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ZEROS OF RANDOM POLYNOMIALS ON $\mathbb{C}^m$

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Abstract. For a regular compact set K in $\mathbb{C}^m$ and a measure μ on K satisfying the Bernstein-Markov inequality, we consider the ensemble $\mathcal{P}_N$ of polynomials of degree N, endowed with the Gaussian probability measure induced by $L^2(\mu)$. We show that for large N, the simultaneous zeros of m polynomials in $\mathcal{P}_N$ tend to concentrate around the Silov boundary of K; more precisely, their expected distribution is asymptotic to $N^m\mu_{eq}$, where μ_{eq} is the equilibrium measure of K. For the case where K is the unit ball, we give scaling asymptotics for the expected distribution of zeros as $N \to \infty$.

1. Introduction

A classical result due to Hammersley [Ha] (see also [SV]), loosely stated, is that the zeros of a random complex polynomial

$$f(z) = \sum_{j=0}^{N} c_j z^j \tag{1}$$

mostly tend towards the unit circle $|z| = 1$ as the degree $N \to \infty$, when the coefficients c_j are independent complex Gaussian random variables of mean zero and variance one. In this paper, we will prove a multivariable result (Theorem 3.1), a special case (Example 3.5) of which shows, loosely stated, that the common zeros of m random complex polynomials in $\mathbb{C}^m$,

$$f_k(z) = \sum_{|J|\le N} c_J^k\, z_1^{j_1} \cdots z_m^{j_m} \qquad \text{for } k = 1, \dots, m\ , \tag{2}$$

tend to concentrate on the product of the unit circles $|z_j| = 1$ $(j = 1, \dots, m)$ as $N \to \infty$, when the coefficients c_J^k are i.i.d. complex Gaussian random variables.

The following is our basic setting: We let K be a compact set in $\mathbb{C}^m$ and let μ be a Borel probability measure on K. We assume that K is non-pluripolar and we let V_K be its pluricomplex Green function. We also assume that K is regular (i.e., $V_K = V_K^*$) and that μ satisfies the Bernstein-Markov inequality (see §2). We give the space $\mathcal{P}_N$ of holomorphic polynomials of degree $\le N$ on $\mathbb{C}^m$ the Gaussian probability measure γ_N induced by the Hermitian inner product

$$(f, g) = \int_K f\bar{g}\, d\mu\ . \tag{3}$$

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The Gaussian measure γ_N can be described as follows: We write $f = \sum_{j=1}^{d(N)} c_j p_j$, where $\{p_j\}$ is an orthonormal basis of $\mathcal{P}_N$ with respect to (3) and $d(N) = \dim \mathcal{P}_N = \binom{N+m}{m}$. Identifying $f \in \mathcal{P}_N$ with $c = (c_1, \ldots, c_{d(N)}) \in \mathbb{C}^{d(N)}$, we have

$$d\gamma_N(s) = \frac{1}{\pi^{d(N)}} e^{-|c|^2} dc\,.$$

(The measure γ_N is independent of the choice of orthonormal basis $\{p_j\}$.) In other words, a random polynomial in the ensemble $(\mathcal{P}_N, \gamma_N)$ is a polynomial $f = \sum_j c_j\, p_j$, where the c_j are independent complex Gaussian random variables with mean 0 and variance 1.

Our main result, Theorem 3.1, gives asymptotics for the expected zero current of k i.i.d. random polynomials with respect to the Gaussian probability measure induced by (3) (where $1 \le k \le m$). In particular, the expected distribution $\mathbf{E}(Z_{f_1,\ldots,f_m})$ of simultaneous zeros of m independent random polynomials in $(\mathcal{P}_N, \gamma_N)$ has the asymptotics

$$\frac{1}{N^m}\, \mathbf{E}(Z_{f_1,\ldots,f_m}) \to \mu_{eq} \quad weak^*\ , \tag{4}$$

where $\mu_{eq} = (\frac{i}{\pi}\partial\bar\partial V_K)^m$ is the equilibrium measure of K (see (8) below). Here, $\mathbf{E}(X)$ denotes the expected value of a random variable X.

We now describe some recent related results on expected distributions of zeros. The one-dimensional case of (4) was given in [Bl2], which generalized the results in [SZ2] for the case where K is a real-analytic domain in $\mathbb{C}$ (or its boundary). After this paper was written, generalizations of (4) to weighted equilibrium measures were given in [Bl3], and generalizations to equilibrium measures on pseudoconcave domains in compact Kähler manifolds were given by R. Berman [Be]. It was also shown in [Bl3] that (4) holds for certain non-Gaussian random polynomials on $\mathbb{C}$. Results on the distribution of zeros of polynomials on $\mathbb{C}$ with random real coefficients were given by Shepp-Vanderbei [SV], Ibragimov-Zeitouni [IZ] and others.

The reader may notice that the distributions of zeros for the measures on $\mathcal{P}_N$ considered here are quite different from those of the $\mathrm{SU}(m+1)$ ensembles studied, for example, in [SZ1, SZ4, BSZ1, BSZ2, DS]. The Gaussian measure on the $\mathrm{SU}(m+1)$ polynomials is based on the inner product

$$\langle f, g\rangle_N = \int_{S^{2m+1}} F_N\, \overline{G_N},$$

where $F_N, G_N \in \mathbb{C}[z_0, z_1, \ldots, z_m]$ denote the degree N homogenizations of f and g respectively. It follows easily from the $\mathrm{SU}(m+1)$-invariance of the inner product that the expected distribution of simultaneous zeros equals $\frac{N^m}{\pi^m}\omega^m$ (exactly), where ω is the Fubini-Study Kähler form (on $\mathbb{C}^m \subset \mathbb{CP}^m$). We note that, unlike (3), this inner product depends on N; indeed, $\|z^J\|_N^2 = \frac{m!(N-|J|)!j_1!\cdots j_m!}{(N+m)!}$ [SZ1, (30)].

In this paper, we also give scaling limits for the expected zero density in the case of the unit ball in $\mathbb{C}^m$ (Theorem 4.1). An open problem is to find scaling limits for more general sets in $\mathbb{C}^m$.

2. Background

We let $\mathcal{L}$ denote the Lelong class of plurisubharmonic (PSH) functions on $\mathbb{C}^m$ of at most logarithmic growth at ∞. That is

$$\mathcal{L} := \{u \in \mathrm{PSH}(\mathbb{C}^m) \mid u(z) \leq \log^+ \|z\| + O(1)\} \tag{5}$$

For K a compact subset of $\mathbb{C}^m$, we define its pluricomplex Green function $V_K(z)$ via

$$V_K(z) = \sup\{u(z) \mid u \in \mathcal{L},\ u \leq 0 \ \text{ on } \ K\}. \tag{6}$$

We will assume K is regular, that is by definition, V_K is continuous on $\mathbb{C}^m$ (and so $V_K = V_K^*$, its uppersemicontinuous regularization). The function V_K is a locally bounded PSH function on $\mathbb{C}^m$ and, in fact

$$V_K - \log^+ \|z\| = O(1)\ . \tag{7}$$

By a basic result of Bedford and Taylor [BT1] (see [Kl]), the complex Monge-Ampère operator $(dd^c)^m = (2i\partial\bar{\partial})^m$ is defined on any locally bounded PSH function $\mathbb{C}^m$ and in particular on V_K. The equilibrium measure of K [BT1, BT2, Le, NZ, Ze] (see also [Kl, Cor. 5.5.3]) is defined by

$$\mu_{eq}(K) := \left(\frac{i}{\pi}\partial\bar{\partial}V_K\right)^m \tag{8}$$

Since V_K satisfies (7), it is a positive Borel measure, here normalized to have mass 1. The support of the measure $\mu_{eq}(K)$ is the Silov boundary of K for the algebra of entire analytic functions [BT2]. In one variable, i.e. $K \subset \mathbb{C}$, V_K is the Green function of the unbounded component of $\mathbb{C} \setminus K$ with a logarithmic pole at ∞, and $\mu_{eq}(K) = \frac{1}{2\pi}\Delta V_K$, where Δ is the Laplacian [Ra].

Let μ be a finite positive Borel measure on K. The measure μ is said to satisfy a Bernstein-Markov (BM) inequality, if, for each $\varepsilon > 0$ there is a constant $C = C(\varepsilon) > 0$ such that

$$\|p\|_K \leq C e^{\varepsilon \deg(p)} \|p\|_{L^2(\mu)} \tag{9}$$

for all holomorphic polynomials p. Essentially, the BM inequality says that the L^2 norms and the sup norms of a sequence of holomorphic polynomials of increasing degrees are "asymptotically equivalent".

The question arises as to which measures actually satisfy the BM inequality. It is a result of Nguyen-Zeriahi [NZ] combined with [Kl, Cor. 5.6.7] that for K regular, $\mu_{eq}(K)$ satisfies BM. This fact is used in Examples 3.5–3.6. In [Bl1, Theorem 2.2], a "mass-density" condition for a measure to satisfy BM was given. (See also [BL].)

Our proof uses the *probabilistic Poincaré-Lelong formula* for the zeros of random functions (Proposition 2.1 below). Considering a slightly more general situation, we let $g_1, \ldots, g_d$ be polynomials with no common zeros on a domain $U \subset \mathbb{C}^m$. (We are interested in the case where $U = \mathbb{C}^m$ and $\{g_j\}$ is an orthonormal basis of $\mathcal{P}_N$ with respect to the inner product (3), as discussed above.) We let $\mathcal{F}$ denote the ensemble of random polynomials of the form $f = \sum c_j\, g_j$, where the c_j are independent complex Gaussian random variables with mean 0 and variance 1. We consider the *Szegő kernel*

$$S_{\mathcal{F}}(z,w) = \sum_{j=1}^{d} g_j(z)\,\overline{g_j(w)}\ .$$

For the case where the g_j are orthonormal with respect to an inner product on $\mathcal{O}(U)$, $S_{\mathcal{F}}(z,w)$ is the kernel for the orthogonal projection onto the span of the g_j.

Under the assumption that the g_j have no common zeros, it is easily shown using Sard's theorem (or Bertini's theorem) that for almost all $f_1,\dots,f_k \in \mathcal{F}$, the differentials $df_1,\dots,df_k$ are linearly independent at all points of the zero set

$$\mathrm{loc}(f_1,\dots,f_k) := \{z \in U : f_1(z) = \cdots = f_k(z) = 0\} .$$

This condition implies that the complex hypersurfaces $\mathrm{loc}(f_j)$ are smooth and intersect transversely, and hence $\mathrm{loc}(f_1,\dots,f_k)$ is a codimension k complex submanifold of U. We then let $Z_{f_1,\dots,f_k} \in \mathcal{D}'^{k,k}(U)$ denote the current of integration over $\mathrm{loc}(f_1,\dots,f_k)$:

$$\left(Z_{f_1,\dots,f_k},\varphi\right) = \int_{\mathrm{loc}(f_1,\dots,f_k)} \varphi \,, \qquad \varphi \in \mathcal{D}^{m-k,m-k}(U) \,.$$

We shall use the following Poincaré-Lelong formula:

Proposition 2.1. *The expected zero current of k independent random polynomials $f_1,\dots,f_k \in \mathcal{F}$ is given by*

$$\mathbf{E}(Z_{f_1,\dots,f_k}) = \left(\frac{i}{2\pi}\partial\bar\partial \log S_{\mathcal{F}}(z,z)\right)^k .$$

A proof of Proposition 2.1 (for sections of holomorphic line bundles) can be found in [SZ4]. The codimension $k=1$ case was given in [SZ1]. The general case follows from the codimension 1 case together with the fact that

$$\mathbf{E}(Z_{f_1,\dots,f_k}) = \mathbf{E}(Z_{f_1}) \wedge \cdots \wedge \mathbf{E}(Z_{f_k}) = \mathbf{E}(Z_f)^k \,, \tag{10}$$

which is a consequence of the independence of the f_j. The wedge product of currents is not always defined, but $Z_{f_1} \wedge \cdots \wedge Z_{f_k}$ is almost always defined (and equals $Z_{f_1,\dots,f_k}$ whenever the hypersurfaces $\mathrm{loc}(f_j)$ are smooth and intersect transversely), and a short argument given in [SZ3] or [SZ4] yields (10). The point case $k=m$ of Proposition 2.1 was given by Edelman-Kostlan [EK, Th. 8.1]. We note that the expectations in (10) are smooth forms.

3. Asymptotics of expected zero currents

We now state our main result on the expected distribution of simultaneous zeros of random polynomials orthonormalized on a compact set:

Theorem 3.1. *Let μ be a Borel probability measure on a regular compact set $K \subset \mathbb{C}^m$, and suppose that (K,μ) satisfies the Bernstein-Markov inequality. Let $1 \le k \le m$, and let $(\mathcal{P}_N^k, \gamma_N^k)$ denote the ensemble of k-tuples of i.i.d. Gaussian random polynomials of degree $\le N$ with the Gaussian measure $d\gamma_N$ induced by $L^2(\mu)$. Then*

$$\frac{1}{N^k}\mathbf{E}_{\gamma_N^k}(Z_{f_1,\dots,f_k}) \to \left(\frac{i}{\pi}\partial\bar\partial V_K\right)^k \qquad \textit{weak}^*, \quad \textit{as } N \to \infty \,,$$

where V_K is the pluricomplex Green function of K with pole at infinity.

Here we say that a sequence T_N of currents of order 0 converges weak* to T if $(T_N, \varphi) \to (T, \varphi)$ for all compactly supported test forms φ with continuous coefficients. Using general variance estimates given in [Sh] together with Theorem 3.1, one can show that with probability one, a sequence $\{f_1^N, \dots, f_k^N\}_{N=1,2,\dots}$ of k-tuples of random polynomials of increasing degree satisfies:

$$\frac{1}{N^k} Z_{f_1^N,\dots,f_k^N} \to \left(\frac{i}{\pi}\partial\bar{\partial}V_K\right)^k \qquad weak^* \; . \tag{11}$$

(See [Sh]; the one-dimensional case of (11) was given in [Bl2] using potential theory.)

To prove Theorem 3.1, we consider the Szegő kernels

$$S_N(z,w) := S_{(\mathcal{P}_N,\gamma_N)}(z,w) = \sum_{j=1}^{d(N)} p_j(z)\overline{p_j(w)} \; ,$$

where $\{p_j\}$ is an $L^2(\mu)$-orthonormal basis for $\mathcal{P}_N$. Our proof is based on approximating the extremal function V_K by the (normalized) logarithms of the Szegő kernels $S_N(z,z)$ (Lemma 3.4) and then applying the Poincaré-Lelong formula of Proposition 2.1.

We begin by considering the polynomial suprema

$$\Phi_N^K(z) = \sup\{|f(z)| : f \in \mathcal{P}_N, \; \|f\|_K \le 1\} \; . \tag{12}$$

Since $\frac{1}{N}\log f \in \mathcal{L}$, for $f \in \mathcal{P}_N$, it is clear that $\frac{1}{N}\log\Phi_N^K \le V_K$, for all N. Pioneering work of Zaharjuta [Za] and Siciak [Si1, Si2] established the convergence of $\frac{1}{N}\log\Phi_N^K$ to V_K. The uniform convergence when K is regular seems not to have been explicitly stated and we give the proof below.

Lemma 3.2. *Let K be a regular compact set in $\mathbb{C}^m$. Then*

$$\frac{1}{N}\log\Phi_N^K(z) \to V_K(z)$$

uniformly on compact subsets of $\mathbb{C}^m$.

Proof. We first note that $1 \le \Phi_j \le \Phi_j\Phi_k \le \Phi_{j+k}$, for $j,k \ge 0$. By a result of Siciak [Si1] and Zaharjuta [Za] (see [Kl, Theorem 5.1.7]),

$$V_K(z) = \lim_{N\to\infty}\frac{1}{N}\log\Phi_N^K(z) = \sup_N \frac{1}{N}\log\Phi_N^K(z) \; , \tag{13}$$

for all $z \in \mathbb{C}^m$.

We use the regularity of K to show that the convergence is uniform: let

$$\psi_N = \frac{1}{N}\log\Phi_N^K \ge 0 \; .$$

Thus for $N,k \ge 1$, $j \ge 0$, we have

$$Nk\,\psi_{Nk} + j\,\psi_j \le (Nk+j)\,\psi_{Nk+j} \; .$$

Since $\psi_N \le \psi_{Nk}$, we then obtain the inequality

$$\psi_{Nk+j} \ge \frac{Nk}{Nk+j}\psi_N + \frac{j}{Nk+j}\psi_j \ge \frac{Nk}{Nk+j}\psi_N \; . \tag{14}$$

Fix $\varepsilon > 0$. For each $a \in \mathbb{C}^m$, we choose $N_a \in \mathbb{Z}^+$ such that

$$V_K(a) - \psi_{N_a}(a) < \varepsilon \qquad \text{and} \qquad \frac{V_K(a)}{N_a} < \varepsilon \,,$$

and then choose a neighborhood U_a of a such that

$$|V_K(z) - V_K(a)| < \varepsilon, \quad \psi_{N_a}(z) \geq \psi_{N_a}(a) - \varepsilon, \quad \frac{V_K(z)}{N_a} < \varepsilon, \qquad \text{for } \; z \in U_a \,.$$

Now let $N \geq N_a^2$, and write $N = N_a k + j$, where $k \geq N_a,\ 0 \leq j < N_a$. By (13)–(14), we have

$$(15)\quad 0 \leq V_K - \psi_N \leq V_K - \frac{N_a k}{N_a k + j}\psi_{N_a} \leq V_K - \frac{N_a}{N_a + 1}\psi_{N_a} \leq V_K - \psi_{N_a} + \frac{1}{N_a + 1} V_K \,.$$

Hence, for all $N \geq N_a^2$ and for all $z \in U_a$, we have

$$\begin{aligned} (16) \qquad 0 \;&\leq\; V_K(z) - \psi_N(z) \\ &<\; V_K(z) - \psi_{N_a}(z) + \varepsilon \\ &=\; [V_K(a) - \psi_{N_a}(a)] + [V_K(z) - V_K(a)] + [\psi_{N_a}(a) - \psi_{N_a}(z)] + \varepsilon \\ &<\; 4\varepsilon \,. \end{aligned}$$

Hence for each compact $A \subset \mathbb{C}^m$, we can cover A with finitely many U_{a_i}, so that we have by (16),

$$\|V_K - \psi_N\|_A \leq 4\varepsilon \qquad \forall\ N \geq \max_i N_{a_i}^2 \,.$$

□

Lemma 3.3. *For all $\varepsilon > 0$, there exists $C = C_\varepsilon > 0$ such that*

$$\frac{1}{d(N)} \leq \frac{S_N(z,z)}{\Phi_N^K(z)^2} \leq C\, e^{\varepsilon N} d(N).$$

Proof. Let $f \in \mathcal{P}_N$ with $\|f\|_K \leq 1$. Then

$$\begin{aligned} |f(z)| \;&=\; \left| \int_K S_N(z,w) f(w)\, d\mu(w) \right| \;\leq\; \int_K |S_N(z,w)|\, d\mu(w) \\ &\leq\; \int_K S_N(z,z)^{\frac12} S_N(w,w)^{\frac12}\, d\mu(w) \;=\; S_N(z,z)^{\frac12}\, \|S_N(w,w)^{\frac12}\|_{L^1(\mu)} \\ &\leq\; S_N(z,z)^{\frac12}\, \|1\|_{L^2(\mu)}\, \|S_N(w,w)^{\frac12}\|_{L^2(\mu)} \;=\; S_N(z,z)^{\frac12}\, d(N)^{\frac12} \,. \end{aligned}$$

Taking the supremum over $f \in \mathcal{P}_N$ with $\|f\|_K \leq 1$, we obtain the left inequality of the lemma.

To verify the right inequality, we let $\{p_j\}$ be a sequence of $L^2(\mu)$-orthonormal polynomials, obtained by applying Gram-Schmid to a sequence of monomials of non-decreasing degree, so that $\{p_1, \ldots p_{d(N)}\}$ is an orthonormal basis of $\mathcal{P}_N$ (for each $N \in \mathbb{Z}^+$). By the Bernstein-Markov inequality (9), we have

$$\|p_j\|_K \leq C\, e^{\varepsilon \deg p_j}$$

and hence

$$|p_j(z)| \leq \|p_j\|_K\, \Phi^K_{\deg p_j}(z) \leq C\, e^{\varepsilon \deg p_j}\, \Phi^K_{\deg p_j}(z) \leq C\, e^{\varepsilon N}\, \Phi_N^K(z)\,, \quad \text{for } \; j \leq d(N).$$

Therefore,

$$S_N(z,z) = \sum_{j=1}^{d(N)} |p_j(z)|^2 \le d(N)\, C^2\, e^{2\varepsilon N} \Phi_N^K(z)^2 \ .$$

□

Lemma 3.4. *Under the hypotheses of Theorem 3.1, we have*

$$\frac{1}{2N} \log S_N(z,z) \to V_K(z)$$

uniformly on compact subsets of $\mathbb{C}^m$.

Proof. Let $\varepsilon > 0$ be arbitrary. Recalling that $d(N) = \binom{N+m}{m}$, we have by Lemma 3.3,

$$-\frac{m}{N}\log(N+m) \le \frac{1}{N}\log\left(\frac{S_N(z,z)}{\Phi_N^K(z)^2}\right) \le \frac{\log C}{N} + \varepsilon + \frac{m}{N}\log(N+m) \ .$$

Since $\varepsilon > 0$ is arbitrary, we then have

$$\frac{1}{N}\log\left(\frac{S_N(z,z)}{\Phi_N^K(z)^2}\right) \to 0 \ . \tag{17}$$

The conclusion follows from Lemma 3.2 and (17). □

Remark: The asymptotic behavior of the orthonormal polynomials $\{p_j\}$ was first studied by A. Zeriahi [Ze], who showed that

$$\limsup_{j\to\infty} \frac{1}{\deg p_j}\ \log|p_j(z)| = V_K(z)\ , \qquad \text{for all}\ \ z \in \mathbb{C}^m \smallsetminus \widehat{K}\ , \tag{18}$$

where $\widehat{K}$ denotes the polynomially convex hull of K. Zeriahi's formula (18) follows immediately from Lemma 3.4.

Proof of Theorem 3.1: It follows from Lemma 3.4, together with continuity of the complex Monge-Àmpere operator under uniform limits [BT1], that

$$\left(\frac{i}{2\pi N}\partial\bar\partial \log S_N(z,z)\right)^k \to \left(\frac{i}{\pi}\partial\bar\partial V_K(z)\right)^k \qquad \mathit{weak}^*\ .$$

The conclusion then follows from Proposition 2.1. □

Example 3.5. Let K be the unit polydisk in $\mathbb{C}^m$. Then $V_K = \max_{j=1}^m \log^+|z_j|$, the Silov boundary of K is the product of the circles $|z_j| = 1$ ($j = 1,\dots,m$), and $d\mu_{eq} = (\frac{1}{2\pi})^m d\theta_1 \cdots d\theta_m$ where $d\theta_j$ is the angular measure on the circle $|z_j| = 1$.

The monomials $z^J := z_1^{j_1}\cdots z_m^{jm}$, for $|J| \le N$, form an orthonormal basis for $\mathcal{P}_N$. A random polynomial in the ensemble is of the form

$$f(z) = \sum_{|J|\le N} c_J z^J$$

where the c_J are independent complex Gaussian random variables of mean zero and variance one. By Theorem 3.1, $\mathbf{E}_{\gamma_N^m}(Z_{f_1,\dots,f_m}) \to (\frac{1}{2\pi})^m d\theta_1\cdots d\theta_m$ weak*, as $N\to\infty$. In particular, the common zeros of m random polynomials tend to the product of the unit circles $|z_j| = 1$ for $j = 1,\dots,m$.

Example 3.6. Let K be the unit ball $\{\|z\| \le 1\}$ in $\mathbb{C}^m$. Then the Silov boundary of K is its topological boundary $\{\|z\| = 1\}$, $V_K(z) = \log^+ \|z\|$, and μ_{eq} is the invariant hypersurface measure on $\|z\| = 1$ normalized to have total mass one.

4. Scaling limit zero density for orthogonal polynomials on S^{2m-1}

Examples 3.5 and 3.6 both reduce to the unit disk in the one variable case. In that case, detailed scaling limits are known (see, for example, [IZ]). For a more general compact set $K \subset \mathbb{C}$ with an analytic boundary, scaling limits are found in [SZ2].

In this section, we consider the case where $K = \{z \in \mathbb{C}^m : \|z\| \le 1\}$ is the unit ball and μ is its equilibrium measure, i.e. invariant measure on the unit sphere S^{2m-1}. We have the following scaling asymptotics for the expected distribution of zeros of m random polynomials orthonormalized on the sphere:

Theorem 4.1. *Let $(\mathcal{P}_N^m, \gamma_N^m)$ denote the ensemble of m-tuples of i.i.d. Gaussian random polynomials of degree $\le N$ with the Gaussian measure $d\gamma_N$ induced by $L^2(S^{2m-1}, \mu)$, where μ is the invariant measure on the unit sphere $S^{2m-1} \subset \mathbb{C}^m$. Then*

$$\mathbf{E}_{\gamma_N^m}(Z_{f_1,\dots,f_m}) = D_N\left(\log\|z\|^2\right)\left(\frac{i}{2}\partial\bar\partial\|z\|^2\right)^m ,$$

where

$$\frac{1}{N^{m+1}} D_N\left(\frac{u}{N}\right) = \frac{1}{\pi^m} F_m''(u)\, F_m'(u)^{m-1} + O\left(\frac{1}{N}\right) ,$$

$$F_m(u) = \log\left[\frac{d^{m-1}}{du^{m-1}}\left(\frac{e^u - 1}{u}\right)\right] .$$

Proof. We write

$$z^J = z_1^{j_1} \cdots z_m^{j_m} , \qquad z = (z_1, \dots, z_m),\ J = (j_1, \dots, j_m) .$$

An easy computation yields

$$\int_{S^{2m-1}} |z^J|^2\, d\mu(z) = \frac{(m-1)! j_1! \cdots j_m!}{(|J| + m - 1)!} = \frac{1}{\binom{|J|+m-1}{m-1}\binom{|J|}{J}} , \tag{19}$$

where

$$|J| = j_1 + \cdots + j_m , \qquad \binom{|J|}{J} = \frac{|J|!}{j_1! \cdots j_m!} .$$

Thus an orthonormal basis for $\mathcal{P}_N$ on S^{2m-1} is:

$$\varphi_J(z) = \binom{|J| + m - 1}{m-1}^{\frac{1}{2}} \binom{|J|}{J}^{\frac{1}{2}} z^J , \qquad |J| \le N . \tag{20}$$

We have

$$\begin{aligned} S_N(z,z) = \sum_{|J|\le N} |\varphi_J(z)|^2 &= \sum_{k=0}^{N} \binom{k+m-1}{m-1} \sum_{|J|=k} \binom{k}{J} |z_1|^{2j_1} \cdots |z_m|^{2j_m} \\ &= \sum_{k=0}^{N} \binom{k+m-1}{m-1} \|z\|^{2k} . \end{aligned}$$

Hence

$$S_N(z,z) = g_N(\|z\|^2)\,, \quad \text{where} \quad g_N(x) = \sum_{k=0}^{N} \binom{k+m-1}{m-1} x^k\,. \tag{21}$$

We note that

$$g_N = \frac{1}{(m-1)!}\, G_N^{(m-1)}\,, \quad \text{where} \quad G_N(x) = \frac{1-x^{N+m}}{1-x}\,.$$

We denote by $O(\frac{1}{N})$ any function $\lambda(N,u) = \lambda_N(u) : \mathbb{Z}^+ \times \mathbb{R} \to \mathbb{R}$ satisfying:

$$\forall R > 0,\ \forall j \in \mathbb{N},\ \exists C_{Rj} \in \mathbb{R}^+ \quad \text{such that} \ \sup_{|u|<R} |\lambda_N^{(j)}(u)| < \frac{C_{Rj}}{N}\,. \tag{22}$$

We note that

$$N \log\left(1+\frac{u}{N}\right) = u + u^2\, O\left(\frac{1}{N}\right) \qquad (\text{for} \ \ |u| < N)\,,$$

and hence

$$\left(1+\frac{u}{N}\right)^N = e^u + u^2\, O\left(\frac{1}{N}\right)\,.$$

Thus we have

$$\frac{1}{N}\, G_N\left(1+\frac{u}{N}\right) = \frac{e^u-1}{u} + O\left(\frac{1}{N}\right)\,. \tag{23}$$

Hence

$$\frac{1}{N^m}\, g_N\left(1+\frac{u}{N}\right) = \frac{1}{(m-1)!}\, \frac{d^{m-1}}{du^{m-1}}\left(\frac{e^u-1}{u}\right) + O\left(\frac{1}{N}\right)\,. \tag{24}$$

Therefore

$$\log\left[\frac{(m-1)!}{N^m}\, g_N\left(1+\frac{u}{N}\right)\right] = F_m(u) + O\left(\frac{1}{N}\right)\,, \tag{25}$$

where F_m is given in the statement of the theorem.

Since the zero distribution is invariant under the SO$(2m)$-action on $\mathbb{C}^m$, we can write

$$\mathbf{E}_{\gamma_N^m}(Z_{f_1,\dots,f_m}) = D_N\left(\log\|z\|^2\right)\left(\frac{i}{2}\partial\bar\partial\|z\|^2\right)^m\,. \tag{26}$$

Then $D_N(\frac{u}{N})$ is the density at the point

$$z^N := \left(\frac{1}{\sqrt{m}}\, e^{u/2N}, \dots, \frac{1}{\sqrt{m}}\, e^{u/2N}\right) \in \mathbb{C}^m\,, \qquad \|z^N\|^2 = e^{u/N}\,.$$

We shall compute using the local coordinates $\zeta_j = \rho_j + i\theta_j = \log z_j$. Let

$$\Omega = \left(\frac{i}{2}\partial\bar\partial \sum |\zeta_j|^2\right)^m\,.$$

By Proposition 2.1 and (21), we have

$$\mathbf{E}_{\gamma_N^m}(Z_{f_1,\dots,f_m}) = \left(\frac{1}{2\pi}\right)^m \det\left(\frac{1}{2}\, \frac{\partial^2}{\partial\rho_j\partial\rho_k} \log g_N\left(\sum e^{2\rho_j}\right)\right)\Omega\,. \tag{27}$$

We note that

$$\Omega = m^m \left[1 + O\left(\frac{1}{N}\right)\right] \left(\frac{i}{2}\partial\bar\partial\|z\|^2\right)^m \qquad \text{at the point} \quad z^N\ . \tag{28}$$

We let $\mathbf{1}$ denote the $m \times m$ matrix all of whose entries are equal to 1 (and we let I denote the $m \times m$ identity matrix). By (25) and (27)–(28), we have

$$\begin{aligned} D_N\left(\frac{u}{N}\right) &= \left(\frac{m}{2\pi}\right)^m \left[1 + O\left(\frac{1}{N}\right)\right] \\ &\quad \times \det\left(2\,m^{-2}\,e^{2u/N}(\log g_N)''(e^{u/N})\,\mathbf{1} + 2\,m^{-1}\,e^{u/N}(\log g_N)'(e^{u/N})\,I\right) \\ &= \frac{1}{\pi^m}\left[1 + O\left(\frac{1}{N}\right)\right] \det\left(m^{-1}N^2\,F''_m(u)\,\mathbf{1} + N\,F'_m(u)\,I\right)\ . \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{1}{N^{m+1}}\,D_N\left(\frac{u}{N}\right) &= \frac{1}{N^{m+1}\,\pi^m}\left[1 + O\left(\frac{1}{N}\right)\right] \\ &\quad \times \left\{\left[N\,F'_m(u)\right]^m + \ m\left[m^{-1}N^2\,F''_m(u)\right]\left[N\,F'_m(u)\right]^{m-1}\right\} \\ &= \frac{1}{\pi^m}\,F''_m(u)\,F'_m(u)^{m-1} + O\left(\frac{1}{N}\right)\ . \end{aligned}$$

□

Remark: There is a similarity between the scaling asymptotics of Theorem 4.1 and that of the one-dimensional $SU(1,1)$ ensembles in [BR] with the norms $\|z^j\| = \binom{L-1+j}{j}^{-1/2}$, for $L \in \mathbb{Z}^+$. Then the expected distribution of zeros of random $SU(1,1)$ polynomials of degree N has the asymptotics [BR, Th. 2.1]:

$$\mathbf{E}_N(Z_f) = \widetilde{D}_N\left(\log|z|^2\right)\,\frac{i}{2}dz \wedge d\bar z\ ,$$

where (in our notation)

$$\frac{1}{N^2}\widetilde{D}_N\left(\frac{u}{N}\right) = \frac{1}{\pi}\,F''_{L-1}(u) + O\left(\frac{1}{N}\right)\ .$$

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