sum_{\substack{i,j \\ i \leq j-2}} \Delta_i f \Delta_j g = \sum_{j=1}^{\infty} S_{j-1} f \Delta_j g.$$

We use $R(f, g)$ to denote the remainder. $R(f, g)$ is given by the following bilinear operator:

$$R(f, g) = \sum_{\substack{i,j \\ |i-j| \leq 1}} \Delta_i f \Delta_j g.$$

Bony's decomposition gives

$$fg = T_f g + T_g f + R(f, g).$$

3. Proof of Theorem 2

It is known (see [10]) that if $||v(t)||_{LL} \in L^1_{loc}(\mathbb{R}^+)$, then the flow $g(t)$ exists, is unique, is a measure-preserving homeomorphism from $\mathbb{R}^2$ to $\mathbb{R}^2$, and satisfies the equation

$$g(t,x) = x + \int_0^t v(\tau, g(\tau,x))d\tau. \tag{4}$$

Furthermore,

$$g(t) - Id \in C^{exp(-\int_0^t ||v(\tau)||_{LL} d\tau)}. \tag{5}$$

From (5), we see that the Holder exponent of $g(t) - Id$ is determined by the log-Lipschitz norm of v. One can characterize log-Lipschitz functions using the following inequality (see [1]):

$$C^{-1}||f||_{LL} \leq ||S_0 f||_{L^\infty} + \sup_{q\geq 1} \frac{||\nabla S_q f||_{L^\infty}}{q+1} \leq C||f||_{LL} \tag{6}$$

for a constant $C > 0$. When computing the Holder exponent of $g(t) - Id$, (5) and (6) motivate us to study the behavior of the quantity $||\nabla S_q v(t)||_{L^\infty}$. In this section, we assume $\omega_0 \in C_c(\mathbb{R}^2)$, and we show that, given $\epsilon > 0$ and $T > 0$, $||\nabla S_q v(t)||_{L^\infty} \leq \epsilon(q+1)$ for sufficiently large q and for all $t \in [0,T]$. We then conclude that $g(t)$ is locally Holder continuous with Holder exponent arbitrarily close to 1. We begin with the following lemma:

Lemma 2. *Let $u \in C_c(\mathbb{R}^2)$. Given $\epsilon > 0$, there exists an $N > 0$ such that $||\Delta_j u||_{L^\infty} < \epsilon$ for all $j > N$.*

Proof. Since $u \in C_c(\mathbb{R}^2)$, we can construct a sequence $(u_k) \subset C_c^\infty(\mathbb{R}^2)$ such that $u_k \to u$ uniformly. Also, since $u_k \in C_c^\infty(\mathbb{R}^2)$, there exists an $N > 0$ so that $||\Delta_j u_k||_{L^\infty} < \frac{\epsilon}{2}$ for $j > N$. Therefore, for k sufficiently large, and for all $j > N$, we have

$$\begin{aligned} ||\Delta_j u||_{L^\infty} &\leq ||\Delta_j(u_k - u)||_{L^\infty} + ||\Delta_j u_k||_{L^\infty} \\ &< \epsilon. \end{aligned}$$

This completes the proof. □

In dimension two, we can rewrite the vorticity equation given in (2) as $\omega(t,x) = \omega_0(g(t)^{-1}(x))$. Therefore, if we assume that $\omega_0 \in C_c$, then $\omega(t) \in C_c$ for all $t \in \mathbb{R}$. We now apply Lemma 2 to $\omega(t)$ and conclude that, for fixed t, given $\epsilon > 0$, there exists N_t such that $\sup_{j>N_t} ||\Delta_j \omega(t)||_{L^\infty} < \epsilon$. In what follows, we need N_t to be time independent. We therefore prove the following lemma:

Lemma 3. *Let v be a solution to* (1) *such that $\omega(v_0) = \omega_0 \in C_c(\mathbb{R}^2)$. Given $\epsilon > 0$ and $T > 0$, there exists an $N = N(T,\epsilon)$ so that $\sup_{j>N} ||\Delta_j \omega(t)||_{L^\infty} < \epsilon$ for all $t \in [0,T]$.*

Proof. To prove the lemma, we use the Ascoli-Arzela Theorem to show that the set $A = \{\omega(t) : t \in [0,T]\}$ is compact in $C(M)$, where M is the compact subset of $\mathbb{R}^2$ containing the supports of all elements of A. The set A is clearly equibounded as

$||\omega(t)||_{L^\infty} = ||\omega_0||_{L^\infty}$ for all $t \in [0,T]$. To show equicontinuity, we observe that, for x_1 and x_2 in $\mathbb{R}^2$ satisfying $g(t_1,x_1) = g(t_2,x_2) = x$, we have,

$$\begin{aligned}|x_1 - x_2| &\le |t_1 - t_2| \sup_{\tau\in[t_1,t_2]} ||v(\tau)||_{L^\infty} \\ &\le C|t_1 - t_2|(||\omega_0||_{L^p} + ||\omega_0||_{L^\infty}),\end{aligned}$$

where we used the bound $||v(t)||_{L^\infty} \le C(||\omega_0||_{L^p} + ||\omega_0||_{L^\infty})$ for $p \in (1,2)$. (For a proof of this bound, we refer the reader to [8], Theorem 3.1.) Therefore, by uniform continuity of ω_0, given $\epsilon > 0$, there exists $\delta > 0$ such that, for $|t_1 - t_2| < \delta$ and for all $x \in \mathbb{R}^2$,

$$\begin{aligned}|\omega(t_1,x) - \omega(t_2,x)| &= |\omega_0(x_1) - \omega_0(x_2)| \\ &< \frac{\epsilon}{2}.\end{aligned}$$

We conclude that, for $|t_1 - t_2| < \delta$,

$$\begin{aligned}&||\omega(t_1,x_1) - \omega(t_1,x_2)| - |\omega(t_2,x_1) - \omega(t_2,x_2)|| \\ &\le |\omega(t_1,x_1) - \omega(t_2,x_1)| + |\omega(t_1,x_2) - \omega(t_2,x_2)| < \epsilon.\end{aligned} \tag{7}$$

Equicontinuity follows from (7). Finally, we observe that A is closed in $C(M)$ by sequential continuity of ω in time and compactness of $[0,T]$. We conclude that A is compact in $C(M)$. From this compactness, it follows that for any $t \in [0,T]$ and $\epsilon > 0$, there exists $t_i \in [0,T]$, $i = 1,2,...,M$, such that $||\Delta_j\omega(t)||_{L^\infty} < \frac{\epsilon}{2} + ||\Delta_j\omega(t_i)||_{L^\infty}$ for all j. Furthermore, given $\omega(t_i)$, there exists an N_{t_i} such that $\sup_{j>N_{t_i}} ||\Delta_j\omega(t_i)||_{L^\infty} < \frac{\epsilon}{2}$. Let $N = \max\{N_{t_1}, ...N_{t_M}\}$. Then, for all $t \in [0,T]$,

$$\sup_{j>N} ||\Delta_j\omega(t)||_{L^\infty} < \epsilon.$$

This completes the proof. □

We bound $||\Delta_j\nabla v||_{L^\infty}$ by $C||\Delta_j\omega||_{L^\infty}$ if $j \ge 0$, and we bound $||\Delta_{-1}\nabla v(t)||_{L^\infty}$ with $C||\omega_0||_{L^p}$ for $p \in (1,\infty)$ using Bernstein's inequality. From Lemma 3, we conclude that, for N sufficiently large, and for all $t \in [0,T]$,

$$||S_N\nabla v(t)||_{L^\infty} \le \epsilon(N+1). \tag{8}$$

We now use (8) to compute the Holder exponent for the flow corresponding to the velocity of a fluid with initial vorticity in $C_c(\mathbb{R}^2)$, which will complete the proof of Theorem 2. We show that, given $\epsilon > 0$ and $T > 0$, $g(t)^{-1} - Id$ belongs to $C^{\sigma(t)}(\mathbb{R}^2)$ for all $t \in [0,T]$, where $\sigma(t) = e^{-Ct\epsilon}$, and C is an absolute constant. We then let $\delta = 1 - e^{-CT\epsilon}$, and we make δ as small as we would like by our choice of ϵ.

Fix $\epsilon > 0$. Write $v = v_{1,N} + v_{2,N}$, where $v_{1,N} = S_{N-1}v$, and $v_{2,N} = (Id - S_{N-1})v$. By (8), it follows that $|v_{1,N}(t,x) - v_{1,N}(t,y)| \le ||\nabla S_{N-1}v(t)||_{L^\infty}|x-y| \le CN\epsilon|x-y|$ for large enough N. Similarly, for large N we can conclude that $|v_{2,N}(t,x) - v_{2,N}(t,y)| \le C\sum_{j=N-1}^{\infty} 2^{-j}||\Delta_j\nabla v(t)||_{L^\infty} \le C2^{-N}\epsilon$. Letting $N = -\log_2|x-y|$, we have that for $|x-y|$ sufficiently small,

$$\begin{aligned}|v(t,x) - v(t,y)| &\le |(v_{1,N}(t,x) + v_{2,N}(t,x)) - (v_{1,N}(t,y) + v_{2,N}(t,y))| \\ &\le C(\epsilon(-\log_2|x-y|)|x-y| + \epsilon|x-y|) \\ &\le C\epsilon(1 - \log_2|x-y|)|x-y|.\end{aligned} \tag{9}$$

We now use (9) and Osgood's Lemma (see [3], Lemma 5.2.1) to compute properties of the flow. We write

$$|v(t,g(t,x)) - v(t,g(t,y))| \leq C\epsilon(1-\log_2|g(t,x)-g(t,y)|)|g(t,x)-g(t,y)|$$

whenever $|g(t,x)-g(t,y)|<\delta$. From (4), this gives

$$|g(t,x)-g(t,y)| \leq |x-y| + \int_0^t C\epsilon(1-\log_2|g(\tau,x)-g(\tau,y)|)|g(\tau,x)-g(\tau,y)|d\tau.$$

By Osgood's Lemma, we conclude that

$$-\log(1-\log|g(t,x)-g(t,y)|) + \log(1-\log|x-y|) \leq Ct\epsilon.$$

Taking the exponential twice, we get

$$\frac{|g(t,x)-g(t,y)|}{|x-y|^{e^{-Ct\epsilon}}} \leq e^{1-e^{-Ct\epsilon}} \leq e$$

whenever $|x-y|<\delta$, which gives

$$\frac{|(g(t,x)-x)-(g(t,y)-y)|}{|x-y|^{e^{-Ct\epsilon}}} \leq e+1.$$

In the case $|x-y|\geq\delta$, we have

$$\frac{|(g(t,x)-x)-(g(t,y)-y)|}{|x-y|^{e^{-Ct\epsilon}}} \leq 2\delta^{-e^{-Ct\epsilon}}||g(t)-Id||_{L^\infty}.$$

To see that $||g(t)^{-1}-Id||_{C^{1-\delta}} \in L^\infty_{loc}(\mathbb{R}^+)$, we observe that

$$\begin{aligned}\sup_{x,y\in\mathbb{R}^2} \frac{|(g(t,x)-x)-(g(t,y)-y)|}{|x-y|^{e^{-Ct\epsilon}}} &\leq (e+1) + 2\delta^{-e^{-Ct\epsilon}}||\int_0^t v(\tau,g(\tau,\cdot))d\tau||_{L^\infty}\\ &\leq (e+1)+2\delta^{-1}T(||\omega_0||_{L^p}+||\omega_0||_{L^\infty})\end{aligned}$$

for all $t\in[0,T]$. This completes the proof of Theorem 2.

4. Paradifferential Estimates for the Transport Equation

In this section, we consider an initial value problem for the vorticity equation corresponding to the two-dimensional Euler Equation. When $n=2$, the vorticity equation is given by

$$\begin{aligned}\partial_t\omega + v\cdot\nabla\omega &= 0,\\ \omega|_{t=0} &= \omega_0.\end{aligned} \tag{10}$$

Note that, if ω satisfies (10), then $\Delta_q\omega$ satisfies the following equation:

$$\begin{aligned}&\partial_t\Delta_q\omega + v\cdot\nabla\Delta_q\omega = [v\cdot\nabla,\Delta_q]\omega,\\ &\Delta_q\omega|_{t=0} = \Delta_q\omega_0.\end{aligned} \tag{11}$$

We want to prove the following estimate:

Proposition 4. *Let $p\in(1,\infty)$ and $\sigma>0$ be fixed. Then there exists two positive constants $C_1(\sigma)$ and C_2 such that*

$$||[v\cdot\nabla,\Delta_q]\omega||_{L^p} \leq C_1(\sigma)(C_2+||S_{q-1}\nabla v||_{L^\infty})2^{-q\sigma}||\omega||_{B^\sigma_{p,\infty}}.$$

Proof. We consider the cases $q \geq 4$ and $q < 4$ separately. We first assume $q \geq 4$ and use Bony's decomposition to write

$$[v \cdot \nabla, \Delta_q]\omega = \sum_{j=1}^{2} [T_{v_j}\partial_j, \Delta_q]\omega + [T_{\partial_j \cdot} v_j, \Delta_q]\omega + [\partial_j R(v_j, \cdot), \Delta_q]\omega.$$

We address each piece of the sum separately. We start with $[T_{v_j}\partial_j, \Delta_q]\omega$. Write

$$[T_{v_j}\partial_j, \Delta_q]\omega = \sum_{q'=q-4}^{q+4} [S_{q'-1}(v_j), \Delta_q]\Delta_{q'}\partial_j\omega.$$

Letting $u = \Delta_{q'}\partial_j\omega$, letting $h = \check{\phi}$ (recall we are assuming $q \geq 4$ here), and keeping in mind that $|q' - q| \leq 4$, we have

$$\begin{aligned}
&||[S_{q'-1}(v_j), \Delta_q]u||_{L^p} = \\
&|| \int_{\mathbb{R}^2} h(y)(S_{q'-1}(v_j)(x - 2^{-q}y) - S_{q'-1}(v_j)(x))u(x - 2^{-q}y)dy||_{L^p} \\
&\leq C||S_{q'-1}\nabla v||_{L^\infty} 2^{-q}||u||_{L^p} \\
&\leq C2^{4\sigma}||S_{q'-1}\nabla v||_{L^\infty} 2^{-q\sigma}||\omega||_{B^\sigma_{p,\infty}},
\end{aligned}$$

where we used the fact that $h \in S$ and therefore $zh(z)$ is integrable, as well as Bernstein's inequality. We now sum over q' to get

$$\begin{aligned}
||[T_{v_j}\partial_j, \Delta_q]\omega||_{L^p} &\leq C2^{4\sigma}2^{-q\sigma}||\omega||_{B^\sigma_{p,\infty}} \sum_{q'=q-4}^{q+4} ||S_{q'-1}\nabla v||_{L^\infty} \\
&\leq C2^{4\sigma}2^{-q\sigma}||\omega||_{B^\sigma_{p,\infty}}(||S_{q-1}\nabla v||_{L^\infty} + ||\nabla v||_{C^0_*}).
\end{aligned} \tag{12}$$

We now consider $[T_{\partial_j \cdot} v_j, \Delta_q]\omega$. To bound $||T_{\partial_j \Delta_q \omega} v_j||_{L^p}$, we use Bernstein's inequality and our assumption that $q \geq 4$, as well as properties of our partition of unity, to write

$$\begin{aligned}
||T_{\partial_j \Delta_q \omega} v_j||_{L^p} &\leq \sum_{q'=q}^{\infty} C2^q 2^{-q'}||S_{q'-1}\Delta_q\omega||_{L^p}||\Delta_{q'}\nabla v||_{L^\infty} \\
&\leq C||\Delta_q\omega||_{L^p} \sup_{q' \geq q} ||\Delta_{q'}\nabla v||_{L^\infty} \\
&\leq C||\nabla v||_{C^0_*} 2^{-q\sigma}||\omega||_{B^\sigma_{p,\infty}}.
\end{aligned} \tag{13}$$

Furthermore, since the Fourier support of $S_{q'-1}\partial_j\omega\Delta_{q'}v_j$ is contained in an annulus with inner and outer radius $C_1 2^{q'}$ and $C_2 2^{q'}$ respectively, we can write

$$\begin{aligned}
||\Delta_q(T_{\partial_j\omega}v_j)||_{L^p} &\leq \sum_{q'=q-4}^{q+4} ||S_{q'-1}\partial_j\omega||_{L^\infty}||\Delta_{q'}v||_{L^p} \\
&\leq \sum_{q'=q-4}^{q+4} C2^{q'}2^{-q'}||S_{q'-1}\omega||_{L^\infty}||\Delta_{q'}\nabla v||_{L^p} \\
&\leq \sum_{q'=q-4}^{q+4} C||S_{q'-1}\nabla v||_{L^\infty}||\Delta_{q'}\omega||_{L^p} \\
&\leq C2^{4\sigma}(||S_{q-1}\nabla v||_{L^\infty} + ||\nabla v||_{C^0_*})2^{-q\sigma}||\omega||_{B^\sigma_{p,\infty}}.
\end{aligned} \tag{14}$$

Once again, we used Bernstein's inequality in the second inequality. It is in this inequality that our assumption that $q \geq 4$ is necessary. For the third inequality, we used the fact that Calderon-Zygmund operators are bounded on L^p for $p \in (1, \infty)$. Combining (13) and (14), we see that

$$||[T_{\partial_j \cdot} v_j, \Delta_q]\omega||_{L^p} \leq C 2^{4\sigma}(||S_{q-1}\nabla v_j||_{L^\infty} + ||\nabla v||_{C^0_*}) 2^{-q\sigma}||\omega||_{B^\sigma_{p,\infty}} \tag{15}$$

for $q \geq 4$.

We now study the remainder term, $[\partial_j R(v_j, \cdot), \Delta_q]\omega$. We need the following lemma:

Lemma 5. *If $s + \sigma > 0$, then $||R(a,b)||_{B^{s+\sigma}_{p,\infty}} \leq C(s,\sigma)||a||_{B^s_{\infty,\infty}}||b|||_{B^\sigma_{p,\infty}}$, where $C(s,\sigma) = C 2^{M\sigma}(\frac{1}{1-(\frac{1}{2})^{s+\sigma}} + 2^{N(s+\sigma)} + ... + 2^{1(s+\sigma)})$ for fixed positive integers M and N.*

Proof. For a proof of the lemma, we refer the reader to [9]. □

To handle the remainder term, we will consider low and high frequencies of v_j separately. We begin with $[\partial_j R((Id - \Delta_{-1})v_j, \cdot), \Delta_q]\omega$. Using Lemma 5 with $s = 1$, Bernstein's inequality, and the fact that the Fourier transform of $(Id - \Delta_{-1})\nabla v$ vanishes in a neighborhood of the origin, we write

$$\begin{aligned} ||\Delta_q \partial_j R((Id - \Delta_{-1})v_j, \omega)||_{L^p} &\leq 2^{-q\sigma}||R((Id - \Delta_{-1})v_j, \omega)||_{B^{\sigma+1}_{p,\infty}} \\ &\leq C(\sigma) 2^{-q\sigma}||(Id - \Delta_{-1})v||_{B^1_{\infty,\infty}}||\omega||_{B^\sigma_{p,\infty}} \\ &\leq C(\sigma) 2^{-q\sigma}||\nabla v||_{C^0_*}||\omega||_{B^\sigma_{p,\infty}}. \end{aligned} \tag{16}$$

Here $C(\sigma) = C(1,\sigma)$ from Lemma 5. To bound $||\partial_j R((Id - \Delta_{-1})v_j, \Delta_q\omega)||_{L^p}$, note that

$$\begin{aligned} ||\partial_j R((Id - \Delta_{-1})v_j, \Delta_q\omega)||_{L^p} &\leq \sum_{\substack{q', q'' \\ |q'-q''| \leq 1 \\ |q'-q| \leq 1}} 2^q ||\Delta_{q''}(Id - \Delta_{-1})v||_{L^\infty}||\Delta_{q'}\Delta_q\omega||_{L^p} \\ &\leq C||\nabla v||_{C^0_*} 2^{-q\sigma}||\omega||_{B^\sigma_{p,\infty}}. \end{aligned} \tag{17}$$

In the first inequality above, we used the fact that the support of the Fourier transform of $\Delta_{q''}(Id - \Delta_{-1})v\Delta_{q'}\Delta_q\omega$ is contained in a ball with radius $C2^q$, along with Bernstein's inequality, to get the factor 2^q. In the second inequality, we used the inequality $||(Id - \Delta_{-1})v||_{B^1_{\infty,\infty}} \leq C||\nabla v||_{C^0_*}$. We now combine (17) with (16) to conclude that

$$||[\partial_j R((Id - \Delta_{-1})v_j, \cdot), \Delta_q]\omega||_{L^p} \leq C(\sigma)||\nabla v||_{C^0_*} 2^{-q\sigma}||\omega||_{B^\sigma_{p,\infty}}. \tag{18}$$

We now estimate $||[\partial_j R(\Delta_{-1}v_j, \cdot), \Delta_q]\omega||_{L^p}$. Using the definition of the remainder operator, as well as the properties of our partition of unity, we write

$$\begin{aligned} [\partial_j R(\Delta_{-1}v_j, \cdot), \Delta_q]\omega &= \partial_j R(\Delta_{-1}v_j, \Delta_q\omega) - \Delta_q(\partial_j R(\Delta_{-1}v_j, \omega)) \\ &= \partial_j(\sum_{\substack{i,k \\ |i-k| \leq 1}} \Delta_k\Delta_{-1}v_j\Delta_i\Delta_q\omega) - \Delta_q\partial_j(\sum_{\substack{i,k \\ |i-k| \leq 1}} \Delta_k\Delta_{-1}v_j\Delta_i\omega). \end{aligned}$$

We begin by estimating $\Delta_q\partial_j(\sum_{\substack{i,k\\|i-k|\le 1}}\Delta_k\Delta_{-1}v_j\Delta_i\omega)$. We first reintroduce the sum over j, allowing us to use the fact that div $v=0$ to move ∂_j inside the parentheses and differentiate ω. This, along with properties of our partition of unity, gives

(19)
$$\sum_{j=1}^{2}||\Delta_q\partial_j(\sum_{\substack{i,k\\|i-k|\le 1}}\Delta_k\Delta_{-1}v_j\Delta_i\omega)||_{L^p}\le C\sum_{j=1}^{2}\sum_{l=-1}^{1}\sum_{k=-1}^{0}||\Delta_q(\Delta_k\Delta_{-1}v_j\Delta_{k-l}\partial_j\omega)||_{L^p}.$$

Note that in the second line of (19), the Fourier support of $\Delta_k\Delta_{-1}v_j\Delta_{k-l}\partial_j\omega$ is contained in a ball with radius $C2^k$. Therefore, the sum in the second line is 0 if $q\ge k+M$, for a constant M. Furthermore, $k\le 0$. Therefore, we are only considering $q\le M$. We then write

$$\begin{aligned}&\sum_{j=1}^{2}\sum_{l=-1}^{1}\sum_{k=-1}^{0}||\Delta_q(\Delta_k\Delta_{-1}v_j\Delta_{k-l}\partial_j\omega)||_{L^p}\\&\le C2^{M\sigma}||v||_{L^\infty}2^{-q\sigma}||\omega||_{B^\sigma_{p,\infty}}.\end{aligned}\tag{20}$$

To bound $||\partial_j(\sum_{\substack{i,k\\|i-k|\le 1}}\Delta_k\Delta_{-1}v_j\Delta_i\Delta_q\omega)||_{L^p}$, we use the fact that the Fourier transform of $\Delta_{-1}v_j$ is supported in the neighborhood of the origin, and we recognize that div $v=0$ allows us to move ∂_j inside the parentheses. Therefore,

$$\begin{aligned}&\sum_{j=1}^{2}||\partial_j(\sum_{\substack{i,k\\|i-k|\le 1}}\Delta_k\Delta_{-1}v_j\Delta_i\Delta_q\omega)||_{L^p}\\&\le C\sum_{j=1}^{2}\sum_{l=-1}^{1}\sum_{k=-1}^{0}||\Delta_k\Delta_{-1}v||_{L^\infty}2^{k-l}||\Delta_{k-l}\Delta_q\omega||_{L^p}\\&\le C||v||_{L^\infty}2^{-q\sigma}||\omega||_{B^\sigma_{p,\infty}},\end{aligned}\tag{21}$$

where we used Bernstein's inequality and Holder's inequality to get the first inequality in (21). We now combine (19) through (21) to conclude that

$$||[\partial_jR(\Delta_{-1}v_j,\cdot),\Delta_q]\omega||_{L^p}\le C2^{M\sigma}||v||_{L^\infty}2^{-q\sigma}||\omega||_{B^\sigma_{p,\infty}}.\tag{22}$$

Combining (12), (15), (18), and (22), we conclude that for $q\ge 4$,

$$\begin{aligned}&||[v\cdot\nabla,\Delta_q]\omega||_{L_p}\\&\le C(2^{M\sigma}+C(\sigma))(||S_{q-1}\nabla v_j||_{L^\infty}+||\nabla v||_{C^0_*}+||v||_{L^\infty})2^{-q\sigma}||\omega||_{B^\sigma_{p,\infty}}.\end{aligned}$$

To complete the proof for the case $q\ge 4$, we bound $||\nabla v||_{C^0_*}+||v||_{L^\infty}$ by $C(||\omega_0||_{L^\infty}+||\omega_0||_{L^p})$ for fixed $p<2$. This completes the proof for the case $q\ge 4$.

For the case $q<4$, write:

$$[v\cdot\nabla,\Delta_q]\omega=v\cdot\nabla\Delta_q\omega-\Delta_q(v\cdot\nabla\omega).\tag{23}$$

Keeping in mind that $q\le 3$, it is easy to see that

$$||v\cdot\nabla\Delta_q\omega||_{L^p}\le C(||\omega_0||_{L^{p_0}}+||\omega_0||_{L^\infty})2^{-q\sigma}||\omega||_{B^\sigma_{p,\infty}},\tag{24}$$

where again we used the bound $||v||_{L^\infty} \leq C(||\omega_0||_{L^{p_0}} + ||\omega_0||_{L^\infty})$ for fixed $p_0 \in (1,2)$. We now write the second term of (23) as

$$\Delta_q(v\cdot\nabla\omega) = \sum_{j=1}^{2} \Delta_q(T_{v_j}\partial_j\omega + T_{\partial_j\omega}v_j + R(v_j,\partial_j\omega)). \tag{25}$$

We successfully bounded the L^p norm of the remainder term of (25) in (16), (19), and (20) of the proof for the $q \geq 4$ case (note that (16), (19), and (20) hold for all q). Therefore, we are only concerned with $\sum_{j=1}^{2}\Delta_q(T_{v_j}\partial_j\omega + T_{\partial_j\omega}v_j)$. Using the fact that $S_{q'-1}v_j\Delta_{q'}\partial_j\omega$ has Fourier support in an annulus with inner radius $C_1 2^{q'}$ and outer radius $C_2 2^{q'}$, and, once again, keeping in mind that $q \leq 3$, we have

$$\begin{aligned} ||\Delta_q(T_{v_j}\partial_j\omega)||_{L^p} &\leq \sum_{q'=q-4}^{q+4} C||S_{q'-1}v||_{L^\infty}2^{q'}||\Delta_{q'}\omega||_{L^p} \\ &\leq C(||\omega_0||_{L^{p_0}} + ||\omega_0||_{L^\infty})2^{-q\sigma}||\omega||_{B^\sigma_{p,\infty}} \end{aligned} \tag{26}$$

for $p_0 \in (1,2)$. Furthermore, since $q \leq 3$, we write

$$\begin{aligned} ||\Delta_q(T_{\partial_j\omega}v_j)||_{L^p} &\leq \sum_{q'=q-4}^{q+4} ||S_{q'-1}\partial_j\omega\Delta_{q'}v_j||_{L^p} \\ &\leq C\sum_{q'=q-4}^{q+4}\sum_{k=-1}^{q'-2} 2^{-k\sigma}||\omega||_{B^\sigma_{p,\infty}}||v||_{L^\infty} \\ &\leq C2^{3\sigma}(||\omega_0||_{L^{p_0}} + ||\omega_0||_{L^\infty})2^{-q\sigma}||\omega||_{B^\sigma_{p,\infty}}. \end{aligned}$$

This completes the proof of the $q < 4$, and therefore completes the proof of the estimate for all q. We conclude that, for all q,

$$||[v\cdot\nabla,\Delta_q]\omega||_{L_p} \leq C_1(\sigma)(||S_{q-1}\nabla v_j||_{L^\infty} + ||\omega_0||_{L^{p_0}} + ||\omega_0||_{L^\infty})2^{-q\sigma}||\omega||_{B^\sigma_{p,\infty}}.$$

□

Note that, if we define $h(t,x) = g(t)^{-1}(x) - x$, then h satisfies an equation similar to (10). We have the following:

$$\begin{aligned} &\partial_t h_i + v\cdot\nabla h_i + v_i = 0, \\ &h_i|_{t=0} = 0. \end{aligned} \tag{27}$$

Since h_i satisfies (27), it also satisfies

$$\begin{aligned} &\partial_t\Delta_q h_i + v\cdot\nabla\Delta_q h_i = -\Delta_q v_i + [v\cdot\nabla,\Delta_q]h_i, \\ &\Delta_q h_i|_{t=0} = 0. \end{aligned} \tag{28}$$

This motivates us to prove a similar commutator estimate with h in place of ω. We prove that the following estimate holds:

Proposition 6. *Let $p \in (1,\infty)$, $\delta > 0$, and $\sigma > 0$ be fixed. Then there exists two positive constants $C_1(\sigma)$ and C_2 such that*

$$||[v\cdot\nabla,\Delta_q]h||_{L_p} \le C_1(\sigma)(C_2 + ||S_{q-1}\nabla v||_{L^\infty})2^{-q\sigma}||h||_{B^\sigma_{p,\infty}} + \sum_{q'=q-4}^{q+4} C(\delta)2^{q'\delta}||\Delta_{q'}v||_{L^p}||h||_{C^{1-\delta}}.$$

Proof. The proof of Proposition 6 is identical to the proof of Proposition 4 with h in place of ω for every term except $\Delta_q(T_{\partial_j h}v_j)$. Therefore, we restrict our attention to this term. This portion of the proof will result in the second piece on the right-hand side in Proposition 6.

Note that, in the proof of Proposition 4, we use the assumption that $q \ge 4$ only when bounding $||\Delta_q(T_{\partial_j\omega}v_j)||_{L^p}$. For all other terms, $q \ge 0$ suffices. This observation, combined with the fact that we will only need to assume $q \ge 0$ to bound $||\Delta_q(T_{\partial_j h}v_j)||_{L^p}$, leads us to consider the cases $q = -1$ and $q \ge 0$ separately.

We first assume $q \ge 0$, and we write $\Delta_q(T_{\partial_j h}v_j) = \Delta_q(\sum_{q'=1}^\infty S_{q'-1}\partial_j h\Delta_{q'}v_j)$. Using the fact that $S_{q'-1}\partial_j h\Delta_{q'}v_j$ has Fourier support in an annulus with inner radius $C_1 2^{q'}$ and outer radius $C_2 2^{q'}$, we apply Bernstein's inequality and Holder's inequality to get

$$\begin{aligned} ||\Delta_q(T_{\partial_j h}v_j)||_{L^p} &\le \sum_{q'=q-4}^{q+4}\sum_{k=-1}^{q'-2} 2^{k\delta}2^{k(1-\delta)}||\Delta_k h||_{L^\infty}||\Delta_{q'}v||_{L^p} \\ &\le \sum_{q'=q-4}^{q+4} C(\delta)2^{q'\delta}||h||_{C^{1-\delta}}||\Delta_{q'}v||_{L^p}. \end{aligned} \tag{29}$$

We now consider the case $q = -1$. As in the proof of Proposition 4 when assuming $q < 4$, we begin by writing:

$$[v\cdot\nabla,\Delta_q]h = v\cdot\nabla\Delta_q h - \Delta_q(v\cdot\nabla h). \tag{30}$$

For the first term of (30), we use the fact that $q = -1$ to get

$$||v\cdot\nabla\Delta_q h||_{L^p} \le C(||\omega_0||_{L^{p_0}} + ||\omega_0||_{L^\infty})2^{-q\sigma}||h||_{B^\sigma_{p,\infty}} \tag{31}$$

for $p_0 \in (1,2)$. The second term of (30) can be written as

$$\Delta_q(v\cdot\nabla h) = \sum_{j=1}^{2}\Delta_q(T_{v_j}\partial_j h + T_{\partial_j h}v_j + R(v_j,\partial_j h)). \tag{32}$$

The proof of the bound on the L^p norm of the remainder term in (32) is identical to the proofs of (16), (19), and (20), (which hold for all q), with h in place of ω. Furthermore, we handled the L^p norm of $\Delta_q(T_{\partial_j h}v_j)$ (see (29), which also holds for all q). Therefore, in (32), we are only concerned with $\Delta_q(T_{v_j}\partial_j h)$. For this term, we refer the reader to the proof for ω with $q < 4$, given in (26). This completes the proof of the case $q = -1$, and therefore completes the proof of the estimate for all q. □

5. Main Theorem

In this section, we prove the following theorem:

Theorem 3. *Let $v_0 \in B^{s+1}_{p,\infty}(\mathbb{R}^2)$, div $v_0 = 0$, and let $\omega(v_0) = \omega_0 \in C_c(\mathbb{R}^2)$, where $sp \leq 2$, $s \in (0,2)$, and $p \in (1,\infty)$. Let $\epsilon > 0$. There exists a unique solution to* (1) *such that $||v(t)||_{B^{s+1-\epsilon}_{p,\infty}}$ belongs to $L^\infty_{loc}(\mathbb{R}^+)$.*

Furthermore, let $g(t,x)$ be the measure-preserving homeomorphism satisfying $\partial_t g(t,x) = v(t, g(t,x))$. Define $h(t,x) = g(t)^{-1}(x) - x$. Then, under the above assumptions on v_0 and ω_0, it follows that, for fixed $\delta > 0$, $||h(t)||_{B^{s+1-\delta}_{p,\infty}}$ belongs to $L^\infty_{loc}(\mathbb{R}^+)$.

Remark. Uniqueness in Theorem 3 follows from [10].

Proof. We begin by proving the first part of the theorem. Our approach is as follows: we fix $\epsilon' > 0$ and $T > 0$, and we define $\sigma(t) = s\exp\{-\frac{C}{s}\epsilon' t\}$. We then show that $||v(t)||_{B^{\sigma(t)+1}_{p,\infty}} \in L^\infty([0,T])$, where C is an absolute constant. Letting $\epsilon = s - s\exp\{-\frac{C}{s}\epsilon' T\}$, we make ϵ as small as we would like by our choice of ϵ'.

We first prove the theorem on a sufficiently small time interval $[t_0, t]$. We then use a bootstrapping argument to show that the theorem holds on any finite time interval $[0,T]$. From (11) and Proposition 4, it follows that

$$||\Delta_q \omega(t)||_{L^p} \leq ||\Delta_q \omega(t_0)||_{L^p} + \int_{t_0}^t C_1(\sigma(\tau))(C_2 + ||S_{q-1}\nabla v(\tau)||_{L^\infty}) 2^{-q\sigma(\tau)} ||\omega(\tau)||_{B^{\sigma(\tau)}_{p,\infty}} d\tau.$$

We see from the proof of Proposition 4 that $C_1(\sigma(t))$ can be bounded by an absolute constant for all $\sigma(t) \in (0,2)$. Therefore, for the remainder of the proof, we drop the dependence of C_1 on $\sigma(t)$. We multiply both sides of the equation by $2^{q\sigma(t)}$ and take the supremum over q to get

$$||\omega(t)||_{B^{\sigma(t)}_{p,\infty}} \leq ||\omega(t_0)||_{B^{\sigma(t_0)}_{p,\infty}} + \sup_q \{\int_{t_0}^t C_1(C_2 + ||S_{q-1}\nabla v(\tau)||_{L^\infty}) 2^{q\sigma(t) - q\sigma(\tau)} ||\omega(\tau)||_{B^{\sigma(\tau)}_{p,\infty}} d\tau\}.$$

We now show that the supremum over q on the right-hand side is finite. We claim that the loss of regularity in the Besov exponent, resulting in the term $2^{q(\sigma(t)-\sigma(\tau))}$, is enough to combat the growth of $||S_{q-1}\nabla v(\tau)||_{L^\infty}$.

When taking the supremum over q of the time integral, we consider two cases separately: the supremum over $q \leq N$, and the supremum over $q > N$. We then use (8) to handle the supremum over $q > N$. We write

$$||\omega(t)||_{B^{\sigma(t)}_{p,\infty}} \leq ||\omega(t_0)||_{B^{\sigma(t_0)}_{p,\infty}} + I_1 + I_2,$$

where

$$I_1 = \sup_{q \leq N} \{\int_{t_0}^t C_1(C_2 + ||S_{q-1}\nabla v(\tau)||_{L^\infty}) 2^{q\sigma(t) - q\sigma(\tau)} ||\omega(\tau)||_{B^{\sigma(\tau)}_{p,\infty}} d\tau\},$$

and

$$I_2 = \sup_{q>N}\{\int_{t_0}^{t} C_1(C_2 + ||S_{q-1}\nabla v(\tau)||_{L^\infty})2^{q\sigma(t)-q\sigma(\tau)}||\omega(\tau)||_{B^{\sigma(\tau)}_{p,\infty}} d\tau\}.$$

To bound I_2, we apply (8), and we conclude that

$$I_2 \leq \int_{t_0}^{t} C||\omega(\tau)||_{B^{\sigma(\tau)}_{p,\infty}} d\tau + \sup_{q>N}\{\int_{t_0}^{t} C_1\epsilon q 2^{q\sigma(t)-q\sigma(\tau)}||\omega(\tau)||_{B^{\sigma(\tau)}_{p,\infty}} d\tau\}. \tag{33}$$

To handle the second integral in (33), we integrate by parts. Letting $\sigma(t) = s\exp(-\frac{2C_1}{s}\epsilon t)$ (this is the definition of $\sigma(t)$ with $C = 2C_1$), we let $u = e^{\frac{2C_1}{s}\epsilon\tau}$ and $dv = C_1\epsilon q e^{-\frac{2C_1}{s}\epsilon\tau}2^{q\sigma(t)-q\sigma(\tau)}d\tau$. Then, substituting u and dv into the second integral in (33), and recognizing that du and v are positive for all $\tau \in [t_0, t]$, we write

$$\begin{aligned}\sup_{q>N}\{\sup_{\tau\in[t_0,t]} ||\omega(\tau)||_{B^{\sigma(\tau)}_{p,\infty}} \int_{t_0}^{t} udv\} &\leq \sup_{q>N}\{\sup_{\tau\in[t_0,t]} ||\omega(\tau)||_{B^{\sigma(\tau)}_{p,\infty}} (uv|_{t_0}^{t})\} \\ &= \sup_{\tau\in[t_0,t]} ||\omega(\tau)||_{B^{\sigma(\tau)}_{p,\infty}} e^{\frac{2C_1}{s}\epsilon t}\frac{1}{2\ln 2}.\end{aligned} \tag{34}$$

We now bound the first time integral on the right-hand side of (33) by

$$C \sup_{\tau\in[t_0,t]} ||\omega(\tau)||_{B^{\sigma(\tau)}_{p,\infty}} (t - t_0). \tag{35}$$

Combining (35) with (34), we conclude that

$$I_2 \leq \sup_{\tau\in[t_0,t]} ||\omega(\tau)||_{B^{\sigma(\tau)}_{p,\infty}} \{e^{\frac{C}{s}\epsilon t}\frac{1}{2\ln 2} + C(t - t_0)\}. \tag{36}$$

To bound I_1, we first observe that $||S_{q-1}\nabla v(\tau)||_{L^\infty} \leq q||\nabla v(\tau)||_{C^0_*}$. Then, bounding $||\nabla v(\tau)||_{C^0_*}$ by $C(||\omega_0||_{L^\infty} + ||\omega_0||_{L^{p_0}})$, for $p_0 \in (1, \infty)$, and recognizing that $2^{q(\sigma(t)-\sigma(\tau))} \leq 1$ for all q, we conclude that

$$I_1 \leq CN(t - t_0) \sup_{\tau\in[t_0,t]} ||\omega(\tau)||_{B^{\sigma(\tau)}_{p,\infty}}. \tag{37}$$

We now combine our estimates for I_1 and I_2 given in (36) and (37), which gives

$$\sup_{\tau\in[t_0,t]} ||\omega(\tau)||_{B^{\sigma(\tau)}_{p,\infty}} \leq ||\omega(t_0)||_{B^{\sigma(t_0)}_{p,\infty}} + C^* \sup_{\tau\in[t_0,t]} ||\omega(\tau)||_{B^{\sigma(\tau)}_{p,\infty}},$$

where we let

$$C^* = e^{\frac{C}{s}\epsilon t}\frac{1}{2\ln 2} + CN(t - t_0). \tag{38}$$

To complete the proof, we must make the constant $C^* < 1$. Fix $t > 0$. Given this t, choose $\epsilon > 0$ small enough to ensure that $\frac{e^{\frac{C}{s}\epsilon t}}{2\ln 2} < 1$. Depending on our choice of t and ϵ, $N = N(t, \epsilon)$ may be very large. Given this N, make $t - t_0$ small enough so that $C^* < 1$. Note that, under these assumptions, $C^* < 1$ when we are working on an interval of length less than or equal to $t - t_0$, as long as the right endpoint of the interval is less than or equal to t. We therefore break $[0, t]$ into a finite number $M = M(t, \epsilon)$ of intervals of length $t - t_0$, and we apply a bootstrapping argument. This gives

$$\sup_{\tau\in[0,t]} ||\omega(\tau)||_{B^{\sigma(\tau)}_{p,\infty}} \leq C^M||\omega_0||_{B^s_{p,\infty}},$$

where $C = \frac{1}{1-C^*}$, and C^M depends on t and ϵ. More precisely, larger initial choice of t and smaller choice of ϵ result in larger M and thus larger C^M.

This completes the proof for regularity of vorticity. To show that this implies regularity of the velocity, we need the following estimate:

Lemma 7. *Let $v_0 \in B^{s+1}_{p,\infty}(\mathbb{R}^2)$. Then there exists two positive constants C_0 and C_1 such that*
$$||v(t)||_{B^{\sigma(t)+1}_{p,\infty}} \leq C_0 e^{C_1 t} + ||\omega(t)||_{B^{\sigma(t)}_{p,\infty}}.$$

Proof. We refer the reader to the proof of Lemma 6.2 in [7]. □

This completes the proof of the first part of the theorem.

We now prove properties of h. We show that $h(t) \in B^{\sigma'(t)}_{p,\infty}(\mathbb{R}^2)$, where $\sigma'(t) = \sigma(t) + 1 - \delta$, and δ is the Holder exponent of $h(t)$ (see Theorem 2). The proof of this part of the theorem is similar to that for ω. However, we must deal with the extra term which shows up in the commutator estimate given in Proposition 6.

We begin by applying Proposition 6 to (28), where, once again, we drop the dependence of C_1 on σ'. This gives
$$||\Delta_q h(t)||_{L^p} \leq ||\Delta_q h(t_0)||_{L^p} + \int_{t_0}^t C_1(C_2 + ||S_{q-1}\nabla v(\tau)||_{L^\infty}) 2^{-q\sigma'(\tau)} ||h(\tau)||_{B^{\sigma'(\tau)}_{p,\infty}} d\tau$$
$$+ \int_{t_0}^t \{ \sum_{q'=q-4}^{q+4} (||h(\tau)||_{C^{1-\delta}} ||\Delta_{q'} v(\tau)||_{L^p} C(\delta) 2^{q'\delta}) + ||\Delta_q v(\tau)||_{L^p} \} d\tau.$$

We now multiply both sides of the inequality by $2^{q\sigma'(t)}$ and take the supremum over q to get
$$||h(t)||_{B^{\sigma'(t)}_{p,\infty}} \leq ||h(t_0)||_{B^{\sigma'(t_0)}_{p,\infty}}$$
$$+ \sup_q \{ \int_{t_0}^t C_1(C_2 + ||S_{q-1}\nabla v(\tau)||_{L^\infty}) 2^{q(\sigma'(t)-\sigma'(\tau))} ||h(\tau)||_{B^{\sigma'(\tau)}_{p,\infty}} d\tau \}$$
$$+ \int_{t_0}^t \{ C(\delta) ||h(\tau)||_{C^{1-\delta}} ||v(\tau)||_{B^{\sigma(\tau)+1}_{p,\infty}} + ||v(\tau)||_{B^{\sigma(\tau)+1}_{p,\infty}} \} d\tau.$$

Here we used the fact that $\sigma'(t) = \sigma(t) + 1 - \delta$, with $\delta > 0$. The constant $C(\delta)$ now depends on $\sigma(\tau)$, but it is uniformly bounded for all $\sigma(\tau) \in (0, 2)$.

We replace $\sigma'(t) - \sigma'(\tau)$ with $\sigma(t) - \sigma(\tau)$ in the first time integral, and we get
$$||h(t)||_{B^{\sigma'(t)}_{p,\infty}} \leq ||h(t_0)||_{B^{\sigma'(t_0)}_{p,\infty}} + J_1 + J_2,$$

where
$$J_1 = \sup_q \{ \int_{t_0}^t C_1(C_2 + ||S_{q-1}\nabla v(\tau)||_{L^\infty}) 2^{q(\sigma(t)-\sigma(\tau))} ||h(\tau)||_{B^{\sigma'(\tau)}_{p,\infty}} d\tau \},$$

and
$$J_2 = \int_{t_0}^t \{ C(\delta) ||h(\tau)||_{C^{1-\delta}} ||v(\tau)||_{B^{\sigma(\tau)+1}_{p,\infty}} + ||v(\tau)||_{B^{\sigma(\tau)+1}_{p,\infty}} \} d\tau.$$

The argument for dealing with J_1 is identical to the argument we used to handle I_1 and I_2 when proving the first part of Theorem 3. Following this approach, we conclude that

$$\sup_{\tau\in[t_0,t]} ||h(\tau)||_{B^{\sigma'(\tau)}_{p,\infty}} \leq ||h(t_0)||_{B^{\sigma'(t_0)}_{p,\infty}} + C^* \sup_{\tau\in[t_0,t]} ||h(\tau)||_{B^{\sigma'(\tau)}_{p,\infty}} + J_2,$$

where C^* is given by (38). Arguing as we did with ω, we make $C^* < 1$ on a sufficiently short time interval and use a bootstrapping argument, as well as the fact that $h(0,x) = 0$, to conclude that

$$\text{(39)} \quad \sup_{\tau\in[0,t]} ||h(\tau)||_{B^{\sigma'(\tau)}_{p,\infty}} \leq C\int_0^t \{C(\delta)||h(\tau)||_{C^{1-\delta}}||v(\tau)||_{B^{\sigma(\tau)+1}_{p,\infty}} + ||v(\tau)||_{B^{\sigma(\tau)+1}_{p,\infty}}\}d\tau.$$

We now observe that the right hand side of (39) is finite by Theorem 2 and by the first part of Theorem 3. This completes the proof of the theorem. □

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