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ON A CONSTANT ARISING IN MANIN'S CONJECTURE FOR DEL PEZZO SURFACES

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Abstract. For split smooth Del Pezzo surfaces, we analyse the structure of the effective cone and prove a recursive formula for the value of α, appearing in the leading constant as predicted by Peyre of Manin's conjecture on the number of rational points of bounded height on the surface. Furthermore, we calculate α for all singular Del Pezzo surfaces of degree ≥ 3.

1. Introduction

Over the field $\mathbb{Q}$ of rational numbers, a split smooth Del Pezzo surface S is $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, or, for $1 \leq r \leq 8$, the blow-up S_r of $\mathbb{P}^2$ in r points which are defined over $\mathbb{Q}$ and are in *general*[1] position. For the degree of S, we have $\deg(\mathbb{P}^2) = 9$, $\deg(\mathbb{P}^1 \times \mathbb{P}^1) = 8$, and $\deg(S_r) = 9 - r$.

An important object associated to S is the *effective cone* $\Lambda_{\mathrm{eff}}(S)$, i.e., the convex cone in

$$\mathrm{Pic}(S)_{\mathbb{R}} := \mathrm{Pic}(S) \otimes_{\mathbb{Z}} \mathbb{R}$$

which is generated by the classes of effective divisors. Note that we identify divisors and their classes in $\mathrm{Pic}(S)$ when this cannot lead to confusion.

On $\mathrm{Pic}(S) \cong \mathbb{Z}^{10-\deg(S)}$, we have the *intersection form* $(\cdot, \cdot)$, which is a non-degenerate symmetric bilinear form. The *anticanonical map* $\phi : S \to \mathbb{P}^{\deg(S)}$ is given by the *anticanonical class* $-K_S = \phi^*(\mathcal{O}_{\mathbb{P}^{\deg(S)}}(1))$. It is an embedding for $\deg(S) \geq 3$.

For $S = S_r$, the anticanonical class is

$$-K_r := -K_{S_r} = 3H - (E_1 + \cdots + E_r),$$

where H is the transform of a general line in $\mathbb{P}^2$, and $E_1, \ldots, E_r$ are the exceptional divisors obtained by blowing up the r points in $\mathbb{P}^2$. By [2, Corollary 3.3], for $r \geq 2$, $\Lambda_{\mathrm{eff}}(S_r)$ is generated by the (-1)-*curves*, i.e., prime divisors D whose self-intersection number (D, D) is -1. For $r \leq 6$, the anticanonical embedding $\phi_r = \phi$ maps the (-1)-curves on S_r exactly to the lines on $\phi_r(S_r) \subset \mathbb{P}^{9-r}$. Note that $H, E_1, \ldots, E_r$ give a basis of $\mathrm{Pic}(S_r) \cong \mathbb{Z}^{r+1}$.

Starting in 1989, Manin initiated a program to study the number of rational points on certain varieties which can be stated in case of a split smooth Del Pezzo surface S of degree d as follows:

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[1] I.e., no three points on a line, no six on a conic, no eight on a cubic curve with one of them a singularity of that curve.

Question 1. *Let U be the complement of the (-1)-curves in S, and let*

$$H : S(\mathbb{Q}) \to \mathbb{Z}_{>0}$$

be the anticanonical height, i.e., $H(\mathbf{x}) = \max\{|x_0|, \dots, |x_d|\}$ where the image in $\mathbb{P}^d(\mathbb{Q})$ of $\mathbf{x} \in S(\mathbb{Q})$ under the anticanonical map $\phi : S \to \mathbb{P}^d$ is represented by integral and coprime coordinates $(x_0, \dots, x_d)$.

What is the asymptotic behavior of the number of rational points of bounded height

$$N_{U,H}(B) := \#\{\mathbf{x} \in U(\mathbb{Q}) \mid H(\mathbf{x}) \le B\}$$

as $B \to \infty$?

By Manin's conjecture [15], which was formulated for the much larger class of Fano varieties, the following answer is expected:

$$N_{U,H}(B) \sim c_{S,H} \cdot B \cdot (\log B)^{9-d}.$$

The leading constant $c_{S,H}$ has received a conjectural interpretation by Peyre [18]: it is expected to be the product

$$c_{S,H} = \alpha(S) \cdot \beta(S) \cdot \omega_H(S),$$

where $\alpha(S)$ is a constant related to the geometry of S, while $\beta(S)$ is a cohomological constant which is always 1 for split Del Pezzo surfaces, and $\omega_H(S)$ is related to the densities of rational points on S over $\mathbb{R}$ and modulo p for all primes p.

So far, Manin's conjecture has been proved for split smooth Del Pezzo surfaces in the cases $d \ge 6$ in the context of the more general proof for toric varieties [3], and for a specific surface in the case $d = 5$ [4].

The purpose of this note is to look more closely at the constant $\alpha(S)$. Its definition is due to Peyre ([18, Definition 2.4]; see [19, Section 6] for more details):

Definition 2. Let $\Lambda_{\mathrm{eff}}(S)$ be the effective cone, $\Lambda_{\mathrm{eff}}^\vee(S)$ its dual cone (with respect to the intersection form) of *nef* divisor classes, and $-K_S$ the anticanonical class on S. Then we define

$$\alpha(S) := \mathrm{Vol}(P(S)),$$

where

$$P(S) := \{x \in \Lambda_{\mathrm{eff}}^\vee(S) \mid (-K_S, x) = 1\}$$

is a polytope whose volume is calculated using the Lebesgue measure on the hyperplane $\{x \in \mathrm{Pic}(S)_{\mathbb{R}}^\vee \mid (-K_S, x) = 1\}$ which is defined by the $(9-d)$-form $\mathrm{d}\mathbf{x}$ such that $\mathrm{d}\mathbf{x} \wedge \mathrm{d}\omega = \mathrm{d}\mathbf{y}$, where $\mathrm{d}\mathbf{y}$ is the form corresponding to the natural Lebesgue measure on $\mathrm{Pic}(S)_{\mathbb{R}}^\vee$ and $\mathrm{d}\omega$ is the linear form defined by $-K_S$ on $\mathrm{Pic}(S)_{\mathbb{R}}^\vee$.

For large d, the calculation of $\alpha(S)$ can be carried out directly by hand (see [4, Section 1.3] for the case $d = 5$). For small d, especially for S_8 of degree $d = 1$, a direct calculation seems to be currently impossible even with the help of software like `Polymake` [17]. In this case, the cone $\Lambda_{\mathrm{eff}}(S_8)$ has 240 generators, while $\Lambda_{\mathrm{eff}}^\vee(S_8)$ has 19440 generators. A direct calculation of $\alpha(S_8)$ would require a triangulation of $\Lambda_{\mathrm{eff}}^\vee(S_8)$, which seems to be out of reach for today's software and hardware.

Therefore, we need a more detailed knowledge of $\Lambda_{\mathrm{eff}}^\vee(S)$. For $S = S_r$ and $r \ge 3$, we have an action of a Weyl group W_r on $\mathrm{Pic}(S_r)$; see Table 1 for the type of W_r and [2, Section 2] for details. Our main result which will allow us to compute $\alpha(S_r)$ recursively is:

Theorem 3. *Let $r \geq 3$. The nef cone $\Lambda_{\mathrm{eff}}^{\vee}(S_r)$ has N_r faces, where N_r is the number of (-1)-curves on S_r. Each face is isomorphic to $\Lambda_{\mathrm{eff}}^{\vee}(S_{r-1})$. The Weyl group W_r acts transitively on the faces and leaves $-K_r$ in the interior of $\Lambda_{\mathrm{eff}}^{\vee}(S_r)$ fixed.*

This observation is a crucial step in the proof of the following recursive formula for $\alpha(S_r)$; see Table 1 for the values of $\alpha(S_r)$ and N_r:

Theorem 4. *Let S_r be a split smooth Del Pezzo surface of degree $9-r$ which is the blow-up of $\mathbb{P}^2$ in r points in general position. Let N_r be the number of (-1)-curves on S_r. We have $\alpha(S_2) = 1/24$ and*

$$\alpha(S_r) = \frac{N_r \cdot \alpha(S_{r-1})}{r \cdot (9-r)}$$

for $3 \leq r \leq 8$. Furthermore, $\alpha(S_1) = 1/6$, $\alpha(\mathbb{P}^1 \times \mathbb{P}^1) = 1/4$, and $\alpha(\mathbb{P}^2) = 1/3$.

r	2	3	4	5	6	7	8
type of W_r		$\mathbf{A}_2 \times \mathbf{A}_1$	$\mathbf{A}_4$	$\mathbf{D}_5$	$\mathbf{E}_6$	$\mathbf{E}_7$	$\mathbf{E}_8$
N_r	3	6	10	16	27	56	240
$\alpha(S_r)$	1/24	1/72	1/144	1/180	1/120	1/30	1

Table 1. Smooth Del Pezzo surfaces

Next, we consider split singular Del Pezzo surfaces S whose singularities are rational double points. Besides the case where S is the Hirzebruch surface F_2 of degree 8, their *minimal desingularizations* $\widetilde{S}$ are obtained as follows: we perform a series of $r \leq 8$ blow-ups of $\mathbb{P}^2$ resulting in $\widetilde{S} = \widetilde{S}_r$, where in at least one step, we blow up a point on a (-1)-curve (resulting in (-2)-curves, i.e., prime divisors with self intersection number -2), where the only restriction for the choice of the blown-up point is that we never blow up a point on a (-2)-curve (therefore, no prime divisors with self intersection number smaller than -2 can occur). Contracting the (-2)-curves on $\widetilde{S}_r$ results in $S = S_r$ of degree $9-r$.

With some minor modifications in its formulation (mostly replacing S by $\widetilde{S}$ where appropriate), Manin's conjecture is expected to hold for singular Del Pezzo surfaces as well; see [9] and the introduction of [14] for details and an overview of the current progress. The definition of $\alpha(S)$ is also almost the same: we must consider $-K_{\widetilde{S}}$, $\Lambda_{\mathrm{eff}}(\widetilde{S})$, and $\Lambda_{\mathrm{eff}}^{\vee}(\widetilde{S})$ in $\mathrm{Pic}(\widetilde{S})$ of rank $10-\deg(S)$. Note that $\Lambda_{\mathrm{eff}}(\widetilde{S})$ is generated by the *negative curves* (i.e., the (-1)- and (-2)-curves) in the singular case if $\deg(S) \leq 7$. The value of $\alpha(S)$ depends not only on the degree of S, but also on the type of singularities on S.

In Section 3, we list $\alpha(S)$ for each type of singular Del Pezzo surface of degree at least 3; see Tables 2, 3, 4, 5, and 6. For most types, the calculation was performed with the help of the data given in [12] and [13, Chapter 6].

For some examples of the calculation of $\alpha(S_r)$ for non-split Del Pezzo surfaces, see [18], [19, Section 6], [8], and [6].

2. Smooth Del Pezzo surfaces

Let S_r be the blow-up of $\mathbb{P}^2$ in r points in general position.

Lemma 5. *Let $2 \leq r \leq 8$. The effective cone $\Lambda_{\mathrm{eff}}(S_r)$ is generated over $\mathbb{R}$ by the (-1)-curves, whose number is N_r as listed in Table 1.*

Proof. See [2, Corollary 3.3]. Their number N_r can be found in [2, Theorem 2.1]. (Note that, for $r = 8$, the semigroup of classes of effective divisors is generated by the (-1)-curves together with $-K_8$.) □

Lemma 6. *Let E be a (-1)-curve on S_r for $r \geq 3$. If $D \in \mathrm{Pic}(S_r)$ fulfills $(D, E) = 0$ and $(D, E') \geq 0$ for all (-1)-curves E' such that $(E, E') = 0$, then D is nef.*

Proof. As the (-1)-curves generate the effective cone, we must show that $(D, E') \geq 0$ also holds for all (-1)-curves E', regardless of the value of (E, E').

If $(E, E') < 0$, then $E' = E$, and $(D, E) = 0$. If $(E, E') = 0$, then $(D, E') \geq 0$ by assumption.

We proceed by induction on $n = (E, E')$. If $n = 1$, then $E + E'$ is a ruling as in [2, Definition 4.6]. The case $n = 2$ occurs for $r \in \{7, 8\}$, and $n = 3$ is possible only for $r = 8$; furthermore, $n \geq 4$ is impossible. The divisor $E + E'$ can be written in at least two ways as the sum of two negative curves (see [2, Section 4] for rulings, and [13, Sections 3.4, 3.5] for $n \in \{2, 3\}$), say $E + E' = E_1 + E_2$, where $E \notin \{E_1, E_2\}$. Then

$$(E, E_1) + (E, E_2) = (E, E') + (E, E) = n - 1,$$

where (E, E_1) and (E, E_2) are both non-negative. Therefore, the induction hypothesis holds for E_1, E_2, and

$$(D, E') = (D, E + E') = (D, E_1 + E_2) = (D, E_1) + (D, E_2) \geq 0$$

completes the induction. □

For any $D \in \mathrm{Pic}(S_r)$ and $c \in \mathbb{R}$, we define

$$D^{=c} := \{D' \in \mathrm{Pic}(S_r)_{\mathbb{R}} \mid (D, D') = c\},$$

and similarly,

$$D^{\geq c} := \{D' \in \mathrm{Pic}(S_r)_{\mathbb{R}} \mid (D, D') \geq c\}.$$

Proof of Theorem 3. By definition, $\Lambda^\vee_{\mathrm{eff}}(S_r)$ is the intersection of the half spaces $E^{\geq 0}$ for all generators E of $\Lambda_{\mathrm{eff}}(S_r)$, which are exactly the (-1)-curves by Lemma 5. By [16, Lemma 5.3], W_r acts transitively on the (-1)-curves. This symmetry implies that each (-1)-curve E defines a proper face $F_E := \Lambda_{\mathrm{eff}}(S_r) \cap E^{=0}$, and that W_r acts transitively on the set of faces $\{F_E \mid E \text{ is a } (-1)\text{-curve}\}$.

Consider S_r as the blow-up of S_{r-1} in one point, resulting in the exceptional divisor E_r. Then

$$\mathrm{Pic}(S_r) = \mathrm{Pic}(S_{r-1}) \oplus \mathbb{Z} \cdot E_r$$

is an orthogonal sum.

We claim that $F_{E_r} = \Lambda_{\mathrm{eff}}(S_{r-1})$, where we regard $\Lambda_{\mathrm{eff}}(S_{r-1}) \subset \mathrm{Pic}(S_{r-1})$ as embedded into $\mathrm{Pic}(S_r)$.

Indeed, if $D \in \Lambda_{\mathrm{eff}}(S_{r-1})$, then $(D, E_r) = 0$, and $(D, E) \geq 0$ for all (-1)-curves E of S_{r-1}, which are exactly the (-1)-curves of S_r with $(E, E_r) = 0$. By Lemma 6, we have $(D, E) \geq 0$ for all (-1)-curves of S_r.

On the other hand, if $D \in E_r^{=0}$, then $D \in \mathrm{Pic}(S_{r-1})$. If $D \in \Lambda_{\mathrm{eff}}(S_r)$, then $(D, E) \geq 0$ for all (-1)-curves of S_r, which includes the (-1)-curves of S_{r-1}, proving the other direction.

The root system corresponding to W_r is

$$R_r = \{D \in \mathrm{Pic}(S_r) \mid (D, D) = -2, (D, -K_r) = 0\}.$$

Since W_r is generated by the reflections $E \mapsto E + (D, E) \cdot D$ corresponding to the roots $D \in R_r$, the anticanonical class $-K_r$ is fixed under W_r. This completes the proof of Theorem 3. □

Proof of Theorem 4. The polytope $P_r := P(S_r)$ whose volume is $\alpha(S_r)$ (see Definition 2) is the intersection of the N_r half-spaces $E^{\geq 0}$ (where E runs through the (-1)-curves of S_r) in the r-dimensional space $-K_r^{=1}$.

Note that $(-K_r, -K_r) = 9 - r$. Therefore, $Q := \frac{1}{9-r} \cdot (-K_r) \in -K_r^{=1}$, and since $(-K_r, E) = 1$ for any (-1)-curve E, the point Q is in the interior of P_r.

Consider the convex hull P_E of Q and the face $P_r \cap E^{=0}$ of P_r corresponding to E. Then P_r is the union of the P_E for all (-1)-curves E, and since their intersections are lower-dimensional,

$$\mathrm{Vol}(P_r) = \sum_E \mathrm{Vol}(P_E).$$

As the intersection form and $-K_r$ are invariant under the Weyl group W_r, it acts on $-K_r^{=1}$ and therefore on P_r. As in Theorem 3, it permutes the faces of P_r transitively. As Q is fixed under W_r and the volume is invariant under W_r, we have $\mathrm{Vol}(P_r) = N_r \cdot \mathrm{Vol}(P_E)$ for any (-1)-curve E.

As in the proof of Theorem 3, we consider S_r as the blow-up of S_{r-1} in one point, resulting in the exceptional divisor E_r, with the orthogonal sum

$$\mathrm{Pic}(S_r) = \mathrm{Pic}(S_{r-1}) \oplus \mathbb{Z} \cdot E_r.$$

We claim that $P_r \cap E_r^{=0} = P_{r-1}$. In view of Theorem 3, it remains to prove that $(D, -K_{r-1}) = 1$ is equivalent to $(D, -K_r) = 1$ on $E_r^{=0}$. This follows directly from $-K_r = -K_{r-1} - E_r$.

Therefore, P_{E_r} is a cone over the $(r-1)$-dimensional polytope P_{r-1} in the r-dimensional space $-K_r^{=1}$. A cone of height 1 over P_{r-1} has volume $\mathrm{Vol}(P_{r-1})/r$. As E_r is orthonormal to $\mathrm{Pic}(S_{r-1})$, and $(-K_r, E_r) = 1$, the distance of Q to P_{r-1} is $1/(9-r)$. Therefore,

$$\mathrm{Vol}(P_{E_r}) = \frac{\mathrm{Vol}(P_{r-1})}{r \cdot (9-r)}.$$

Together with $\alpha(S_{r-1}) = \mathrm{Vol}(P_{r-1})$ and $\mathrm{Vol}(P_r) = N_r \cdot \mathrm{Vol}(P_{E_r})$, this completes the proof of the recursive formula.

For $r = 2$, we have $\Lambda_{\text{eff}} = \langle E_1, E_2, H - E_1 - E_2 \rangle$ and $-K_2 = 3H - E_1 - E_2$. Therefore, $\alpha(S_2)$ is the volume of

$$\begin{aligned}&\{(a_0, a_1, a_2) \in \mathbb{R}^3 \mid 3a_0 - a_1 - a_2 = 1, a_1 \geq 0, a_2 \geq 0, a_0 - a_1 - a_2 \geq 0\}\\ =&\{(a_0, a_1) \in \mathbb{R}^2 \mid a_1 \geq 0, 3a_0 - a_1 - 1 \geq 0, -2a_0 + 1 \geq 0\}\\ =&\text{convex hull of } (1/3, 0),\ (1/2, 0),\ (1/2, 1/2),\end{aligned}$$

which is a rectangular triangle whose legs have length $1/6$ and $1/2$. Hence, $\alpha(S_2) = 1/24$, while $\alpha(S_1) = 1/6$, $\alpha(\mathbb{P}^1 \times \mathbb{P}^1) = 1/4$, and $\alpha(\mathbb{P}^2) = 1/3$ can also be calculated directly, which completes the proof of Theorem 4. □

Remark 7. By the proof of [18, Lemme 9.4.2], $\alpha(S_1) = 1/6$, and by the proof of [18, Lemme 10.4.2],

$$\alpha(S_2) = 1/3 \cdot \mathrm{Vol}\{(x_1, x_2) \in \mathbb{R}^2_{>0} \mid x_1 + x_2 \leq 1/2\},$$

which is clearly $\alpha_2 = 1/24$ and therefore agrees with our result. Note that the recursion formula does not hold for $r = 2$:

$$\alpha(S_2) = \frac{1}{24} \neq \frac{N_2 \cdot \alpha(S_1)}{2 \cdot (9-2)} = \frac{1}{28}.$$

The value $\alpha(S_4) = 1/(6 \cdot 4!)$ was previously calculated in [4, Section 1.3].

3. Singular Del Pezzo surfaces

For the classification of singular Del Pezzo surfaces, see [10], [11], [1]. It turns out that in each degree, the surfaces can be divided into different types according to the number and types of their singularities and the number of lines.

For each type, we might have more than one isomorphism class (e.g., two for type $\mathbf{D}_4$ in degree 3, and an infinite family for type $\mathbf{A}_1$ in degree 3). However, the degrees in $\mathrm{Pic}(\widetilde{S}_r)$ of the negative curves, which generate the effective cone, and of $-K_r$ only depend on the type. Therefore, $\alpha(S_r)$ depends only on the type.

More precisely, all this information is encoded for each type in its extended Dynkin diagram of negative curves. For degree ≥ 4, these diagrams can be found in [11, Propositions 6.1, 8.1, 8.3, 8.4]. In [1], the information for each degree is encoded in a smaller number of diagrams. Also see [12] and [13, Chapter 6] for detailed information on several singular types of various degrees.

From the extended Dynkin diagram, we can derive a basis of $\mathrm{Pic}(\widetilde{S}_r)$, and $-K_r$ and all effective divisors in terms of this basis as explained in [12, Section 3].

With this information, it is straightforward to calculate $\alpha(S_r)$ for minimal desingularizations of all split types of degree ≥ 3. In practice, this task is significantly simplified by the use of software such as `Polymake`.

For the Hirzebruch surface F_2, which is the unique singular Del Pezzo surface of degree 8, we have $\alpha(F_2) = 1/8$.

For S_r of types $\mathbf{D}_4$ and $\mathbf{D}_5$ of degree 4 and $\mathbf{E}_6$ of degree 3, the constant $\alpha(S_r)$ has been calculated while proving Manin's conjecture for these surfaces; see [14], [5], [7], respectively. Note that in these three cases, the calculation of $\alpha(S_r)$ is particularly simple as the effective cone is *simplicial*, i.e., the number of generators of Λ_{eff} equals the rank of $\mathrm{Pic}(\widetilde{S}_r)$.

type	singularities	lines	generators of $\Lambda_{\mathrm{eff}}^\vee$	α
0	–	3	3	1/24
i	$\mathbf{A}_1$	2	3	1/48

Table 2. Del Pezzo surfaces of degree 7

type	singularities	lines	generators of $\Lambda_{\mathrm{eff}}^\vee$	α
0	–	6	5	1/72
i	$\mathbf{A}_1$	4	5	1/144
ii	$\mathbf{A}_1$	3	4	1/144
iii	$2\mathbf{A}_1$	2	4	1/288
iv	$\mathbf{A}_2$	2	4	1/432
v	$\mathbf{A}_2 + \mathbf{A}_1$	1	4	1/864

Table 3. Del Pezzo surfaces of degree 6

type	singularities	lines	generators of $\Lambda_{\mathrm{eff}}^\vee$	α
0	–	10	10	1/144
i	$\mathbf{A}_1$	7	9	1/288
ii	$2\mathbf{A}_1$	5	8	1/576
iii	$\mathbf{A}_2$	4	7	1/864
iv	$\mathbf{A}_2 + \mathbf{A}_1$	3	7	1/1728
v	$\mathbf{A}_3$	2	5	1/3456
vi	$\mathbf{A}_4$	1	5	1/17280

Table 4. Del Pezzo surfaces of degree 5

Remark 8. We have two different types with the same singularities in three cases ($\mathbf{A}_1$ in degree 6, $2\mathbf{A}_1$ and $\mathbf{A}_3$ in degree 4). The two types in each pair can be distinguished by their number of lines. Therefore, the two different types have different effective cones. However, both types in each pair have the same constant α. It is unclear whether this is more than a coincidence.

Remark 9. For singular Del Pezzo surfaces S_r of degree 2 and 1, we do not calculate $\alpha(S_r)$ since the number of different types is much larger than in degree ≥ 3.

For surfaces of degree 1 whose effective cone has many generators, the triangulation of the nef cone might be too complicated for a computation of α using `Polymake`. We expect this to happen in case of "mild" singularities (e.g., of type $\mathbf{A}_1$, $\mathbf{A}_2$ or $2\mathbf{A}_1$). Here, the nef cone should be almost as complicated as in the smooth case, and some help "by hand" might be necessary.

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type	singularities	lines	generators of $\Lambda_{\mathrm{eff}}^{\vee}$	α
0	$-$	16	26	$1/180$
i	$\mathbf{A}_1$	12	22	$1/360$
ii	$2\mathbf{A}_1$	9	19	$1/720$
iii	$2\mathbf{A}_1$	8	17	$1/720$
iv	$\mathbf{A}_2$	8	16	$1/1080$
v	$3\mathbf{A}_1$	6	15	$1/1440$
vi	$\mathbf{A}_2+\mathbf{A}_1$	6	15	$1/2160$
vii	$\mathbf{A}_3$	5	11	$1/4320$
$viii$	$\mathbf{A}_3$	4	10	$1/4320$
ix	$4\mathbf{A}_1$	4	4	$1/2880$
x	$\mathbf{A}_2+2\mathbf{A}_1$	4	13	$1/4320$
xi	$\mathbf{A}_3+\mathbf{A}_1$	3	10	$1/8640$
xii	$\mathbf{A}_4$	3	9	$1/21600$
$xiii$	$\mathbf{D}_4$	2	6	$1/34560$
xiv	$\mathbf{A}_3+2\mathbf{A}_1$	2	10	$1/17280$
xv	$\mathbf{D}_5$	1	6	$1/345600$

Table 5. Del Pezzo surfaces of degree 4

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type	singularities	lines	generators of $\Lambda_{\text{eff}}^{\vee}$	α
0	–	27	99	1/120
i	$\mathbf{A}_1$	21	78	1/240
ii	$2\mathbf{A}_1$	16	62	1/480
iii	$\mathbf{A}_2$	15	52	1/720
iv	$3\mathbf{A}_1$	12	49	1/960
v	$\mathbf{A}_2 + \mathbf{A}_1$	11	43	1/1440
vi	$\mathbf{A}_3$	10	32	1/2880
vii	$4\mathbf{A}_1$	9	38	1/1920
viii	$\mathbf{A}_2 + 2\mathbf{A}_1$	8	35	1/2880
ix	$\mathbf{A}_3 + \mathbf{A}_1$	7	27	1/5760
x	$2\mathbf{A}_2$	7	31	1/4320
xi	$\mathbf{A}_4$	6	21	1/14400
xii	$\mathbf{D}_4$	6	16	1/23040
xiii	$\mathbf{A}_3 + 2\mathbf{A}_1$	5	23	1/11520
xiv	$2\mathbf{A}_2 + \mathbf{A}_1$	5	18	1/28800
xv	$\mathbf{A}_4 + \mathbf{A}_1$	4	18	1/28800
xvi	$\mathbf{A}_5$	3	13	1/86400
xvii	$\mathbf{D}_5$	3	11	1/230400
xviii	$3\mathbf{A}_2$	3	21	1/25920
xix	$\mathbf{A}_5 + \mathbf{A}_1$	2	13	1/172800
xx	$\mathbf{E}_6$	1	7	1/6220800

TABLE 6. Del Pezzo surfaces of degree 3

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