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SCALAR CURVATURE OF MINIMAL HYPERSURFACES IN A SPHERE

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Abstract. We first extend the well-known scalar curvature pinching theorem due to Peng-Terng, and prove that if M a closed minimal hypersurface in S^{n+1} $(n = 6, 7)$, then there exists a positive constant $\delta(n)$ depending only on n such that if $n \le S \le n + \delta(n)$, then $S \equiv n$, i.e., M is one of the Clifford torus $S^k(\sqrt{\frac{k}{n}}) \times S^{n-k}(\sqrt{\frac{n-k}{n}})$, $k = 1, 2, ..., n-1$. Secondly, we point out a mistake in Ogiue and Sun's paper in which they claimed that they had solved the open problem proposed by Peng and Terng.

1. Introduction

Let M be an n-dimensional closed minimal hypersurface in an $(n+1)$-dimensional unit sphere S^{n+1}. Denote by S the squared length of the second fundamental form of M and R its scalar curvature. So $R = n(n-1) - S$. The famous rigidity theorem due to Simons, Lawson, Chern, do Carmo and Kobayashi [4, 5, 10] says that if $S \le n$, then $S \equiv 0$, or $S \equiv n$. i.e., M is the great sphere S^n, or the Clifford torus $S^k(\sqrt{\frac{k}{n}}) \times S^{n-k}(\sqrt{\frac{n-k}{n}})$. Further discussions in this direction have been carried out by many other authors [1, 3, 6, 11, 12, 13, 14], etc.. On the other hand, many geometers have been interested in the question whether there are several scalar curvature pinching phenomena for closed minimal hypersurfaces in a unit sphere. In [8], Peng and Terng proved that if the scalar curvature of M is a constant, then there exists a positive constant $\alpha(n)$ depending only on n such that if $n \le S \le n + \alpha(n)$, then $S = n$. Later Cheng and Yang [2] improved the pinching constant $\alpha(n)$ to $n/3$. More general, Peng and Terng [9] obtained an important pinching theorem for minimal hypersurfaces without assumption that the scalar curvature is a constant. Precisely, they proved that if $M^n (n \le 5)$ is a closed minimal hypersurface in S^{n+1}, then there exists a positive constant $\delta(n)$ depending only on n such that if $n \le S \le n + \delta(n)$, then $S \equiv n$. The following problem proposed by Peng and Terng [9] is very attractive.

Open Problem. *Let M be an n-dimensional closed minimal hypersurface in $S^{n+1}, n \ge 6$. Does there exist a positive constant $\delta(n)$ depending only on n such that if $n \le S \le n + \delta(n)$, then $S \equiv n$, i.e., M is one of the Clifford torus $S^k(\sqrt{\frac{k}{n}}) \times S^{n-k}(\sqrt{\frac{n-k}{n}}), k = 1, 2, ..., n-1$?*

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In this note, we solve the open problem for $n = 6, 7$, and prove the following pinching theorem for minimal hypersurfaces in unit spheres of dimensions 7 and 8.

Theorem. *Let M be an n-dimensional closed minimal hypersurface in $S^{n+1}, n = 6, 7$. Then there exists a positive constant $\delta(n)$ depending only on n such that if $n \le S \le n + \delta(n)$, then $S \equiv n$, i.e., M is one of the Clifford torus $S^k(\sqrt{\frac{k}{n}}) \times S^{n-k}(\sqrt{\frac{n-k}{n}}), k = 1, 2, ..., n-1$. Here $\delta(6) = \frac{1}{76}$ and $\delta(7) = \frac{1}{1126}$.*

Our theorem generalize the scalar curvature pinching theorem due to Peng and Terng [9] from the case $n \le 5$ to $n \le 7$. Up to now, the open problem for $n \ge 8$ is still open.

In [7], Ogiue and Sun claimed that they had solved the open problem for arbitrary n. Unfortunately, there is a fatal mistake in their proof. In section 4, we point out their mistake.

2. Fundamental formulas for minimal hypersurfaces in a sphere

Throughout this paper let M be an n-dimensional closed minimal hypersurface in an $(n + 1)$-dimensional unit sphere S^{n+1}. We shall make use of the following convention on the range of indices:

$$1 \le A, B, C, \cdots \le n + 1, \qquad 1 \le i, j, k, \cdots \le n.$$

Choose a local orthonormal frame field $\{e_A\}$ in S^{n+1} such that, restricted to M, the e_i's are tangent to M. Let $\{\omega_A\}$ be the dual frame fields of $\{e_A\}$ and $\{\omega_{AB}\}$ the connection 1-forms of S^{n+1} respectively. Restricting these forms to M, we have $\omega_{n+1\,i} = \sum_j h_{ij}\,\omega_j$, $h_{ij} = h_{ji}$. Let R and h be the scalar curvature and the second fundamental form of M respectively. Denote by S the squared length of h and H the mean curvature of M. Then we have

$$h = \sum_{i,j} h_{ij}\,\omega_i \otimes \omega_j, \qquad S = \sum_{i,j} h_{ij}^2. \tag{2.1}$$

$$H = \frac{1}{n}\sum_i h_{ii} = 0, \qquad R = n(n-1) - S. \tag{2.2}$$

Denote by h_{ijk}, h_{ijkl} and h_{ijklm} the first, second and third covariant derivatives of the second fundamental form tensor h_{ij}. Then

$$\nabla h = \sum_{i,j,k} h_{ijk}\,\omega_i \otimes \omega_j \otimes \omega_k, \qquad h_{ijk} = h_{ikj}. \tag{2.3}$$

$$h_{ijkl} = h_{ijlk} + \sum_m h_{mj}R_{mikl} + \sum_m h_{im}R_{mjkl}. \tag{2.4}$$

$$h_{ijklm} = h_{ijkml} + \sum_r h_{rjk}R_{rilm} + \sum_r h_{irk}R_{rjlm} + \sum_r h_{ijr}R_{rklm}. \tag{2.5}$$

For an arbitrary fixed point $x \in M$, we take orthonormal frames such that $h_{ij} = \lambda_i\delta_{ij}$ for all i, j. Then

$$\sum_i \lambda_i = 0, \qquad \sum_i \lambda_i^2 = S. \tag{2.6}$$

Following [4, 9], we have

$$\frac{1}{2}\triangle S = |\nabla h|^2 + S(n-S). \tag{2.7}$$

$$\frac{1}{2}\triangle(|\nabla h|^2) = |\nabla^2 h|^2 + (2n+3-S)|\nabla h|^2 + 3(2B-A) - \frac{3}{2}|\nabla S|^2. \tag{2.8}$$

$$\lambda_k^2 - 4\lambda_i\lambda_k < \frac{\sqrt{17}+1}{2}S, \quad 1 \le i,\, k \le n. \tag{2.9}$$

$$3(A-2B) \le \frac{\sqrt{17}+1}{2}S|\nabla h|^2, \tag{2.10}$$

$$\int_M [(S-2n-\frac{3}{2})|\nabla h|^2 + \frac{3}{2}(A-2B) + \frac{9}{8}|\nabla S|^2]dM \ge 0, \tag{2.11}$$

where $A = \sum\limits_{i,j,k} h_{ijk}^2\lambda_i^2$ and $B = \sum\limits_{i,j,k} h_{ijk}^2\lambda_i\lambda_j$.

3. Proof of Theorem

The crucial point in our proof is to give a sharper pointwise estimate of $3(A-2B)$ in terms of S and $|\nabla h|^2$ by using new method. The following lemmas will be used in the proof of the theorem.

Lemma 3.1. *Let M be an n-dimensional closed minimal hypersurface in the unit sphere $S^{n+1}, n \ge 6$. Suppose that*

$$3(A-2B) \le t(n)S|\nabla h|^2, \tag{3.1}$$

where $t(n)$ is a number depending only on n satisfying $0 \le t(n) < 2 + \frac{3}{n}$. Then there exists a positive constant $\delta(n)$ such that if $n \le S \le n+\delta(n)$, then $S \equiv n$.

Proof. By the assumption, we have

$$\frac{6-[2t(n)-4]n}{9} > 0.$$

We choose a positive constant $\delta(n)$ depending only on n satisfying

$$0 < \delta(n) < \frac{6-[2t(n)-4]n}{9}. \tag{3.2}$$

It follows from the assumption $n \le S(x) \le n+\delta$ that

$$\int_M |\nabla S|^2 dM = 2\int_M [S^2(S-n) - S|\nabla h|^2]dM \le 2\int_M (n+\delta-S)|\nabla h|^2 dM. \tag{3.3}$$

From (2.11), (3.1) and (3.3), we obtain

$$0 \le \int_M (\frac{2t(n)-5}{4}S - \frac{6-n-9\delta(n)}{4})|\nabla h|^2 dM. \tag{3.4}$$

Since

$$t(n) < 2 + \frac{3}{n} \le \frac{5}{2},$$

we get

$$\begin{aligned} 0 &\leq \int_M [(2t(n)-5)S - (6-n-9\delta(n)])|\nabla h|^2 dM \\ &\leq \int_M [(2t(n)-4)n + (9\delta(n)-6)]|\nabla h|^2 dM, \end{aligned} \tag{3.5}$$

which implies

$$\int_M |\nabla h|^2 dM \leq 0. \tag{3.6}$$

Hence $|\nabla h|^2 = 0$. It's easy to see from (2.7) that $S \equiv n$. □

Remark 1. Under the assumption of Lemma 3.1, if $t(n) = 2$, then the pinching constant $\delta(n) = \frac{2}{3}$, which is a universal positive constant independent of n.

Lemma 3.2. *Let M be a closed minimal hypersurface in a 7-dimensional unit sphere S^7. Then*

$$\sum_i h_{iik}^2 \Phi(i,k) \leq 2.49S \cdot \sum_i h_{iik}^2, \ 1 \leq k \leq 6, \tag{3.7}$$

where

$$\Phi(i,k) = \begin{cases} \lambda_k^2 - 4\lambda_i\lambda_k, & i \neq k, \\ 2S, & i = k. \end{cases}$$

Proof. Without loss of generality, we suppose that $k = 1$. If $\Phi(i,1) \leq 2.49S$ for any i, or $\sum\limits_i h_{ii1}^2 = 0$, it is easy to get (3.7). Otherwise, without loss of generality, we suppose that $\Phi(2,1) > 2.49S$. Then

$$\begin{aligned} \frac{\sqrt{17}+1}{2}S &\geq \frac{\sqrt{17}+1}{2}(\lambda_1^2+\lambda_2^2) \\ &\geq \lambda_1^2 - 2\Big(\sqrt{\frac{\sqrt{17}-1}{2}}\lambda_1\Big)\Big(\sqrt{\frac{\sqrt{17}+1}{2}}\lambda_2\Big) \\ &= \Phi(2,1) > 2.49S. \end{aligned} \tag{3.8}$$

This implies

$$\lambda_m^2 \leq S - (\lambda_1^2+\lambda_2^2) < S - \frac{2.49}{2.57}S = \frac{8}{257}S, \ m = 3,\ 4,\ 5,\ 6. \tag{3.9}$$

By (3.9) we have

$$\Phi(m,1) = \lambda_1^2 - 4\lambda_1\lambda_m < S + 4\cdot\sqrt{\frac{8}{257}S}\cdot\sqrt{S} < 2S, \ m = 3,\ 4,\ 5,\ 6. \tag{3.10}$$

Since M is a minimal hypersurface, we have $\sum_i h_{ii} = 0$. Hence

$$h_{221} = -\sum_{i\neq 2} h_{ii1},$$

which implies

$$\sum_{i\neq 2} h_{ii1}^2 \geq \frac{h_{221}^2}{5}. \tag{3.11}$$

It follows from (3.8), (3.10) and (3.11) that

$$2.49S\sum_i h_{ii1}^2 \geq 2.49Sh_{221}^2 + 0.49S\cdot\frac{h_{221}^2}{5} + \sum_{i\neq 2} h_{ii1}^2\Phi(i,1) \geq \sum_i h_{ii1}^2\Phi(i,1). \tag{3.12}$$

□

Lemma 3.3. *Let M be a closed minimal hypersurface in an 8-dimensional unit sphere S^8. Then*

$$\sum_i h_{iik}^2\Phi(i,k) \leq 2.428S\cdot\sum_i h_{iik}^2,\ 1\leq k\leq 7, \tag{3.13}$$

where

$$\Phi(i,k) = \begin{cases} \lambda_k^2 - 4\lambda_i\lambda_k, & i\neq k,\\ 1.62S, & i=k. \end{cases}$$

Proof. Without loss of generality, we suppose that $k=1$. If $\Phi(i,1)\leq 2.428S$ for any i, or $\sum\limits_i h_{ii1}^2 = 0$, it is easy to get (3.13). Otherwise, without loss of generality, we suppose that $\Phi(2,1) > 2.428S$. Then

$$\frac{\sqrt{17}+1}{2}S \geq \frac{\sqrt{17}+1}{2}(\lambda_1^2+\lambda_2^2) \geq \lambda_1^2 - 4\lambda_1\lambda_2 > 2.428S. \tag{3.14}$$

It follows from the above that

$$\lambda_m^2 \leq S - (\lambda_1^2+\lambda_2^2) < S - \frac{2.428}{2.562}S = \frac{67}{1281}S, \tag{3.15}$$

where $3\leq m\leq 7$. On the other hand, we have

$$\lambda_1^2 + (\lambda_1^2 + 4\lambda_2^2) \geq \lambda_1^2 - 4\lambda_1\lambda_2 > 2.428S.$$

This implies

$$\lambda_1^2 \leq S - \lambda_2^2 < S - \frac{2.428S - 2(\lambda_1^2+\lambda_2^2)}{2} \leq 0.786S. \tag{3.16}$$

From (3.15) and (3.16) we have

$$\Phi(m,1) = \lambda_1^2 - 4\lambda_1\lambda_m < 0.786S + 4\cdot\sqrt{\frac{67}{1281}S}\cdot\sqrt{0.786S} \leq 1.62S, \tag{3.17}$$

where $3\leq m\leq 7$. Since M is a minimal hypersurface, we have $\sum_i h_{ii} = 0$, which implies

$$\sum_{i\neq 2} h_{ii1}^2 \geq \frac{h_{221}^2}{6}. \tag{3.18}$$

From (3.14), (3.17) and (3.18) we obtain

$$\begin{aligned} 2.428S\sum_i h_{ii1}^2 &\geq 2.428Sh_{221}^2 + 0.808S\cdot\frac{h_{221}^2}{6} + \sum_{i\neq 2} h_{ii1}^2\Phi(i,1) \\ &\geq \sum_i h_{ii1}^2\Phi(i,1). \end{aligned} \tag{3.19}$$

□

Now we are in a position to give the proof of our theorem.

Proof of Theorem. (i) When $n=6$, it follows from Lemma 3.2 that

$$\begin{aligned} &3\sum_{i\neq k} h_{iik}^2(\lambda_k^2 - 4\lambda_k\lambda_i) + \sum_i h_{iii}^2(-3\lambda_i^2) \\ &= 3\sum_k(\sum_i h_{iik}^2\phi(i,k)) - 3\sum_k h_{kkk}^2\cdot 2S + \sum_i h_{iii}^2(-3\lambda_i^2) \\ &\leq 2.49S\cdot\sum_{i,k} 3h_{iik}^2 - 2.49S\sum_k 2h_{kkk}^2 \\ &= 2.49S(3\sum_{i\neq k} h_{iik}^2 + \sum_i h_{iii}^2). \end{aligned} \tag{3.20}$$

This together with (2.11) implies

$$\begin{aligned} 3(A-2B) = &\sum_{\substack{i,j,k\\ dinstinct}} h_{ijk}^2[2(\lambda_i^2+\lambda_j^2+\lambda_k^2) - (\lambda_i+\lambda_j+\lambda_k)^2] \\ &+ 3\sum_{i\neq k} h_{iik}^2(\lambda_k^2 - 4\lambda_k\lambda_i) + \sum_i h_{iii}^2(-3\lambda_i^2) \\ &\leq 2S\sum_{\substack{i,j,k\\ dinstinct}} h_{ijk}^2 + 2.49S(3\sum_{i\neq k} h_{iik}^2 + \sum_i h_{iii}^2) \\ &\leq 2.49S|\nabla h|^2. \end{aligned} \tag{3.21}$$

Notice that $\delta(6) = \frac{1}{76}$ and $t(6) = 2.49$, we conclude from Lemma 3.1 and (3.21) that $S \equiv 6$, i.e., M is one of the Clifford torus $S^k(\sqrt{\frac{k}{6}}) \times S^{6-k}(\sqrt{\frac{6-k}{6}}), k = 1, 2, ..., 5$.

(ii) When $n = 7$, it follows from Lemma 3.3 that

$$\begin{aligned} &3\sum_{i\neq k} h_{iik}^2(\lambda_k^2 - 4\lambda_k\lambda_i) + \sum_i h_{iii}^2(-3\lambda_i^2) \\ &= 3\sum_k(\sum_i h_{iik}^2\phi(i,k)) - 3\sum_k h_{kkk}^2\cdot 1.62S + \sum_i h_{iii}^2(-3\lambda_i^2) \\ &\leq 2.428\cdot\sum_{i,k} 3h_{iik}^2 - 2.428S\sum_k 2h_{kkk}^2 \\ &= 2.428(3\sum_{i\neq k} h_{iik}^2 + \sum_i h_{iii}^2). \end{aligned} \tag{3.22}$$

This together with (2.11) implies

$$
\begin{aligned}
3(A-2B) = &\sum_{\substack{i,j,k\\ dinstinct}} h_{ijk}^2[2(\lambda_i^2+\lambda_j^2+\lambda_k^2)-(\lambda_i+\lambda_j+\lambda_k)^2] \\
&+3\sum_{i\neq k} h_{iik}^2(\lambda_k^2-4\lambda_k\lambda_i)+\sum_i h_{iii}^2(-3\lambda_i^2) \\
&\leq 2S\sum_{\substack{i,j,k\\ dinstinct}} h_{ijk}^2+2.428S(3\sum_{i\neq k} h_{iik}^2+\sum_i h_{iii}^2) \\
&\leq 2.428S|\nabla h|^2.
\end{aligned} \tag{3.23}
$$

Notice that $\delta(7)=\frac{1}{1126}$ and $t(7)=2.428$, we conclude from Lemma 3.1 and (3.23) that $S\equiv 7$, i.e., M is a Clifford torus. This completes the proof of the theorem. □

4. Notes on Ogiue and Sun's proof

In [7], Ogiue and Sun claimed that they improved Peng and Terng's pinching theorem for $n(\leq 5)$-dimensional minimal hypersurfaces [9] to the case of arbitrary n:

Let M be an n-dimensional closed minimally immersed hypersurface in S^{n+1}. Then there exists a constant $\varepsilon(n)=2n^2(n+4)/[3(n+2)^2]$ such that if $n\leq S\leq n+\varepsilon(n)$, then $S\equiv n$ so that M is a Clifford torus.

If the claim were true, definitely it would have been an important contribution to the theory of minimal submanifolds. Unfortunately, there is a fatal mistake in the proof of the key lemma in [7]. Put $g_3=\sum_{i,j,k} h_{ij}h_{jk}h_{ki}$, $g_4=\sum_{i,j,k,l} h_{ij}h_{jk}h_{kl}h_{li}$. This lemma and the sketch of its proof is cited as follows.

Lemma([7]). *Let M be an n-dimensional closed minimally immersed hypersurface in S^{n+1}. If $S\geq n$, then we have*

$$\sum_{i,j,k,l} h_{ijkl}^2 \geq \frac{3(n+2)}{n(n+4)}S(S-n)^2-\frac{3}{n}S^2(S-n)+3(Sg_4-g_3^2-S^2).$$

Proof. Since M is minimal, we have $\sum_i h_{ii}=0$ and $\sum_{i,j} h_{ijij}h_{jj}=0$.

From(1.1)[7] we get $\Delta h_{ii}=(n-S)h_{ii}$ and $\sum_{i,j} h_{iijj}h_{ii}=S(n-S)$.

Let $f_{ij}=h_{ijij}$. We consider $f=\sum_i f_{ii}^2+3\sum_{i\neq j} f_{ij}^2+6\sum_{i,j}(h_{jj}^2h_{ii}-h_{ii}^2h_{jj})f_{ij}$ as a function of f_{ij}. Solve the following problem for the conditional extremum:

$$F=\sum_i f_{ii}^2+3\sum_{i\neq j} f_{ij}^2+6\sum_{i,j}(h_{jj}^2h_{ii}-h_{ii}^2h_{jj})f_{ij}+\lambda[\sum_{i,j} f_{ij}h_{ii}-S(n-S)]+\mu\sum_{i,j} f_{ij}h_{ij}, \tag{2.1)[7}$$

where λ and μ are the Lagrange multipliers. It is clear that the critical point of F is the minimum point of f. Taking derivatives of F with respect to f_{ij}, we get

$$F_{f_{ij}}=2f_{ii}+\lambda h_{ii}+\mu h_{ii}=0,\quad i=j, \tag{2.2)[7}$$

$$F_{f_{ij}}=6f_{ij}+6(h_{jj}^2h_{ii}-h_{ii}^2h_{jj})+\lambda h_{ii}+\mu h_{jj}=0,\quad i\neq j, \tag{2.3)[7}$$

and they satisfy

$$\sum_{i,j} h_{jj} f_{ij} = 0, \sum_{i,j} h_{ii} f_{ij} = S(n-S), \sum_i h_{ii}^2 = S, \sum_i h_{ii} = 0. \qquad (2.4)[7]$$

. . . .

and so, in view of (2.4)[7]

$$\begin{aligned} &\sum_i f_{ii}^2 + 3\sum_{i\neq j} f_{ij}^2 + 6\sum_{i\neq j}(h_{jj}^2 h_{ii} - h_{ii}^2 h_{jj}) f_{ij} \\ &= 3\sum_{i\neq j}(h_{jj}^2 h_{ii} - h_{ii}^2 h_{jj}) f_{ij} - \frac{1}{2}\lambda \sum_{i,j} h_{ii} f_{ij} \\ &= 3\sum_{i\neq j}(h_{jj}^2 h_{ii} - h_{ii}^2 h_{jj}) f_{ij} + \frac{\lambda}{2} S(S-n). \qquad (2.10)[7] \end{aligned}$$

. . . .

$$\lambda = \frac{6(n+2)}{n(n+4)}(S-n) - \frac{6}{n}S. \qquad (2.15)[7]$$

(2.10)[7] and (2.15)[7] show that

$$\begin{aligned} &\sum_i f_{ii}^2 + 3\sum_{i\neq j} f_{ij}^2 + 6\sum_{i\neq j}(h_{jj}^2 h_{ii} - h_{ii}^2 h_{jj}) f_{ij} \\ &= \frac{3(n+2)}{n(n+4)} S(S-n)^2 - \frac{3}{n} S^2(S-n) + 3\sum_{i\neq j}(h_{jj}^2 h_{ii} - h_{ii}^2 h_{jj}) f_{ij}, \end{aligned}$$

and so,

$$\begin{aligned} \sum_i h_{iiii}^2 + 3\sum_{i\neq j} h_{ijij}^2 &\geq \frac{3(n+2)}{n(n+4)} S(S-n)^2 - \frac{3}{n} S^2(S-n) - 3\sum_{i\neq j}(h_{jj}^2 h_{ii} - h_{ii}^2 h_{jj}) f_{ij} \\ &= \frac{3(n+2)}{n(n+4)} S(S-n)^2 - \frac{3}{n} S^2(S-n) - 3(Sg_4 - g_3^2 - S^2). \qquad (2.16)[7] \end{aligned}$$

Combining (1.4)[7] and (2.16)[7], we get the Lemma. □

We see from the above sketch that the key lemma in [7] is derived by computing the minimal value of the function

$$f = \sum_i f_{ii}^2 + 3\sum_{i\neq j} f_{ij}^2 + 6\sum_{i,j}(h_{jj}^2 h_{ii} - h_{ii}^2 h_{jj}) f_{ij}$$

in the domain

$$\{(f_{11}, f_{12}, \ldots f_{1n}, f_{21}, \ldots f_{nn}) \mid \sum_{i,j} h_{ij} f_{ij} = 0, \sum_{i,j} h_{ii} f_{ij} = S(n-S)\}.$$

Let $P_0 = ((f_{11})_0, (f_{12})_0, \ldots (f_{1n})_0, (f_{21})_0, \ldots (f_{nn})_0)$ be the point where f attains it's minimal value. We see that the exact meaning of the equation above (2.16)[7] is:

$$f|_{p_0} = C(n,S) + 3\sum_{i\neq j}(h_{jj}^2 h_{ii} - h_{ii}^2 h_{jj})(f_{ij})_0, \qquad (4.1)$$

where

$$C(n,S) = \frac{3(n+2)}{n(n+4)}S(S-n)^2 - \frac{3}{n}S^2(S-n).$$

This implies

$$\begin{aligned} f &= \sum_i f_{ii}^2 + 3\sum_{i\neq j} f_{ij}^2 + 6\sum_{i,j}(h_{jj}^2 h_{ii} - h_{ii}^2 h_{jj}) f_{ij} \\ &\geq C(n,S) + 3\sum_{i\neq j}(h_{jj}^2 h_{ii} - h_{ii}^2 h_{jj})(f_{ij})_0. \end{aligned} \tag{4.2}$$

We notice that f_{ij} on the left hand side is different from $(f_{ij})_0$ on the right hand side. Unfortunately, (2.16)[7] is derived from (4.2) under the additional assumption that $f_{ij} = (f_{ij})_0$. This is a fatal mistake. In fact, the key lemma [7] is derived from the following assertion.

For $f_{11}, f_{12}, \ldots f_{1n}, f_{21}, \ldots f_{nn}$ and $h_{11}, h_{22}, \ldots h_{nn}$ satisfying the conditions

$$\sum_{i,j} h_{jj} f_{ij} = 0,\ \sum_{i,j} h_{ii} f_{ij} = S(n-S),\ \sum_i h_{ii}^2 = S,\ \sum_i h_{ii} = 0, \tag{4.3}$$

we always have

$$\sum_i f_{ii}^2 + 3\sum_{i\neq j} f_{ij}^2 \geq C(n,S) + 3(Sg_4 - g_3^2 - S^2). \tag{4.4}$$

Unfortunately, we have the following counter example for the assertion above.
Example 4.1. Set

$$h_{11} = -h_{22} = -\sqrt{\frac{S}{2}}; \quad h_{ii} = 0,\ i \geq 3.$$

$$f_{ij} = \frac{1}{2}(h_{ii} - h_{jj})(1 + h_{ii}h_{jj}) - \frac{S-n}{2(n+4)}(h_{ii} + h_{jj}),\ i \neq j.$$

$$f_{ii} = \frac{3}{n+4}(S-n)h_{ii}.$$

It is easy to see that f_{ij} and h_{ii} satisfy (4.3). On the other hand, we have

$$\sum_i f_{ii}^2 + 3\sum_{i\neq j} f_{ij}^2 = C(n,S) + \frac{3}{2}(Sg_4 - g_3^2 - \frac{S^3}{n}) < C(n,S) + 3(Sg_4 - g_3^2 - S^2).$$

This contradicts with (4.4).

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