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ON THE INTEGRAL SYSTEMS RELATED TO HARDY-LITTLEWOOD-SOBOLEV INEQUALITY

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Abstract. We prove all the maximizers of the sharp Hardy-Littlewood-Sobolev inequality are smooth. More generally, we show all the nonnegative critical functions are smooth, radial with respect to some points and strictly decreasing in the radial direction. In particular, we resolve all the cases left open by previous works of Chen, Li and Ou on the corresponding integral systems.

1. Introduction

The classical Hardy-Littlewood-Sobolev inequality states that for $0 < \alpha < n$, $1 < p_0, q_0 < \frac{n}{\alpha}$ such that $\frac{1}{p_0} + \frac{1}{q_0} = 1 + \frac{\alpha}{n}$ (see [10, theorem 1 on p119])

$$\left| \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{f(x)\, g(y)}{|x-y|^{n-\alpha}} dxdy \right| \leq c(n, p_0, \alpha) |f|_{L^{p_0}(\mathbb{R}^n)} |g|_{L^{q_0}(\mathbb{R}^n)}.$$

In [9], it was shown that the sharp constant

$$c(n, p_0, \alpha) = \sup \left\{ \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{f(x)\, g(y)}{|x-y|^{n-\alpha}} dxdy : |f|_{L^{p_0}(\mathbb{R}^n)} = 1, |g|_{L^{q_0}(\mathbb{R}^n)} = 1 \right\}$$

is achieved by some functions f and g. Moreover, after multiplying some constants, any maximizer f, g must be radial symmetric with respect to the same point, strictly decreasing in the radial direction and satisfy the integral system

$$f(x)^{p_0-1} = \int_{\mathbb{R}^n} \frac{g(y)}{|x-y|^{n-\alpha}} dy, \quad g(x)^{q_0-1} = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy.$$

It was also shown that when $p_0 = q_0$, we have

$$f(x) = g(x) = c(n, p_0) \left(\frac{\lambda}{|x-x_0|^2 + \lambda^2} \right)^{n/p_0}$$

for some $\lambda > 0$ and $x_0 \in \mathbb{R}^n$.

If we let $p = \frac{1}{p_0-1}$, $q = \frac{1}{q_0-1}$, $u = f^{p_0-1}$, $v = g^{q_0-1}$, then the Euler-Lagrange equation becomes

$$u(x) = \int_{\mathbb{R}^n} \frac{v(y)^q}{|x-y|^{n-\alpha}} dy, \quad v(x) = \int_{\mathbb{R}^n} \frac{u(y)^p}{|x-y|^{n-\alpha}} dy \tag{1.1}$$

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for nonnegative functions $u \in L^{p+1}(\mathbb{R}^n)$ and $v \in L^{q+1}(\mathbb{R}^n)$ and $0 < \alpha < n$, $\frac{\alpha}{n-\alpha} < p, q < \infty$, $\frac{1}{p+1} + \frac{1}{q+1} + \frac{\alpha}{n} = 1$. When $p = q = \frac{n+\alpha}{n-\alpha}$, as observed in [9], it follows from the fact $\frac{1}{|x|^{n-\alpha}} = c(n,\alpha) \frac{1}{|x|^{n-\frac{\alpha}{2}}} * \frac{1}{|x|^{n-\frac{\alpha}{2}}}$ that $u = v$, then the system reduces to

$$u(x) = \int_{\mathbb{R}^n} \frac{u(y)^{\frac{n+\alpha}{n-\alpha}}}{|x-y|^{n-\alpha}} dy. \tag{1.2}$$

In [4], using an integral form of the method of moving planes ([5]), it was shown that any nonzero nonnegative regular solution u of (1.2) must be of the form

$$u(x) = c(n,\alpha) \left(\frac{\lambda}{\lambda^2 + |x - x_0|^2} \right)^{\frac{n-\alpha}{2}}$$

for some $\lambda > 0$ and $x_0 \in \mathbb{R}^n$. This solves an open problem proposed in [9] (see a somewhat different argument in [8] and the clarifications in [4, Remark 1.3 on p332]). In [2, 3], such kinds of analysis were extended to the system (1.1) under the additional constraints $p \geq 1$ and $q \geq 1$. However the analysis does not give the regularity of maximizer for all the Hardy-Littlewood-Sobolev inequalities. On the other hand, it does not seem that we will have nonsmooth maximizers for the Hardy-Littlewood-Sobolev inequality in any case. The main aim of this article is to prove the regularity and radial symmetry of nonnegative solutions of the system (1.1) in its full range. Another motivation comes from the study of regularity issues for a similar integral system in [7].

Theorem 1.1. *Assume* $0 < \alpha < n$, $\frac{\alpha}{n-\alpha} < p, q < \infty$, $\frac{1}{p+1} + \frac{1}{q+1} + \frac{\alpha}{n} = 1$, $u \in L^{p+1}(\mathbb{R}^n)$ *is nonnegative and does not vanish identically. If*

$$v(x) = \int_{\mathbb{R}^n} \frac{u(y)^p}{|x-y|^{n-\alpha}} dy, \quad u(x) = \int_{\mathbb{R}^n} \frac{v(y)^q}{|x-y|^{n-\alpha}} dy.$$

Then $u \in C^\infty(\mathbb{R}^n)$, $v \in C^\infty(\mathbb{R}^n)$. *Moreover, there exists a point* $x_0 \in \mathbb{R}^n$ *such that both* u *and* v *are radial symmetric with respect to* x_0 *and strictly decreasing along radial direction.*

Indeed, the regularity is still true under the relatively weaker assumption $u \in L^{p+1}_{loc}(\mathbb{R}^n)$ (see Proposition 2.2). The method in [2, 3], which is basically linear in nature, does not seem to work for the case when one of the two indices p and q is strictly less than 1. We will develop some nonlinear approaches which work for all p and q at once. In [7], we will apply this technique to derive the regularity for another integral system. In Section 2 below, we will prove a local regularity result which has the regularity part in Theorem 1.1 as a corollary. In Section 3, we will prove all the solutions are radial.

2. Regularity issue

In this section, we will show any solution u, v to the system (1.1) must be smooth if we assume $u \in L^{p+1}_{loc}(\mathbb{R}^n)$. Such a local integrability condition is necessary for the smoothness because as observed in [9], system (1.1) has singular solutions as

$$u(x) = c(n,\alpha,p) |x|^{-\frac{n}{p+1}}, \quad v(x) = c(n,\alpha,p) |x|^{-\frac{n}{q+1}}.$$

This follows from a simple change of variable in the integrals. To achieve the regularity, we start with a local result which has some similarity to [1, theorem 2] and [8, theorem 1.3].

Proposition 2.1. *Given $0<\alpha,\beta<n$, $1<a,b\le\infty$, $1\le r<\infty$ such that*

$$\frac{1}{ra}+\frac{1}{b}=\frac{\alpha}{rn}+\frac{\beta}{n}.$$

Assume

$$\frac{n}{n-\beta}<p<q<\infty,$$

$$\frac{\alpha}{n}<\frac{r}{q}+\frac{1}{a}<\frac{r}{p}+\frac{1}{a}<1,$$

$u,f\in L^{p}(B_R)$, $U\in L^{a}(B_R)$, $V\in L^{b}(B_R)$ are all nonnegative functions with $f|_{B_{R/2}}\in L^{q}\left(B_{R/2}\right)$,

$$|U|_{L^{a}(B_R)}^{1/r}|V|_{L^{b}(B_R)}\le\varepsilon(n,p,q,r,\alpha,\beta,a,b)\ \textit{small}$$

and

$$u(x)\le\int_{B_R}\frac{V(y)}{|x-y|^{n-\beta}}\left[\int_{B_R}\frac{U(z)\,u(z)^{r}}{|y-z|^{n-\alpha}}dz\right]^{1/r}dy+f(x)$$

for $x\in B_R$, then $u\in L^{q}\left(B_{R/4}\right)$, moreover

$$|u|_{L^{q}\left(B_{R/4}\right)}\le c(n,p,q,r,\alpha,\beta,a,b)\left(R^{\frac{n}{q}-\frac{n}{p}}|u|_{L^{p}(B_R)}+|f|_{L^{q}\left(B_{R/2}\right)}\right).$$

Proof. By scaling, we may assume $R=1$. First assume we have $u,f\in L^{q}(B_1)$. Denote

$$v(x)=\int_{B_1}\frac{U(y)\,u(y)^{r}}{|x-y|^{n-\alpha}}dy\text{ for }x\in B_1.$$

Let p_1 and q_1 be the numbers defined by

$$\frac{1}{p_1}=\frac{r}{p}+\frac{1}{a}-\frac{\alpha}{n},\qquad\frac{1}{q_1}=\frac{r}{q}+\frac{1}{a}-\frac{\alpha}{n},$$

then it follows from Hardy-Littlewood-Sobolev inequality that

$$\begin{aligned}|v|_{L^{p_1}(B_1)}&\le c(n,p,r,\alpha,a)\,|U|_{L^{a}(B_1)}\,|u|_{L^{p}(B_1)}^{r},\\|v|_{L^{q_1}(B_1)}&\le c(n,q,r,\alpha,a)\,|U|_{L^{a}(B_1)}\,|u|_{L^{q}(B_1)}^{r}.\end{aligned}$$

Given $0<s<t\le 1/2$. For $x\in B_s$, we have

$$\begin{aligned}u(x)&\le\int_{B_{\frac{s+t}{2}}}\frac{V(y)\,v(y)^{1/r}}{|x-y|^{n-\beta}}dy+\int_{B_1\setminus B_{\frac{s+t}{2}}}\frac{V(y)\,v(y)^{1/r}}{|x-y|^{n-\beta}}dy+f(x)\\&\le\int_{B_{\frac{s+t}{2}}}\frac{V(y)\,v(y)^{1/r}}{|x-y|^{n-\beta}}dy+\frac{c(n,\beta)}{(t-s)^{n-\beta}}\int_{B_1\setminus B_{\frac{s+t}{2}}}V(y)\,v(y)^{1/r}dy+f(x)\\&\le\int_{B_{\frac{s+t}{2}}}\frac{V(y)\,v(y)^{1/r}}{|x-y|^{n-\beta}}dy+\frac{c(n,p,r,\alpha,\beta,a,b)\,|u|_{L^{p}(B_1)}}{(t-s)^{n-\beta}}+f(x).\end{aligned}$$

Hence we have

$$|u|_{L^q(B_s)} \leq c(n,q,r,\beta,b)\,|V|_{L^b(B_1)}\,|v|^{1/r}_{L^{q_1}\left(B_{\frac{s+t}{2}}\right)} + \frac{c(n,p,q,r,\alpha,\beta,a,b)\,|u|_{L^p(B_1)}}{(t-s)^{n-\beta}} + |f|_{L^q\left(B_{1/2}\right)}.$$

On the other hand, for $x \in B_{\frac{s+t}{2}}$, we have

$$\begin{aligned} v(x) &= \int_{B_t} \frac{U(y)\,u(y)^r}{|x-y|^{n-\alpha}}dy + \int_{B_1\setminus B_t} \frac{U(y)\,u(y)^r}{|x-y|^{n-\alpha}}dy \\ &\leq \int_{B_t} \frac{U(y)\,u(y)^r}{|x-y|^{n-\alpha}}dy + \frac{c(n,\alpha)}{(s-t)^{n-\alpha}}\int_{B_1\setminus B_t} U(y)\,u(y)^r\,dy \\ &\leq \int_{B_t} \frac{U(y)\,u(y)^r}{|x-y|^{n-\alpha}}dy + \frac{c(n,p,r,\alpha,a)\,|U|_{L^a(B_1)}\,|u|^r_{L^p(B_1)}}{(s-t)^{n-\alpha}}. \end{aligned}$$

This implies

$$|v|_{L^{q_1}\left(B_{\frac{s+t}{2}}\right)} \leq c(n,q,r,\alpha,a)\,|U|_{L^a(B_1)}\,|u|^r_{L^q(B_t)} + \frac{c(n,p,q,r,\alpha,a)\,|U|_{L^a(B_1)}\,|u|^r_{L^p(B_1)}}{(s-t)^{n-\alpha}}.$$

Combine the two inequalities together, we see

$$\begin{aligned} |u|_{L^q(B_s)} &\leq c(n,q,r,\alpha,\beta,a,b)\,|U|^{1/r}_{L^a(B_1)}\,|V|_{L^b(B_1)}\,|u|_{L^q(B_t)} \\ &\quad + \frac{c(n,p,q,r,\alpha,\beta,a,b)}{(s-t)^{\max\{(n-\alpha)/r,n-\beta\}}}\,|u|_{L^p(B_1)} + |f|_{L^q\left(B_{1/2}\right)} \\ &\leq \frac{1}{2}\,|u|_{L^q(B_t)} + \frac{c(n,p,q,r,\alpha,\beta,a,b)}{(s-t)^{\max\{(n-\alpha)/r,n-\beta\}}}\,|u|_{L^p(B_1)} + |f|_{L^q\left(B_{1/2}\right)} \end{aligned}$$

if ε is small enough. It follows from usual iteration procedure ([6, lemma 4.3 on p.75]) that

$$|u|_{L^q\left(B_{1/4}\right)} \leq c(n,p,q,r,\alpha,\beta,a,b)\left(|u|_{L^p(B_1)} + |f|_{L^q\left(B_{1/2}\right)}\right).$$

To prove the full proposition, we note that for some function $0 \leq \eta(x) \leq 1$,

$$u(x) = \eta(x)\int_{B_R} \frac{V(y)}{|x-y|^{n-\beta}}\left[\int_{B_R} \frac{U(z)\,u(z)^r}{|y-z|^{n-\alpha}}dz\right]^{1/r} dy + \eta(x)\,f(x).$$

We may define a map T by

$$T(\varphi)(x) = \eta(x)\int_{B_R} \frac{V(y)}{|x-y|^{n-\beta}}\left[\int_{B_R} \frac{U(z)\,|\varphi(z)|^r}{|y-z|^{n-\alpha}}dz\right]^{1/r} dy.$$

Note that we have

$$|T(\varphi)|_{L^p(B_1)} \leq c(n,p,r,\alpha,\beta,a,b)\,|U|^{1/r}_{L^a(B_1)}\,|V|_{L^b(B_1)}\,|\varphi|_{L^p(B_1)} \leq \frac{1}{2}\,|\varphi|_{L^p(B_1)}$$

and

$$|T(\varphi)|_{L^q(B_1)} \leq c(n,q,r,\alpha,\beta,a,b)\,|U|^{1/r}_{L^a(B_1)}\,|V|_{L^b(B_1)}\,|\varphi|_{L^q(B_1)} \leq \frac{1}{2}\,|\varphi|_{L^q(B_1)}$$

if ε is small enough. Moreover, for any $\varphi, \psi \in L^p(B_1)$, it follows from Minkowski's inequality that

$$|T(\varphi)(x) - T(\psi)(x)| \le T(|\varphi - \psi|)(x) \text{ for } x \in B_1,$$

hence

$$|T(\varphi) - T(\psi)|_{L^p(B_1)} \le |T(|\varphi - \psi|)|_{L^p(B_1)} \le \frac{1}{2} |\varphi - \psi|_{L^p(B_1)}.$$

Similarly, we have for any $\varphi, \psi \in L^q(B_1)$,

$$|T(\varphi) - T(\psi)|_{L^q(B_1)} \le \frac{1}{2} |\varphi - \psi|_{L^q(B_1)}.$$

For $k \in \mathbb{N}$, let $f_k(x) = \min\{f(x), k\}$, then it follows from contraction mapping theorem that we may find a unique $u_k \in L^q(B_1)$ such that

$$\begin{aligned} u_k(x) &= T(u_k)(x) + \eta(x) f_k(x) \\ &= \eta(x) \int_{B_R} \frac{V(y)}{|x-y|^{n-\beta}} \left[\int_{B_R} \frac{U(z) |u_k(z)|^r}{|y-z|^{n-\alpha}} dz \right]^{1/r} dy + \eta(x) f_k(x). \end{aligned}$$

Applying the apriori estimate to u_k, we see

$$|u_k|_{L^q(B_{1/4})} \le c(n, p, q, r, \alpha, \beta, a, b) \left(|u_k|_{L^p(B_1)} + |f|_{L^q(B_{1/2})} \right).$$

Now observe that

$$u(x) = T(u)(x) + \eta(x) f(x).$$

We see

$$\begin{aligned} |u_k - u|_{L^p(B_1)} &\le |T(u_k) - T(u)|_{L^p(B_1)} + |f_k - f|_{L^p(B_1)} \\ &\le \frac{1}{2} |u_k - u|_{L^p(B_1)} + |f_k - f|_{L^p(B_1)}. \end{aligned}$$

Hence $|u_k - u|_{L^p(B_1)} \le 2 |f_k - f|_{L^p(B_1)} \to 0$ as $k \to \infty$. Taking a limit process in the apriori estimate for u_k, we get the proposition. $\square$

Now we are ready to derive the full regularity for the system (1.1). Such regularity under the additional assumption $p \ge 1$ and $q \ge 1$ was proved in [2, 8].

Proposition 2.2. *Assume* $0 < \alpha < n$, $\frac{\alpha}{n-\alpha} < p, q < \infty$, $\frac{1}{p+1} + \frac{1}{q+1} + \frac{\alpha}{n} = 1$, $u \in L^{p+1}_{loc}(\mathbb{R}^n)$ *is nonnegative and does not vanish identically. If*

$$v(x) = \int_{\mathbb{R}^n} \frac{u(y)^p}{|x-y|^{n-\alpha}} dy, \quad u(x) = \int_{\mathbb{R}^n} \frac{v(y)^q}{|x-y|^{n-\alpha}} dy.$$

Then $u, v \in C^\infty(\mathbb{R}^n)$. *Moreover if we know* $u \in L^{p+1}(\mathbb{R}^n)$, *then* $u(x) \to 0$ *and* $v(x) \to 0$ *as* $|x| \to \infty$.

Proof. Since $u \in L^{p+1}_{loc}(\mathbb{R}^n)$, we see $u(x) < \infty$ a.e. $x \in \mathbb{R}^n$. It follows that $v(x) < \infty$ a.e. $x \in \mathbb{R}^n$. For any $R > 0$, we may find $x_0 \in B_R$ such that $v(x_0) < \infty$. This gives us $\int_{\mathbb{R}^n \setminus B_R} \frac{u(y)^p}{|x_0-y|^{n-\alpha}} dy < \infty$. It follows that $\int_{\mathbb{R}^n \setminus B_R} \frac{u(y)^p}{|y|^{n-\alpha}} dy < \infty$. Now

$$v(x) = \int_{B_R} \frac{u(y)^p}{|x-y|^{n-\alpha}} dy + \int_{\mathbb{R}^n \setminus B_R} \frac{u(y)^p}{|x-y|^{n-\alpha}} dy,$$

it follows from the Hardy-Littlewood-Sobolev inequality that the first term lies in $L^{q+1}(\mathbb{R}^n)$. On the other hand, for $x \in B_{\theta R}$ with $0 < \theta < 1$, we have

$$\int_{\mathbb{R}^n \backslash B_R} \frac{u(y)^p}{|x-y|^{n-\alpha}} dy \leq \frac{1}{(1-\theta)^{n-\alpha}} \int_{\mathbb{R}^n \backslash B_R} \frac{u(y)^p}{|y|^{n-\alpha}} dy.$$

It follows that $v \in L^{q+1}_{loc}(B_R)$. Since R is arbitrary, we have $v \in L^{q+1}_{loc}(\mathbb{R}^n)$.

Let

$$f_R(x) = \int_{\mathbb{R}^n \backslash B_R} \frac{v(y)^q}{|x-y|^{n-\alpha}} dy,$$
$$g_R(x) = \int_{\mathbb{R}^n \backslash B_R} \frac{u(y)^p}{|x-y|^{n-\alpha}} dy,$$

then we know

$$u(x) = \int_{B_R} \frac{v(y)^q}{|x-y|^{n-\alpha}} dy + f_R(x),$$
$$v(x) = \int_{B_R} \frac{u(y)^p}{|x-y|^{n-\alpha}} dy + g_R(x),$$

and $f_R \in L^{p+1}(B_R) \cap L^\infty_{loc}(B_R)$, $g_R \in L^{q+1}(B_R) \cap L^\infty_{loc}(B_R)$.

To continue, we observe that by symmetry, we may assume $p \geq q$, then $p \geq \frac{n+\alpha}{n-\alpha}$ and $p - \frac{\alpha}{n}(p+1) \geq 1$. On the other hand, it follows from $\frac{1}{p+1} + \frac{1}{q+1} + \frac{\alpha}{n} = 1$ that $pq - 1 = \frac{\alpha}{n}(p+1)(q+1)$. Hence

$$\left[p - \frac{\alpha}{n}(p+1)\right] q - 1 = \frac{\alpha}{n}(p+1) > 0,$$

and this implies $q^{-1} < p - \frac{\alpha}{n}(p+1)$. Choose r such that

$$1 \leq r \leq p - \frac{\alpha}{n}(p+1) \text{ and } q^{-1} \leq r,$$

for example, we may take $r = p - \frac{\alpha}{n}(p+1)$, then

$$v(x)^{1/r} \leq \left(\int_{B_R} \frac{u(y)^p}{|x-y|^{n-\alpha}} dy\right)^{1/r} + g_R(x)^{1/r}.$$

We have

$$\begin{aligned} u(x) &= \int_{B_R} \frac{v(y)^{q-r^{-1}} v(y)^{1/r}}{|x-y|^{n-\alpha}} dy + f_R(x) \\ &\leq \int_{B_R} \frac{v(y)^{q-r^{-1}}}{|x-y|^{n-\alpha}} \left(\int_{B_R} \frac{u(z)^{p-r} u(z)^r}{|y-z|^{n-\alpha}} dz\right)^{1/r} dy + h_R(x). \end{aligned}$$

Here

$$h_R(x) = \int_{B_R} \frac{v(y)^{q-r^{-1}} g_R(y)^{1/r}}{|x-y|^{n-\alpha}} dy + f_R(x).$$

It follows from the fact that $g_R \in L^{q+1}(B_R) \cap L^\infty_{loc}(B_R)$ that $h_R \in L^{p+1}(B_R) \cap L^{\overline{q}}_{loc}(B_R)$ for all $\overline{q} < \infty$. Let

$$a = \frac{p+1}{p-r}, \quad b = \frac{q+1}{q-r^{-1}},$$

then calculation shows $\frac{1}{ra}+\frac{1}{b}=\frac{\alpha}{rn}+\frac{\alpha}{n}$, moreover we have

$$\frac{r}{p+1}+\frac{1}{a}=\frac{p}{p+1}<1,$$

and

$$\frac{1}{a}-\frac{\alpha}{n}=\frac{p-\frac{\alpha}{n}(p+1)-r}{p+1}\geq 0.$$

Hence for any $p+1<\overline{q}<\infty$, when R is small enough, it follows from Proposition 2.1 (by choosing α, β, p, q, r, a, b, u, U, V and f in Proposition 2.1 as α, α, $p+1$, $\overline{q}$, r, $\frac{p+1}{p-r}$, $\frac{q+1}{q-r^{-1}}$, u, u^{p-r}, $v^{q-r^{-1}}$ and h_R respectively) that $u\in L^{\overline{q}}\left(B_{R/4}\right)$. Since every point may be viewed as a center, we see $u\in L^{\overline{q}}_{loc}(\mathbb{R}^n)$. This implies $v\in L^{\infty}_{loc}(\mathbb{R}^n)$ and then $u\in L^{\infty}_{loc}(\mathbb{R}^n)$. Now observe that $f_R,g_R\in C^{\infty}(B_R)$, it follows from the usual bootstrap method that $u,v\in C^{\infty}(\mathbb{R}^n)$. The fact $u,v\in L^{\infty}(\mathbb{R}^n)$ under the assumption $u\in L^{p+1}(\mathbb{R}^n)$ follows from carefully going through the above argument and applying Holder's inequality when needed. Note that

$$u=\frac{\chi_{B_1}(x)}{|x|^{n-\alpha}}*v^q+\frac{\chi_{\mathbb{R}^n\backslash B_1}(x)}{|x|^{n-\alpha}}*v^q.$$

By interpolation we know $v\in L^s(\mathbb{R}^n)$ for all $q+1\leq s\leq\infty$, hence $v^q\in L^s$ for $\frac{q+1}{q}\leq s\leq\infty$. Since $\frac{q+1}{q}<\frac{n}{\alpha}$, it follows form the fact $\frac{\chi_{B_1}(x)}{|x|^{n-\alpha}}\in L^{\frac{n}{n-\alpha}-\varepsilon}(\mathbb{R}^n)$ and $\frac{\chi_{\mathbb{R}^n\backslash B_1}(x)}{|x|^{n-\alpha}}\in L^{\frac{n}{n-\alpha}+\varepsilon}(\mathbb{R}^n)$ for $\varepsilon>0$ small that $u(x)\to 0$ as $|x|\to\infty$. The fact $v(x)\to 0$ as $|x|\to\infty$ follows similarly. □

3. All solutions are radial

In this section, we will use the integral form of the method of moving plane (developed in [4]) to prove the radial symmetry of solutions to the integral system. Such radial property was derived in [3, 4, 8] under the further assumptions that both p and q are at least 1. Our approach works for both this case and the case when p or q is strictly less than 1. We will need the following basic inequality: assume $0<\theta\leq 1$, $a\geq b\geq 0$, $c\geq 0$, then

$$(a+c)^{\theta}-(b+c)^{\theta}\leq a^{\theta}-b^{\theta}.$$

Indeed for $x\geq 0$, let $f(x)=(a+x)^{\theta}-(b+x)^{\theta}$, then for $x>0$, $f'(x)=\theta(a+x)^{\theta-1}-\theta(b+x)^{\theta-1}\leq 0$. The inequality follows.

For $\xi\in\mathbb{R}^m$ and $s>0$, we denote

$$|\xi|_{l^s}=\left(\sum_{i=1}^{m}|\xi_i|^s\right)^{1/s}.$$

Proof of Theorem 1.1. By Proposition 2.2, we know $u,v\in C^{\infty}(\mathbb{R}^n)$, $u(x)\to 0$ and $v(x)\to 0$ as $|x|\to\infty$. It follows from Hardy-Littlewood-Sobolev inequality that $v\in L^{q+1}(\mathbb{R}^n)$. Without losing of generality, we may assume $p\geq q$, then we know $p\geq\frac{n+\alpha}{n-\alpha}$ and $p>q^{-1}$. Hence we may find a number r such that $1\leq r<p$ and $q^{-1}\leq r$.

For $\lambda \in \mathbb{R}$, we denote $H_\lambda = \{x \in \mathbb{R}^n : x_1 < \lambda\}$. For $x = (x_1, x') \in \mathbb{R}^n$, let $x_\lambda = (2\lambda - x_1, x')$. We also denote $u_\lambda(x) = u(x_\lambda)$, $v_\lambda(x) = v(x_\lambda)$,

$$\begin{aligned}\mathcal{B}_\lambda^u &= \{x \in H_\lambda : u_\lambda(x) > u(x)\}, \\ \mathcal{B}_\lambda^v &= \{x \in H_\lambda : v_\lambda(x) > v(x)\}.\end{aligned}$$

Note that by a change of variable, we have

$$\begin{aligned}u(x) &= \int_{\mathbb{R}^n} \frac{v(y)^q}{|x-y|^{n-\alpha}} dy \\ &= \int_{H_\lambda} \frac{v(y)^q}{|x-y|^{n-\alpha}} dy + \int_{H_\lambda} \frac{v(y_\lambda)^q}{|x_\lambda - y|^{n-\alpha}} dy.\end{aligned}$$

Hence

$$u(x_\lambda) = \int_{H_\lambda} \frac{v(y_\lambda)^q}{|x-y|^{n-\alpha}} dy + \int_{H_\lambda} \frac{v(y)^q}{|x_\lambda - y|^{n-\alpha}} dy.$$

This implies

$$\begin{aligned}&u(x_\lambda) - u(x) \\ &= \int_{H_\lambda} \left(v(y_\lambda)^q - v(y)^q\right) \left(\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x_\lambda - y|^{n-\alpha}}\right) dy.\end{aligned}$$

In particular, for $x \in \mathcal{B}_\lambda^u$, we have

$$\begin{aligned}0 &\le u(x_\lambda) - u(x) \\ &\le \int_{\mathcal{B}_\lambda^v} \left(v(y_\lambda)^q - v(y)^q\right) \left(\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x_\lambda - y|^{n-\alpha}}\right) dy \\ &\le \int_{\mathcal{B}_\lambda^v} \left(\left(v(y_\lambda)^{1/r}\right)^{qr} - \left(v(y)^{1/r}\right)^{qr}\right) \frac{1}{|x-y|^{n-\alpha}} dy \\ &\le qr \int_{\mathcal{B}_\lambda^v} v(y_\lambda)^{q-r^{-1}} \left(v(y_\lambda)^{1/r} - v(y)^{1/r}\right) \frac{1}{|x-y|^{n-\alpha}} dy.\end{aligned}$$

It follows from Hardy-Littlewood-Sobolev inequality that

$$\begin{aligned}&|u_\lambda - u|_{L^{p+1}\left(\mathcal{B}_\lambda^u\right)} \\ &\le c(n, \alpha, q, r) \left|v_\lambda^{q-r^{-1}} \left(v_\lambda^{1/r} - v^{1/r}\right)\right|_{L^{\frac{q+1}{q}}\left(\mathcal{B}_\lambda^v\right)} \\ &\le c(n, \alpha, q, r) \left|v_\lambda^{q-r^{-1}}\right|_{L^{\frac{q+1}{q-r^{-1}}}\left(\mathcal{B}_\lambda^v\right)} \left|v_\lambda^{1/r} - v^{1/r}\right|_{L^{(q+1)r}\left(\mathcal{B}_\lambda^v\right)} \\ &= c(n, \alpha, q, r) \left|v_\lambda\right|_{L^{q+1}\left(\mathcal{B}_\lambda^v\right)}^{q-r^{-1}} \left|v_\lambda^{1/r} - v^{1/r}\right|_{L^{(q+1)r}\left(\mathcal{B}_\lambda^v\right)}\end{aligned}$$

On the other hand, for $x \in \mathcal{B}_\lambda^v$, we have

$$\begin{aligned}
v(x_\lambda) &= \int_{\mathcal{B}_\lambda^u} \frac{u(y_\lambda)^p}{|x-y|^{n-\alpha}} dy + \int_{\mathcal{B}_\lambda^u} \frac{u(y)^p}{|x_\lambda - y|^{n-\alpha}} dy \\
&\quad + \int_{H_\lambda \setminus \mathcal{B}_\lambda^u} \frac{u(y_\lambda)^p}{|x-y|^{n-\alpha}} dy + \int_{H_\lambda \setminus \mathcal{B}_\lambda^u} \frac{u(y)^p}{|x_\lambda - y|^{n-\alpha}} dy \\
&\le \int_{\mathcal{B}_\lambda^u} \frac{u(y_\lambda)^p}{|x-y|^{n-\alpha}} dy + \int_{\mathcal{B}_\lambda^u} \frac{u(y)^p}{|x_\lambda - y|^{n-\alpha}} dy \\
&\quad + \int_{H_\lambda \setminus \mathcal{B}_\lambda^u} \frac{u(y)^p}{|x-y|^{n-\alpha}} dy + \int_{H_\lambda \setminus \mathcal{B}_\lambda^u} \frac{u(y_\lambda)^p}{|x_\lambda - y|^{n-\alpha}} dy.
\end{aligned}$$

Since

$$\begin{aligned}
v(x) &= \int_{\mathcal{B}_\lambda^u} \frac{u(y)^p}{|x-y|^{n-\alpha}} dy + \int_{\mathcal{B}_\lambda^u} \frac{u(y_\lambda)^p}{|x_\lambda - y|^{n-\alpha}} dy \\
&\quad + \int_{H_\lambda \setminus \mathcal{B}_\lambda^u} \frac{u(y)^p}{|x-y|^{n-\alpha}} dy + \int_{H_\lambda \setminus \mathcal{B}_\lambda^u} \frac{u(y_\lambda)^p}{|x_\lambda - y|^{n-\alpha}} dy,
\end{aligned}$$

it follows that

$$\begin{aligned}
0 &\le v(x_\lambda)^{1/r} - v(x)^{1/r} \\
&\le \left(\int_{\mathcal{B}_\lambda^u} \frac{u(y_\lambda)^p}{|x-y|^{n-\alpha}} dy + \int_{\mathcal{B}_\lambda^u} \frac{u(y)^p}{|x_\lambda - y|^{n-\alpha}} dy \right)^{1/r} \\
&\quad - \left(\int_{\mathcal{B}_\lambda^u} \frac{u(y)^p}{|x-y|^{n-\alpha}} dy + \int_{\mathcal{B}_\lambda^u} \frac{u(y_\lambda)^p}{|x_\lambda - y|^{n-\alpha}} dy \right)^{1/r} \\
&= \left(\int_{\mathcal{B}_\lambda^u} \left| \left(\frac{u(y_\lambda)^{p/r}}{|x-y|^{(n-\alpha)/r}}, \frac{u(y)^{p/r}}{|x_\lambda - y|^{(n-\alpha)/r}} \right) \right|_{l^r}^r dy \right)^{1/r} \\
&\quad - \left(\int_{\mathcal{B}_\lambda^u} \left| \left(\frac{u(y)^{p/r}}{|x-y|^{(n-\alpha)/r}}, \frac{u(y_\lambda)^{p/r}}{|x_\lambda - y|^{(n-\alpha)/r}} \right) \right|_{l^r}^r dy \right)^{1/r} \\
&\le \left(\int_{\mathcal{B}_\lambda^u} \left| \left(\frac{u(y_\lambda)^{p/r} - u(y)^{p/r}}{|x-y|^{(n-\alpha)/r}}, \frac{u(y)^{p/r} - u(y_\lambda)^{p/r}}{|x_\lambda - y|^{(n-\alpha)/r}} \right) \right|_{l^r}^r dy \right)^{1/r} \\
&\le 2 \left(\int_{\mathcal{B}_\lambda^u} \frac{\left(u_\lambda(y)^{p/r} - u(y)^{p/r} \right)^r}{|x-y|^{n-\alpha}} dy \right)^{1/r} \\
&\le \frac{2p}{r} \left(\int_{\mathcal{B}_\lambda^u} \frac{u_\lambda(y)^{p-r} (u_\lambda(y) - u(y))^r}{|x-y|^{n-\alpha}} dy \right)^{1/r}.
\end{aligned}$$

It follows from Hardy-Littlewood-Sobolev inequality that

$$
\begin{aligned}
&\left|v_\lambda^{1/r}-v^{1/r}\right|_{L^{(q+1)r}\left(\mathcal{B}_\lambda^v\right)}\\
&\le \frac{2p}{r}\left|\int_{\mathcal{B}_\lambda^u}\frac{u_\lambda(y)^{p-r}\left(u_\lambda(y)-u(y)\right)^r}{|x-y|^{n-\alpha}}dy\right|_{L^{q+1}\left(\mathcal{B}_\lambda^v\right)}^{1/r}\\
&\le c(n,\alpha,p,r)\left|u_\lambda^{p-r}\left(u_\lambda-u\right)^r\right|_{L^{\frac{p+1}{p}}\left(\mathcal{B}_\lambda^u\right)}^{1/r}\\
&\le c(n,\alpha,p,r)\left|u_\lambda^{p-r}\right|_{L^{\frac{p+1}{p-r}}\left(\mathcal{B}_\lambda^u\right)}^{1/r}\left|\left(u_\lambda-u\right)^r\right|_{L^{(p+1)/r}\left(\mathcal{B}_\lambda^u\right)}^{1/r}\\
&= c(n,\alpha,p,r)\left|u_\lambda\right|_{L^{p+1}\left(\mathcal{B}_\lambda^u\right)}^{\frac{p-r}{r}}\left|u_\lambda-u\right|_{L^{p+1}\left(\mathcal{B}_\lambda^u\right)}.
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
&\left|u_\lambda-u\right|_{L^{p+1}\left(\mathcal{B}_\lambda^u\right)}\\
&\le c(n,\alpha,p,q,r)\left|u_\lambda\right|_{L^{p+1}\left(\mathcal{B}_\lambda^u\right)}^{\frac{p-r}{r}}\left|v_\lambda\right|_{L^{q+1}\left(\mathcal{B}_\lambda^v\right)}^{q-r^{-1}}\left|u_\lambda-u\right|_{L^{p+1}\left(\mathcal{B}_\lambda^u\right)}\\
&= c(n,\alpha,p,q,r)\left|u\right|_{L^{p+1}\left(2\lambda e_1-\mathcal{B}_\lambda^u\right)}^{\frac{p-r}{r}}\left|v\right|_{L^{q+1}\left(2\lambda e_1-\mathcal{B}_\lambda^v\right)}^{q-r^{-1}}\left|u_\lambda-u\right|_{L^{p+1}\left(\mathcal{B}_\lambda^u\right)}\\
&\le c(n,\alpha,p,q,r)\left|u\right|_{L^{p+1}\left(2\lambda e_1-\mathcal{B}_\lambda^u\right)}^{\frac{p-r}{r}}\left|v\right|_{L^{q+1}\left(\mathbb{R}^n\right)}^{q-r^{-1}}\left|u_\lambda-u\right|_{L^{p+1}\left(\mathcal{B}_\lambda^u\right)}.
\end{aligned}
$$

Here $e_1=(1,0,\cdots,0)$.

After these preparations, we will use the method of moving planes to prove the radial symmetry of the solutions.

First, we have to show it is possible to start. Indeed, for λ large enough, we know $|u|_{L^{p+1}\left(2\lambda e_1-\mathcal{B}_\lambda^u\right)}$ can be arbitrary small, this implies that

$$\left|u_\lambda-u\right|_{L^{p+1}\left(\mathcal{B}_\lambda^u\right)}\le\frac{1}{2}\left|u_\lambda-u\right|_{L^{p+1}\left(\mathcal{B}_\lambda^u\right)},$$

and hence $\left|u_\lambda-u\right|_{L^{p+1}\left(\mathcal{B}_\lambda^u\right)}=0$. It follows that $\mathcal{B}_\lambda^u=\emptyset$ when λ is large enough.

Next we let $\lambda_0=\inf\left\{\lambda\in\mathbb{R}:\mathcal{B}_{\lambda'}^u=\emptyset\text{ for all }\lambda'\ge\lambda\right\}$. It follows from the fact $u(x)\to 0$ as $|x|\to\infty$ and $u(x)>0$ for all x that λ_0 must be a finite number. It follows from the definition of λ_0 that $u_{\lambda_0}(x)\le u(x)$ for $x\in H_{\lambda_0}$. We claim that $u_{\lambda_0}=u$. If this is not the case, then since

$$v_{\lambda_0}(x)-v(x)=\int_{H_{\lambda_0}}\left(u_{\lambda_0}(y)^p-u(y)^p\right)\left(\frac{1}{|x-y|^{n-\alpha}}-\frac{1}{|x_{\lambda_0}-y|^{n-\alpha}}\right)dy$$

and

$$u_{\lambda_0}(x)-u(x)=\int_{H_{\lambda_0}}\left(v_{\lambda_0}(y)^q-v(y)^q\right)\left(\frac{1}{|x-y|^{n-\alpha}}-\frac{1}{|x_{\lambda_0}-y|^{n-\alpha}}\right)dy,$$

we see $u_{\lambda_0}(x)<u(x)$ for $x\in H_{\lambda_0}$. This implies $\chi_{2\lambda e_1-\mathcal{B}_\lambda^u}\to 0$ *a.e.* as $\lambda\uparrow\lambda_0$. It follows that $|u|_{L^{p+1}\left(2\lambda e_1-\mathcal{B}_\lambda^u\right)}\to 0$ as $\lambda\uparrow\lambda_0$. Hence

$$\left|u_\lambda-u\right|_{L^{p+1}\left(\mathcal{B}_\lambda^u\right)}\le\frac{1}{2}\left|u_\lambda-u\right|_{L^{p+1}\left(\mathcal{B}_\lambda^u\right)}$$

when λ is close to λ_0. This implies $\mathcal{B}_\lambda^u = \emptyset$ for λ close to λ_0 and it contradicts with the choice of λ_0. Hence when the moving plane process stops, we must have symmetry. Moreover, $u_\lambda(x) < u(x)$ for $x \in H_\lambda$ when $\lambda > \lambda_0$. Indeed, for any $\lambda > \lambda_0$, we can not have $u_\lambda = u$ because otherwise u is periodic in the first direction and can not lie in L^{p+1}. Hence $u_\lambda < u$ in H_λ.

By translation, we may assume $u(0) = \max_{x \in \mathbb{R}^n} u(x)$, then it follows that the moving plane process from any direction must stop at the origin, hence u must be radial symmetric and strictly decreasing in the radial direction. It follows from the equation that v has the same properties. $\square$

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