

DENSITIES IN FREE GROUPS AND \mathbb{Z}^k ,
 VISIBLE POINTS AND TEST ELEMENTS

ILYA KAPOVICH, IGOR RIVIN, PAUL SCHUPP, AND VLADIMIR SHPILRAIN

ABSTRACT. In this article we relate two different densities. Let F_k be the free group of finite rank $k \geq 2$ and let α be the abelianization map from F_k onto \mathbb{Z}^k . We prove that if $S \subseteq \mathbb{Z}^k$ is invariant under the natural action of $SL(k, \mathbb{Z})$ then the asymptotic density of S in \mathbb{Z}^k and the annular density of its full preimage $\alpha^{-1}(S)$ in F_k are equal. This implies, in particular, that for every integer $t \geq 1$, the annular density of the set of elements in F_k that map to t -th powers of primitive elements in \mathbb{Z}^k is equal to $\frac{1}{t^k \zeta(k)}$, where ζ is the Riemann zeta-function.

An element g of a group G is called a *test element* if every endomorphism of G which fixes g is an automorphism of G . As an application of the result above we prove that the annular density of the set of all test elements in the free group $F(a, b)$ of rank two is $1 - \frac{6}{\pi^2}$. Equivalently, this shows that the union of all proper retracts in $F(a, b)$ has annular density $\frac{6}{\pi^2}$. Thus being a test element in $F(a, b)$ is an “intermediate property” in the sense that the probability of being a test element is strictly between 0 and 1.

1. Introduction

The idea of genericity and generic-case behavior in finitely presented groups was introduced by Gromov [17, 18] and is currently the subject of much research. (See, for example [2, 3, 4, 6, 8, 9, 10, 19, 22, 23, 24, 25, 26, 33, 34, 50].) Looking at the properties of random groups led Gromov [19] to a probabilistic proof that there exists a finitely generated group that is not uniformly embeddable in a Hilbert space. It also turns out that random group-theoretic objects exhibit various kinds of algebraic rigidity properties. In particular, Kapovich, Schupp and Shpilrain [24] proved that a random cyclically reduced element of a free group $F = F(A)$ is of minimal length in its $Aut(F)$ -orbit and that such an element has a trivial stabilizer in $Out(F)$. Moreover, it turns out [24] that random one-relator groups satisfy a strong Mostow-type rigidity. Specifically, two random one-relator groups $G_r = \langle a_1, \dots, a_k | r \rangle$ and $G_s = \langle a_1, \dots, a_k | s \rangle$ are isomorphic if and only if their Cayley graphs on the given set of generators $\{a_1, \dots, a_k\}$ are isomorphic as labelled graphs where the graph isomorphism is only allowed to permute the label set $\{a_1, \dots, a_k\}^{\pm 1}$.

The most straightforward definition of “genericity” is based on the notion of “asymptotic density”.

Definition 1.1 (Asymptotic density). Suppose that T is a countable set and that $\ell : T \rightarrow \mathbb{N}$ is a function (referred to as *length*) such that for every $n \in \mathbb{N}$ the set $\{x \in T : \ell(x) \leq n\}$ is finite. If $X \subseteq T$ and $n \geq 0$, we denote $\rho_\ell(n, X) := \#\{x \in X : \ell(x) \leq n\}$ and $\gamma_\ell(n, X) = \#\{x \in X : \ell(x) = n\}$.

Received by the editors February 7, 2006.

2000 *Mathematics Subject Classification*. Primary 20P05, Secondary 11M, 20F, 37A, 60B, 60F.

Let $S \subseteq T$. The *asymptotic density* of S in T is

$$\bar{\rho}_{T,\ell}(S) := \limsup_{n \rightarrow \infty} \frac{\#\{x \in S : \ell(x) \leq n\}}{\#\{x \in T : \ell(x) \leq n\}} = \limsup_{n \rightarrow \infty} \frac{\rho_\ell(n, S)}{\rho_\ell(n, T)},$$

where we treat a fraction $\frac{0}{0}$, if it occurs, as 0.

If the actual limit exists, we denote it by $\rho_{T,\ell}(S)$ and call this limit the *strict asymptotic density* of S in T . We say that S is *generic in T with respect to ℓ* if $\rho_{T,\ell}(S) = 1$ and that S is *negligible in T* if $\rho_{T,\ell}(S) = 0$.

If S is T -generic then the probability that a uniformly chosen element of T of length at most n belongs to S tends to 1 as n tends to infinity. The term ‘‘asymptotic density’’ appears to originally come from number theory, but on occasion this notion appeared in the literature under other names. Thus in [7] this notion is called ‘‘natural density’’.

It turns out that a different density, recording the proportions of a set in two successive spheres, is sometimes more suitable for subsets of a free group.

Definition 1.2 (Annular Density). Let T, S, ℓ be as in Definition 1.1. The *annular density* of S in T with respect to ℓ is:

$$\begin{aligned} \bar{\sigma}_{T,\ell}(S) &:= \limsup_{n \rightarrow \infty} \frac{1}{2} \left(\frac{\#\{x \in S : \ell(x) = n - 1\}}{\#\{x \in T : \ell(x) = n - 1\}} + \frac{\#\{x \in S : \ell(x) = n\}}{\#\{x \in T : \ell(x) = n\}} \right) = \\ &\limsup_{n \rightarrow \infty} \frac{1}{2} \left(\frac{\gamma_\ell(n - 1, S)}{\gamma_\ell(n - 1, T)} + \frac{\gamma_\ell(n, S)}{\gamma_\ell(n, T)} \right), \end{aligned}$$

where we treat a fraction $\frac{0}{0}$, if it occurs, as 0. Again, if the actual limit exists, we denote this limit by $\sigma_{T,\ell}(S)$ and call it the *strict annular density* of S in T with respect to ℓ .

Convention 1.3. Throughout this paper $F = F(A)$ will be a free group of rank $k \geq 2$ with a fixed finite basis $A = \{a_1, \dots, a_k\}$. If $w \in F$ then $|w|_A$ denotes the freely reduced length of w with respect to the basis A . In discussing the density (asymptotic or annular) of subsets of F using the notation above, we will assume that the ambient set is $T = F$ and that the length function $\ell(w)$ is $|w|_A$. If $S \subseteq F$ we denote its asymptotic and annular densities by $\bar{\rho}_A(S)$ and $\bar{\sigma}_A(S)$ respectively, and if the strict asymptotic density or the strict annular density exist we denote them by $\rho_A(S)$ and $\sigma_A(S)$ respectively. Also, we denote $\gamma_A(S) := \gamma_\ell(S)$ and $\rho_A(n, S) := \rho_\ell(n, S)$ in this case.

For subsets of \mathbb{Z}^k a length function $\ell : \mathbb{Z}^k \rightarrow \mathbb{R}$ will usually be the restriction to \mathbb{Z}^k of $\|\cdot\|_p$ -norm from \mathbb{R}^k for some $1 \leq p \leq \infty$. In this case for $S \subseteq \mathbb{Z}^k$ we denote the corresponding asymptotic density of S in $T = \mathbb{Z}^k$ by $\bar{\rho}_p(S)$ and if the strict asymptotic density exists, we denote it by $\rho_p(S)$.

It is not hard to see that if for a subset $S \subseteq F$ the strict asymptotic density $\rho_A(S)$ exists then the strict annular density $\sigma_A(S)$ also exists and in fact $\sigma_A(S) = \rho_A(S)$. Namely, since the sizes of both the balls and the spheres in $F(A)$ grow as constant multiples of $(2k - 1)^n$, if the strict asymptotic density $\rho_A(S)$ exists, then the limit $\lim_{n \rightarrow \infty} \frac{\gamma_A(n, S)}{\gamma_A(n, F)}$ exists and is equal to $\rho_A(S)$. Then the definition of $\sigma_A(S)$ implies that the strict annular density $\sigma_A(S)$ exists and is also equal to $\rho_A(S)$. However, as Example 1.5 below shows, it is possible that $\sigma_A(S)$ exists while $\rho_A(S)$ does not.

Thus there are reasonable situations where the parity of the radius of a sphere or a ball affects the outcome when measuring the relative size of a subset of a free group, and annular density turns out to be a more suitable and relevant quantity. This is the case when we consider a subset of \mathbb{Z}^k and the full preimage of this subset in F under the abelianization map. Moreover, annular density and its “close relatives” also make sense from the computational prospective. A typical experiment for generating a “random” element in a ball $B(n)$ of radius n in $F(A)$ might proceed as follows. First choose a uniformly random integer $m \in [0, n]$ and then choose a uniformly random element x from the m -sphere in $F(A)$ via a simple non-backtracking random walk of length m . It is easy to see that this experiment, while very natural, does not correspond to the uniform distribution on $B(n)$. For example, if F has rank $k = 2$, then for the uniform distribution on $B(n)$ the probability that the element x has length n is approximately $\frac{2}{3}$ for large n while in our experiment described above this probability is $1/(n + 1)$. In fact if $w \in B(n)$ then the probability of choosing the element w in the above experiment is $\frac{1}{(n+1)\#S(m)}$ where m is the freely reduced length of w and where $S(m)$ is the sphere of radius m in F . Thus if $X \subseteq F$ then the probability of choosing an element of X in the above experiment is

$$\frac{1}{n + 1} \sum_{m=0}^n \frac{\#(X \cap S(m))}{\#S(m)}.$$

If in our experiment we choose an element of $S(n - 1) \cup S(n)$ by first randomly and uniformly choosing $m \in \{n - 1, n\}$ and then choosing a uniformly random element of $S(m)$, then for a subset X of F the probability of picking an element of X is

$$\frac{1}{2} \left(\frac{\#(X \cap S(n - 1))}{\#S(n - 1)} + \frac{\#(X \cap S(n))}{\#S(n)} \right),$$

and the formulas from the definition of annular density appear.

For most of the cases where one can actually compute the asymptotic density of the set of elements in a free group having some natural algebraic property, this set turns out to be either generic or negligible. (Of course, a subset is negligible if and only if its complement is generic.) The following subsets are known to be negligible in a free group $F = F(A)$ of rank $k \geq 2$, both in the sense of asymptotic and annular densities: the set of all proper powers [2], a finite union of conjugacy classes, a subgroup of infinite index [48], a finite union of automorphic orbits (e.g. the set of all primitive elements) [16, 24], the set of all elements whose cyclically reduced forms are not automorphically minimal [24], the union of all proper free factors of F (this follows from results of [45] and [7, 6, 24]). Examples of generic sets, again in the sense of both the asymptotic and the annular densities, include: the set of all words whose symmetrizations satisfy the $C'(1/6)$ small cancellation condition [2], the set of words with nontrivial images in the abelianization of $F(A)$ [48] and the set of elements of $F(A)$ with cyclic stabilizers in $Aut(F(A))$ [24]. It is therefore interesting to find examples of natural properties of elements of free groups which are “intermediate” in the sense that they have density different from either 0 or 1. In this article we show that being a test element in the free group of rank two is such an example.

Convention 1.4 (The abelianization map). Recall that F is a free group of rank $k \geq 2$ with free basis $A = \{a_1, \dots, a_k\}$. We identify \mathbb{Z}^k with the abelianization of F

where the abelianization homomorphism $\alpha : F \rightarrow \mathbb{Z}^k$ is given by $a_i \mapsto e_i$, $i = 1, \dots, k$. Here e_i is the i -th unit vector. We also denote $\alpha(w)$ by \bar{w} .

It is easy to construct an example of a subset H of F such that the annular density of H in F and the asymptotic density of $\alpha(H)$ in \mathbb{Z}^k are different. For instance, let $F = F(a, b)$ and consider the subgroup $H = \langle a, b[a, b] \rangle \leq F$. Then H has infinite index in F and hence has both asymptotic and annular density 0 in F . On the other hand, $\alpha(H) = \alpha(F) = \mathbb{Z}^2$ has asymptotic density 1 in \mathbb{Z}^k with respect to any length function on \mathbb{Z}^2 .

Example 1.5. [7] Let $F = F(a, b)$, where $A = \{a, b\}$, be free of rank two. Let $\alpha : F \rightarrow \mathbb{Z}^2$ be the abelianization map. Note that for any $w \in F$ the lengths $|w|_A$ and $\|\alpha(w)\|_1$ have the same parity, since $\|\alpha(w)\|_1 = |w_a| + |w_b|$, where w_a, w_b are the exponent sums on a and b in w . Let $S = \{z \in \mathbb{Z}^2 : \|z\|_1 \text{ is even}\}$ and let $\tilde{S} := \alpha^{-1}(S) \subseteq F$. Then $\tilde{S} = \{w \in F : |w|_A \text{ is even}\}$. It is not hard to see that the strict asymptotic density of S in \mathbb{Z}^2 , with respect to $\|\cdot\|_p$ for any $1 \leq p \leq \infty$, exists and is equal to $1/2$.

Since \tilde{S} is exactly the union of all spheres of even radii in F , and the ratio of the sizes of spheres of radius n and $n - 1$ is equal to 3, it follows that the limits $\lim_{n \rightarrow \infty} \frac{\#\{w \in \tilde{S} : |w|_A = n\}}{\#\{w \in F : |w|_A = n\}}$ and $\lim_{n \rightarrow \infty} \frac{\#\{w \in \tilde{S} : |w|_A \leq n\}}{\#\{w \in F : |w|_A \leq n\}}$ do not exist. However, it is easy to see that for every $n \geq 1$

$$\frac{\#\{w \in \tilde{S} : |w|_A = n - 1\}}{\#\{w \in F : |w|_A = n - 1\}} + \frac{\#\{w \in \tilde{S} : |w|_A = n\}}{\#\{w \in F : |w|_A = n\}} = 1,$$

and therefore $\sigma_A(\tilde{S}) = \frac{1}{2}$. Thus although the strict asymptotic density of $\tilde{S} \subseteq F$ does not exist, the strict annular density does exist and is equal to the strict asymptotic density of $S \subseteq \mathbb{Z}^2$. More examples of a similar nature are discussed in Remark 1.8 of [36].

Example 1.5 demonstrates why the notion of annular density may be more suitable for working with subsets of free groups, while asymptotic density is more suitable for subsets of free abelian groups. Geometrically, this difference comes from the fact that free abelian groups are amenable with balls forming a Folner sequence, while free groups are non-amenable.

Although the counting occurs in very different places, it is interesting to ask how the asymptotic density of a subset $S \subseteq \mathbb{Z}^k$, with respect to some natural length function, and the annular density of its full preimage $\alpha^{-1}(S)$ in F are related. We shall see that there is a reasonable assumption about the set S which guarantees that the two densities are actually equal.

To do this we need to understand the image of the uniform distribution on the sphere of radius n in F under the abelianization map α . There is an explicit formula for the size of the preimage of an element, and there is also a Central Limit Theorem saying that, when appropriately normalized, the distribution converges to a normal distribution. The methods of [38] also give a *Local Limit Theorem* showing that, when working with width-two spherical shells in a free group, the densities of the image distributions in \mathbb{Z}^k converge to a normal density. Such a result was later also shown (by rather different methods, and in greater generality) by Richard Sharp in [41]. Recently Petridis and Risager [36] obtained a similar Local Limit Theorem

for counting conjugacy classes rather than elements of F . On the face of it, studying the annular density of the set $\alpha^{-1}(S)$ in F presents new challenges. The central limit theorem by itself seems too crude a tool and *a priori* it would appear that one would need very sharp error bounds in the local limit theorem. Nevertheless, we produce a short argument solving this problem where one of the key ingredients is the ergodicity of the $SL(k, \mathbb{Z})$ -action on \mathbb{R}^k .

We can now state our main result:

Theorem A. *Let $F = F(A)$ be a free group of rank $k \geq 2$ with free basis $A = \{a_1, \dots, a_k\}$ and let $\alpha : F \rightarrow \mathbb{Z}^k$ be the abelianization homomorphism.*

Let $S \subseteq \mathbb{Z}^k$ be an $SL(k, \mathbb{Z})$ -invariant subset and put $\tilde{S} = \alpha^{-1}(S) \subseteq F$.

Then

- (1) *For every $1 \leq p \leq \infty$ the strict asymptotic density $\rho_p(S)$ exists and, moreover, for every $1 \leq p \leq \infty$ we have $\rho_p(S) = \rho_\infty(S)$.*
- (2) *The strict annular density $\sigma_A(\tilde{S})$ exist and, moreover, $\sigma_A(\tilde{S}) = \rho_\infty(S)$.*

That is,

$$\lim_{n \rightarrow \infty} \frac{1}{2} \left(\frac{\gamma_A(n-1, \{w \in F : \alpha(w) \in S\})}{\gamma_A(n-1, F)} + \frac{\gamma_A(n, \{w \in F : \alpha(w) \in S\})}{\gamma_A(n, F)} \right) = \lim_{n \rightarrow \infty} \frac{\#\{z : z \in \mathbb{Z}^k, \|z\|_\infty \leq n, \text{ and } z \in S\}}{\#\{z : z \in \mathbb{Z}^k, \|z\|_\infty \leq n\}}.$$

The requirement that S be $SL(k, \mathbb{Z})$ -invariant essentially says that the subset S of \mathbb{Z}^k is defined in “abstract” group-theoretic terms, not involving the specific choice of a free basis for \mathbb{Z}^k . Note that Proposition 2.2 below gives an explicit formula for $\rho_\infty(S)$ in Theorem A.

Our main application of Theorem A concerns the case where S is the set of all “visible” points in \mathbb{Z}^k . A nonzero point z of \mathbb{Z}^k is called *visible* if the greatest common divisor of the coordinates of z is equal to 1. This terminology is standard in number theory [37] and reflects the fact that if z is visible then the line segment between the origin and z does not contain any other integer lattice points. For a nonzero point $z \in \mathbb{Z}^k$ being visible is also equivalent to z not being a proper power in \mathbb{Z}^k , that is, to z generating a maximal cyclic subgroup of \mathbb{Z}^k . More generally, if $t \geq 1$ is an integer, we will say that $z \in \mathbb{Z}^k$ is *t-visible* if $z = z_1^t$ for some visible $z_1 \in \mathbb{Z}^k$, that is, if the greatest common divisor of the coordinates of z is equal to t .

We want to “lift” this terminology to free groups.

Definition 1.6 (Visible elements in free groups). Let $F = F(A)$ be a free group of rank $k \geq 2$ with free basis $A = \{a_1, \dots, a_k\}$ and let $\alpha : F \rightarrow \mathbb{Z}^k$ be the abelianization homomorphism, that is, $\alpha(a_i) = e_i \in \mathbb{Z}^k$. We say that an element $w \in F$ is *visible* if $\alpha(w)$ is visible in \mathbb{Z}^k . Let V be the set of visible elements of F . Similarly, for an integer $t \geq 1$ an element $w \in F$ is *t-visible* if $\alpha(w)$ is *t-visible* in \mathbb{Z}^k . We use V_t to denote the set of all *t-visible* elements of F and we use U_t to denote the set of all *t-visible* elements of \mathbb{Z}^k .

Note that $V = V_1$ and that for every $t \geq 1$ the definition of V_t does not depend on the choice of the free basis A of F .

The following proposition giving the asymptotic density of the set of t -visible points in \mathbb{Z}^k in terms of the Riemann zeta-function is well-known in number theory [11].

Proposition 1.7. *For any integer $t \geq 1$ we have*

$$\rho_\infty(U_t) = \frac{1}{t^k \zeta(k)}.$$

The case $k = 2$ and $t = 1$ of Proposition 1.7 was proved by Mertens in 1874 [30]. (See also Theorem 332 of the classic book of Hardy and Wright [20].) Recall that $\zeta(k) = \sum_{n=1}^\infty \frac{1}{n^k}$ and, in particular, $\zeta(2) = \frac{\pi^2}{6}$.

It is therefore natural to investigate the asymptotic density of the set of visible elements in F . As a direct corollary of Theorem A, of Proposition 2.9 below and of Proposition 1.7 we obtain:

Theorem B. *Let $F = F(A)$ be a free group of rank $k \geq 2$ with free basis $A = \{a_1, a_2, \dots, a_k\}$. Let $t \geq 1$ be an integer. Then the strict annular density $\sigma_A(V_t)$ exists and*

$$\sigma_A(V_t) = \frac{1}{t^k \zeta(k)}.$$

Moreover, in this case

$$0 < \frac{4k - 4}{(2k - 1)^2 t^k \zeta(k)} \leq \liminf_{n \rightarrow \infty} \frac{\rho_A(n, V_t)}{\rho_A(n, F)} \leq \limsup_{n \rightarrow \infty} \frac{\rho_A(n, V_t)}{\rho_A(n, F)} \leq 1 - \frac{4k - 4}{(2k - 1)^2} \left(1 - \frac{1}{t^k \zeta(k)}\right) < 1.$$

A result similar to Theorem B for counting conjugacy classes in F_k with primitive images in \mathbb{Z}^k has been recently independently obtained by Petridis and Risager [36].

For the case of the free group of rank two we compute the two “spherical densities” for the density of the set V_1 , corresponding to even and odd n tending to infinity:

Theorem C. *Let $k = 2$. We have*

$$\lim_{m \rightarrow \infty} \frac{\gamma_A(2m, V_1)}{\gamma_A(2m, F)} = \frac{2}{3\zeta(2)} = \frac{4}{\pi^2}$$

and

$$\lim_{m \rightarrow \infty} \frac{\gamma_A(2m - 1, V_1)}{\gamma_A(2m - 1, F)} = \frac{8}{\pi^2}.$$

Moreover, the strict asymptotic density $\rho_A(V_1)$ does not exist.

The above result shows that the statements of Theorem A and Theorem B cannot be substantially improved. This fact underscores the conclusion that annular density is the right kind of notion for measuring the sizes of subsets of free groups, where the abelianization map is concerned.

We apply Theorem B to compute the annular density of test elements in a free group of rank two. Recall that an element $g \in G$ is called a *test element* if every endomorphism of G fixing g is actually an automorphism of G . It is easy to see that for two conjugate elements $g_1, g_2 \in G$ the element g_1 is a test element if and only if g_2 is also a test element and thus the property of being a test element depends only on the conjugacy class of an element g . The notion of a test element was introduced

by Shpilrain [42] and has since become a subject of active research both in group theory and in the context of other algebraic structures such as polynomial algebras and Lie algebras. (See, for example, [15, 21, 27, 31, 32, 35, 40, 43, 46].) The idea of a test element goes back to the classic work of Nielsen who proved that in $F(a, b)$ the commutator $[a, b]$ has this property. It turns out that studying test elements in a particular group G produces interesting information about the automorphism group of G .

Here we prove:

Theorem D. *Let $F = F(a, b)$ be a free group of rank two with free basis $A = \{a, b\}$. Then for the set \mathcal{T} of all test elements in F the strict annular density exists and*

$$\sigma_A(\mathcal{T}) = 1 - \frac{6}{\pi^2}.$$

Moreover,

$$0 < \frac{4}{9}\left(1 - \frac{6}{\pi^2}\right) \leq \liminf_{n \rightarrow \infty} \frac{\rho_A(n, \mathcal{T})}{\rho_A(n, F)} \leq \limsup_{n \rightarrow \infty} \frac{\rho_A(n, \mathcal{T})}{\rho_A(n, F)} \leq 1 - \frac{8}{3\pi^2} < 1.$$

By a result of Turner [47], an element of F is a test element if and only if it does not belong to a proper retract of F . Therefore Theorem D implies that the strict annular density of the union of all proper retracts of $F(a, b)$ is $\frac{6}{\pi^2}$. In Theorem D above we have $\frac{4}{9}\left(1 - \frac{6}{\pi^2}\right) \approx 0.1742$, $1 - \frac{6}{\pi^2} \approx 0.3920$ and $1 - \frac{8}{3\pi^2} \approx 0.7298$. Thus Theorem D shows that being a test element is an “intermediate” property in the free group of rank two. More generally, Theorem B implies that, for every $k \geq 2$ and for N sufficiently large, the set V_N of N -visible elements in F_k has strictly positive annular density arbitrarily close to 0 while the set $S_N = \bigcup_{t=1}^N V_t$ has annular density less than but arbitrarily close to 1. It is well-known that every positive rational number has an “Egyptian fraction” representation as a finite sum of distinct terms of the form $\frac{1}{n}$. It does not seem clear what values can be obtained as finite sums of distinct terms of the form $\frac{1}{n^2}$ and there are excluded intervals. In particular, if such a sum uses 1, it is at least 1 while if 1 is not used then the sum is at most $\frac{\pi^2}{6} - 1$. Multiplying by the scale factor $\frac{1}{\zeta(2)}$, we see that we cannot obtain an annular density in the open interval $\left(1 - \frac{6}{\pi^2}, \frac{6}{\pi^2}\right)$ by taking a finite union of the sets V_t (Proposition 2.2 below shows that the same is true for infinite unions). It is interesting to note that the probabilities of being a test element or of not being a test element are the boundary points of this excluded interval.

Nathan Dunfield and Dylan Thurston recently proved [13] that for a two-generator one-relator group being free-by-cyclic is an intermediate property. While they do not provide an exact value for the asymptotic density (nor do they prove that either the strict asymptotic density or the strict annular density exist), they show that it is strictly between 0 and 1. Computer experiments by Kapovich and Schupp, by Mark Sapir and by Dunfield and Thurston indicate that in the two-generator case this asymptotic density is greater than 0.9.

2. Comparing densities in \mathbb{Z}^k and in F_k

Convention 2.1. Throughout this section let $S \subseteq \mathbb{Z}^k$ be as in Theorem A and let $\delta := \bar{\rho}_\infty(S)$.

We can now prove that for every $SL(k, \mathbb{Z})$ -invariant subset S of \mathbb{Z}^k the strict asymptotic density $\rho_\infty(S)$ exists. Recall that Proposition 1.7 stated in the introduction gives the precise value of the strict asymptotic density of the set of all t -visible elements in \mathbb{Z}^k . The crucial points of the proof below are that the complement of an $SL(k, \mathbb{Z})$ -invariant set is also $SL(k, \mathbb{Z})$ -invariant and that $\sum_{t=1}^\infty \frac{1}{t^k \zeta(k)} = 1$.

Proposition 2.2. *Let $Y \subseteq \mathbb{Z}^k$ be a nonempty $SL(k, \mathbb{Z})$ -invariant subset that does not contain $\mathbf{0} \in \mathbb{Z}^k$. Let I be the set of all integers $t \geq 1$ such that there exists a t -visible element in Y . Then*

- (1) $Y = \cup_{t \in I} U_t$.
- (2) *The strict asymptotic density $\rho_\infty(Y)$ exists and*

$$\rho_\infty(Y) = \sum_{t \in I} \rho(U_t) = \sum_{t \in I} \frac{1}{t^k \zeta(k)}.$$

Proof. Observe first that

$$\sum_{t=1}^\infty \frac{1}{t^k \zeta(k)} = \frac{1}{\zeta(k)} \sum_{t=1}^\infty \frac{1}{t^k} = \frac{\zeta(k)}{\zeta(k)} = 1.$$

Since $k \geq 2$, two nonzero elements $z, z' \in \mathbb{Z}^k$ lie in the same $SL(k, \mathbb{Z})$ -orbit if and only if the greatest common divisors of the coordinates of z and of z' are equal. Thus every $SL(k, \mathbb{Z})$ -orbit of a nonzero element of \mathbb{Z}^k has the form U_t for some $t \geq 1$. This implies part (1) of Proposition 2.2.

Let $I' := \{t \in \mathbb{Z} : t \geq 1, t \notin I\}$. If either I or I' is finite, part (2) of Proposition 2.2 follows directly from proposition 1.7. Suppose now that both I and I' are infinite and let $Y' := \mathbb{Z}^k - (Y \cup \{\mathbf{0}\}) = \cup_{t \in I'} U_t$.

For every finite subset $J \subseteq I$ let $Y_J := \cup_{t \in J} U_t$. Since $Y_J \subseteq Y$, it follows that

$$\liminf_{n \rightarrow \infty} \frac{\#\{z \in Y : \|z\|_\infty \leq n\}}{\#\{z \in \mathbb{Z}^k : \|z\|_\infty \leq n\}} \geq \rho_\infty(Y_J) = \sum_{t \in J} \frac{1}{t^k \zeta(k)}.$$

Since this is true for every finite subset J of I , we conclude that

$$\liminf_{n \rightarrow \infty} \frac{\#\{z \in Y : \|z\|_\infty \leq n\}}{\#\{z \in \mathbb{Z}^k : \|z\|_\infty \leq n\}} \geq \sum_{t \in I} \frac{1}{t^k \zeta(k)}.$$

The same argument applies to the $SL(k, \mathbb{Z})$ -invariant set Y' and therefore:

$$\liminf_{n \rightarrow \infty} \frac{\#\{z \in Y' : \|z\|_\infty \leq n\}}{\#\{z \in \mathbb{Z}^k : \|z\|_\infty \leq n\}} \geq \sum_{t \in I'} \frac{1}{t^k \zeta(k)}.$$

This implies

$$\begin{aligned}
 1 - \liminf_{n \rightarrow \infty} \frac{\#\{z \in Y' : \|z\|_\infty \leq n\}}{\#\{z \in \mathbb{Z}^k : \|z\|_\infty \leq n\}} &\leq 1 - \sum_{t \in I'} \frac{1}{t^k \zeta(k)} \Rightarrow \\
 \limsup_{n \rightarrow \infty} \left(1 - \frac{\#\{z \in Y' : \|z\|_\infty \leq n\}}{\#\{z \in \mathbb{Z}^k : \|z\|_\infty \leq n\}} \right) &\leq 1 - \sum_{t \in I'} \frac{1}{t^k \zeta(k)} \Rightarrow \\
 \limsup_{n \rightarrow \infty} \frac{\#\{z \in Y : \|z\|_\infty \leq n\}}{\#\{z \in \mathbb{Z}^k : \|z\|_\infty \leq n\}} &\leq 1 - \sum_{t \in I'} \frac{1}{t^k \zeta(k)} = \sum_{t \in I} \frac{1}{t^k \zeta(k)}.
 \end{aligned}$$

Hence

$$\lim_{n \rightarrow \infty} \frac{\#\{z \in Y : \|z\|_\infty \leq n\}}{\#\{z \in \mathbb{Z}^k : \|z\|_\infty \leq n\}} = \sum_{t \in I} \frac{1}{t^k \zeta(k)},$$

as required. □

Recall that $S \subseteq \mathbb{Z}^k$ is an $SL(k, \mathbb{Z})$ -invariant subset and that $\delta = \bar{\rho}_\infty(S)$. Proposition 2.2 implies that in fact $\delta = \rho_\infty(S)$.

It is well known that if $\Omega \subseteq \mathbb{R}^k$ is a “nice” bounded open set then the Lebesgue measure $\lambda(\Omega)$ can be computed as

$$\lambda(\Omega) = \lim_{r \rightarrow \infty} \frac{\#\{(\mathbb{Z}^k \cap r\Omega)\}}{r^k}.$$

Here we say that a bounded open subset of \mathbb{R}^k is “nice” if its boundary is piecewise smooth.

We need a similar formula for counting the points of S . For a real number $r \geq 1$ and a nice bounded open set $\Omega \subseteq \mathbb{R}^k$ let

$$\mu_{r,S}(\Omega) := \frac{\#\{S \cap r\Omega\}}{r^k}.$$

Proposition 2.3. *For any nice bounded open set $\Omega \subseteq \mathbb{R}^k$ we have*

$$\lim_{r \rightarrow \infty} \mu_{r,S}(\Omega) = \delta\lambda(\Omega).$$

Proof. Each $\mu_{r,S}$ can be regarded as a measure on \mathbb{R}^k . We prove the theorem by showing that the $\mu_{r,S}$ weakly converge to $\delta\lambda$ as $r \rightarrow \infty$, where λ is the Lebesgue measure.

By Helly’s theorem there exists a sequence $(r_i)_{i=1}^\infty$ with $\lim_{i \rightarrow \infty} r_i = \infty$ such that the sequence $\mu_{r_1,S}, \mu_{r_2,S}, \dots$ is weakly convergent to some limiting measure. We now show that for every such convergent subsequence of $\mu_{r_i,S}$ the limiting measure is indeed equal to $\delta\lambda$, where λ is the Lebesgue measure.

Indeed, suppose that $\sigma = (r_i)_{i=1}^\infty$ is a sequence with $\lim_{i \rightarrow \infty} r_i = \infty$ such that the sequence $\mu_{r_i,S}$ converges to the limiting measure $\mu_\sigma = \lim_{i \rightarrow \infty} \mu_{r_i,S}$. Every $\mu_{r_i,S}$ is invariant with respect to the natural $SL(k, \mathbb{Z})$ -action since this action preserves the set S and also commutes with homotheties of \mathbb{R}^k centered at the origin. Therefore the limiting measure μ_σ is also $SL(k, \mathbb{Z})$ -invariant.

Moreover, the measures $\mu_{r,S}$ are dominated by the measures λ_r defined as $\lambda_r(\Omega) = \frac{\#(\mathbb{Z}^k \cap r\Omega)}{r^k}$. Since, as observed earlier, the measures λ_r converge to the Lebesgue measure λ , it follows that μ_σ is absolutely continuous with respect to λ . It is known that the natural action of $SL(k, \mathbb{Z})$ on \mathbb{R}^k is ergodic with respect to λ . (See, for example, Zimmer’s classic monograph [49].) Therefore μ_σ is a constant multiple $c\lambda$ of λ . The constant c can be computed explicitly for a set such as the open unit ball B in the $\|\cdot\|_\infty$ norm on \mathbb{R}^k defining the length function ℓ on \mathbb{Z}^k .

By assumption we know that

$$\lim_{r \rightarrow \infty} \frac{\#\{z \in \mathbb{Z}^k : z \in S \cap rB\}}{\#\{z \in \mathbb{Z}^k : z \in rB\}} = \delta.$$

We also have

$$\lim_{r \rightarrow \infty} \frac{\#\{z \in \mathbb{Z}^k : z \in rB\}}{r^k} = \lambda(B)$$

and hence

$$\lim_{r \rightarrow \infty} \frac{\#\{z \in \mathbb{Z}^k : z \in S \cap rB\}}{r^k} = \delta\lambda(B).$$

Therefore $c = \delta$ and $\mu_\sigma = \delta\lambda$. The above argument in fact shows that every convergent subsequence, with $r \rightarrow \infty$, of $\mu_{r,S}$ converges to $\delta\lambda$ and therefore $\lim_{r \rightarrow \infty} \mu_{r,S} = \delta\lambda$. □

Remark 2.4. Let $1 \leq p \leq \infty$. Then the open unit ball in \mathbb{R}^k with respect to $\|\cdot\|_p$ is "nice". Proposition 2.3, applied to Ω being this ball, implies that $\rho_p(S) = \rho_\infty(S) = \delta$.

Convention 2.5. As always, $F = F(a_1, \dots, a_k)$ is the free group of rank $k \geq 2$ with free basis $A = \{a_1, \dots, a_k\}$ and $\alpha : F \rightarrow \mathbb{Z}^k$ is the abelianization homomorphism sending a_i to e_i in \mathbb{Z}^k . We will denote $\alpha(w)$ by \bar{w} . For $n \geq 1$, $B_F(n)$ denotes the set of all $w \in F$ with $|w|_A \leq n$. Also, for a point $x = (x_1, \dots, x_k) \in \mathbb{R}^k$ we denote by $\|x\|$ the $\|\cdot\|_2$ -norm of x , that is $\|x\| = \sqrt{\sum_{i=1}^k x_i^2}$.

Notation 2.6. For an integer $n \geq 1$ and a point $x \in \mathbb{R}^k$ let

$$p_n(x) = \frac{\gamma_A(n-1, \{f \in F : \alpha(f) = x\sqrt{n}\})}{2\gamma_A(n-1, F)} + \frac{\gamma_A(n, \{f \in F : \alpha(f) = x\sqrt{n}\})}{2\gamma_A(n, F)}.$$

Thus p_n is a distribution supported on finitely many points of $\frac{1}{\sqrt{n}}\mathbb{Z}^k$.

We need the following facts about the sequence of distributions p_n . Of these the most significant is part (2) which is a local limit theorem in our context. It was obtained by Rivin [38] and, independently and via different methods, by Sharp [41] (specifically, we use Theorem 1 of [41] for part (2) of Proposition 2.7 below).

Proposition 2.7. [38, 41] *Let $k \geq 2$ and let p_n be as above. Then:*

- (1) *The sequence of distributions p_n converges weakly to a normal distribution \mathfrak{N} , with density \mathbf{n} .*
- (2) *We have*

$$\sup_{x \in \mathbb{Z}^k / \sqrt{n}} |p_n(x)n^{k/2} - \mathbf{n}(x)| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

(3) We have

$$\lim_{c \rightarrow \infty} \sum \{p_n(x) : x \in \mathbb{Z}^k/\sqrt{n} \text{ and } \|x\| \geq c\} = 0.$$

Recall that convergence in distribution for a sequence of probability measures μ_n to a probability measure μ means that for any bounded continuous function, the integral with respect to μ_n converges to the integral with respect to μ . A local limit theorem is a statement saying that the probability densities converge pointwise. We refer the reader to the book of Feller [14] for the background information on these notions.

Theorem 2.8. *Let $\Omega \subseteq \mathbb{R}^k$ be a nice bounded open set. Then*

$$\lim_{n \rightarrow \infty} \sum_{x \in S \cap \sqrt{n}\Omega} p_n(x/\sqrt{n}) = \delta \mathfrak{N}(\Omega).$$

Proof. We have

$$\begin{aligned} \sum_{x \in \mathbb{Z}_t^k \cap \sqrt{n}\Omega} p_n(x/\sqrt{n}) &= \sum_{y \in \frac{1}{\sqrt{n}}S \cap \Omega} p_n(y) = \\ n^{-k/2} \sum_{y \in \frac{1}{\sqrt{n}}S \cap \Omega} \mathfrak{n}(y) &+ n^{-k/2} \sum_{y \in \frac{1}{\sqrt{n}}S \cap \Omega} (n^{k/2}p_n(y) - \mathfrak{n}(y)). \end{aligned}$$

The local limit theorem in part (2) of Proposition 2.7 tells us that, as $n \rightarrow \infty$, each summand $n^{k/2}p_n(y) - \mathfrak{n}(y)$ of the second sum in the last line of equation above converges to zero and hence so does their Cesaro mean. Proposition 2.3 implies that, as $n \rightarrow \infty$, the first summand $n^{-k/2} \sum_{y \in \frac{1}{\sqrt{n}}S \cap \Omega} \mathfrak{n}(y)$ converges to

$$\delta \int_{\Omega} \mathfrak{n} d\lambda = \delta \mathfrak{N}(\Omega).$$

□

We can now compute the strict asymptotic density of $\tilde{S} = \alpha^{-1}(S)$ in F and obtain Theorem A.

Proof of Theorem A. Recall that $S \subseteq \mathbb{Z}^k$ is an $SL(k, \mathbb{Z})$ -invariant set and that $\delta = \bar{\rho}_{\infty}(S)$. Proposition 2.2 implies that in fact $\rho_{\infty}(S)$ exists and $\delta = \rho_{\infty}(S)$. Moreover, as we have seen in Remark 2.4, for every $1 \leq p \leq \infty$ the strict asymptotic density $\rho_p(S)$ exists and $\rho_p(S) = \delta = \rho_{\infty}(S)$. This proves part (1) of Theorem A.

To prove part (2) of Theorem A we need to establish that the strict annular density $\sigma_A(S)$ exists and that $\sigma_A(S) = \delta$.

For $c > 0$ denote $\Omega_c := \{x \in \mathbb{R}^k : \|x\| < c\}$. Then $\lim_{c \rightarrow \infty} \mathfrak{N}(\Omega_c) = 1$. Let $\epsilon > 0$ be arbitrary. Choose $c > 0$ such that

$$|\mathfrak{N}(\Omega_c) - 1| \leq \epsilon/3$$

and such that

$$\lim_{n \rightarrow \infty} \sum \{p_n(x) : x \in \mathbb{Z}^k/\sqrt{n} \text{ and } \|x\| \geq c\} \leq \epsilon/6.$$

By Theorem 2.8 and the above formula there is some $n_0 \geq 1$ such that for all $n \geq n_0$ we have

$$\left| \sum_{x \in S \cap \sqrt{n}\Omega_c} p_n(x/\sqrt{n}) - \delta \mathfrak{N}(\Omega_c) \right| \leq \epsilon/3$$

and

$$\sum \{p_n(x) : x \in \mathbb{Z}^k/\sqrt{n} \text{ and } \|x\| \geq c\} \leq \epsilon/3.$$

Let

$$Q(n) := \frac{\gamma_A(n-1, \{w \in F : \bar{w} \in S\})}{2\gamma_A(n-1, F)} + \frac{\gamma_A(n, \{w \in F : \bar{w} \in S\})}{2\gamma_A(n, F)}.$$

For $n \geq n_0$ we have

$$\begin{aligned} Q(n) &= \\ & \frac{\#\{w \in F : \bar{w} \in S, |w|_A = n-1 \text{ and } \|\bar{w}\| < c\sqrt{n}\}}{2\gamma_A(n-1, F)} + \\ & \frac{\#\{w \in F : \bar{w} \in S, |w|_A = n \text{ and } \|\bar{w}\| < c\sqrt{n}\}}{2\gamma_A(n, F)} + \\ & \frac{\#\{w \in F : \bar{w} \in S, |w|_A = n-1 \text{ and } \|\bar{w}\| \geq c\sqrt{n}\}}{2\gamma_A(n-1, F)} + \\ & \frac{\#\{w \in F : \bar{w} \in S, |w|_A = n \text{ and } \|\bar{w}\| \geq c\sqrt{n}\}}{2\gamma_A(n, F)} = \\ & \sum_{x \in S \cap \sqrt{n}\Omega_c} p_n(x/\sqrt{n}) + \sum_{x \in S \cap (\mathbb{R}^k - \sqrt{n}\Omega_c)} p_n(x/\sqrt{n}). \end{aligned}$$

In the last line of the above equation, the first sum differs from $\delta \mathfrak{N}(\Omega_c)$ by at most $\epsilon/3$ since $n \geq n_0$ and the second sum is $\leq \epsilon/3$ by the choice of c and n_0 . Therefore, again by the choice of c , we have $|Q(n) - \delta| \leq \epsilon$. Since $\epsilon > 0$ was arbitrary, this implies that $\lim_{n \rightarrow \infty} Q(n) = \delta$, as claimed. \square

The following observation shows how to estimate the asymptotic density in terms of the annular density.

Proposition 2.9. *Let $Y \subseteq F$ be a subset such that the strict annular density $\delta = \sigma_A(Y)$ exists. Then*

$$\frac{4k-4}{(2k-1)^2} \delta \leq \liminf_{n \rightarrow \infty} \frac{\rho_A(n, S)}{\rho_A(n, F)} \leq \limsup_{n \rightarrow \infty} \frac{\rho_A(n, S)}{\rho_A(n, F)} \leq 1 - \frac{4k-4}{(2k-1)^2} (1-\delta).$$

In particular, if $0 < \delta < 1$ then

$$0 < \liminf_{n \rightarrow \infty} \frac{\rho_A(n, S)}{\rho_A(n, F)} \leq \limsup_{n \rightarrow \infty} \frac{\rho_A(n, S)}{\rho_A(n, F)} < 1.$$

Proof. Note that for $n \geq 1$ we have $\gamma_A(n, F) = 2k(2k-1)^{n-1}$ and that, up to an additive constant, $\rho_A(n, F) = \frac{k}{k-1}(2k-1)^n$. Denote $a_n = \gamma_A(n, Y)$. We have

$$\delta = \lim_{n \rightarrow \infty} \frac{1}{2} \left(\frac{a_{n-1}}{2k(2k-1)^{n-2}} + \frac{a_n}{2k(2k-1)^{n-1}} \right) = \frac{1}{2} \lim_{n \rightarrow \infty} \frac{a_{n-1} \frac{2k-1}{2k-2} + a_n \frac{1}{2k-2}}{\frac{k}{k-1} (2k-1)^{n-1}}.$$

Therefore

$$\begin{aligned} \liminf_{n \rightarrow \infty} \frac{\rho_A(n, Y)}{\rho_A(n, F)} &= \liminf_{n \rightarrow \infty} \frac{a_1 + \dots + a_n}{\frac{k}{k-1}(2k-1)^n} \geq \liminf_{n \rightarrow \infty} \frac{a_{n-1} + a_n}{\frac{k}{k-1}(2k-1)^n} = \\ \frac{2k-2}{2k-1} \liminf_{n \rightarrow \infty} \frac{a_{n-1} \frac{2k-1}{2k-2} + a_n \frac{2k-1}{2k-2}}{\frac{k}{k-1}(2k-1)^n} &\geq \frac{2k-2}{2k-1} \liminf_{n \rightarrow \infty} \frac{a_{n-1} \frac{2k-1}{2k-2} + a_n \frac{1}{2k-2}}{\frac{k}{k-1}(2k-1)^n} = \\ \frac{4k-4}{(2k-1)^2} \liminf_{n \rightarrow \infty} \frac{1}{2} \cdot \frac{a_{n-1} \frac{2k-1}{2k-2} + a_n \frac{1}{2k-2}}{\frac{k}{k-1}(2k-1)^{n-1}} &= \frac{4k-4}{(2k-1)^2} \delta. \end{aligned}$$

Applying the same argument to the set $F - Y$, we get

$$\liminf_{n \rightarrow \infty} \frac{\rho_A(n, F - Y)}{\rho_A(n, F)} \geq \frac{4k-4}{(2k-1)^2} (1 - \delta).$$

Therefore

$$\limsup_{n \rightarrow \infty} \frac{\rho_A(n, Y)}{\rho_A(n, F)} = 1 - \liminf_{n \rightarrow \infty} \frac{\rho_A(n, F - Y)}{\rho_A(n, F)} \leq 1 - \frac{4k-4}{(2k-1)^2} (1 - \delta).$$

□

3. Spherical densities

In this section we will prove Theorem C from the Introduction and, for the case of $k = 2$, compute the “spherical densities”

$$\lim_{m \rightarrow \infty} \frac{\gamma_A(2m, V_1)}{\gamma_A(2m, F)}$$

and

$$\lim_{m \rightarrow \infty} \frac{\gamma_A(2m-1, V_1)}{\gamma_A(2m-1, F)}.$$

This is done by computing the “spherical densities” for the set $V_1^{ev} \subseteq F$ consisting of all points of V_1 of even length and comparing it with the strict asymptotic density of the set $U_1^{ev} \subseteq \mathbb{Z}^k$ of all elements of \mathbb{Z}^k of even $\|\cdot\|_\infty$ -length. The key point is that for $n = 2m$ we have $\gamma_A(2m-1, V_1^{ev}) = 0$ and $\gamma_A(2m, V_1) = \gamma_A(2m, V_1^{ev})$. Therefore for the quantities from the definition of annular density we have

$$\frac{1}{2} \left(\frac{\gamma_A(2m, V_1^{ev})}{\gamma_A(2m, F)} + \frac{\gamma_A(2m-1, V_1^{ev})}{\gamma_A(2m-1, F)} \right) = \frac{\gamma_A(2m, V_1^{ev})}{2\gamma_A(2m, F)} = \frac{\gamma_A(2m, V_1)}{2\gamma_A(2m, F)}.$$

This allows us to essentially repeat the proof of Theorem A, applied to the sets U_1^{ev} and $V_1^{ev} = \alpha^{-1}(U_1^{ev})$, except that instead of ergodicity of the action of $SL(k, \mathbb{Z})$ we use the ergodicity of the action on \mathbb{R}^k of a congruence subgroup of $SL(k, \mathbb{Z})$ that leaves U_1^{ev} invariant.

Convention 3.1. We say that an element $z = (z_1, \dots, z_k) \in \mathbb{Z}^k$ is *even* if $\|z\|_1 = |z_1| + \dots + |z_k|$ is even and that z is *odd* if $\|z\|_1$ is odd. Similarly, $w \in F$ is *even* if $|w|_A$ is even and $w \in F$ is *odd* if $|w|_A$ is odd. Note that $w \in F$ is even if and only if $\alpha(w) \in \mathbb{Z}^k$ is even.

Let G_k be the set of all $M \in SL(k, \mathbb{Z})$ such that $M = I_k$ in $SL(k, \mathbb{Z}/2\mathbb{Z})$. Thus G_k is a finite index subgroup of $SL(k, \mathbb{Z})$ also known as the 2-congruence subgroup. Denote by $\mathbb{Z}^{k, ev}$ the set of all even elements in \mathbb{Z}^k . Also denote $U_1^{ev} := U_1 \cap \mathbb{Z}^{k, ev}$.

Observe that $\mathbb{Z}^{k, ev}$ and U_1^{ev} are G_k -invariant and that $V_1^{ev} = \alpha^{-1}(U_1^{ev})$. (The actual set-wise stabilizer of $\mathbb{Z}^{k, ev}$ in $SL(k, \mathbb{Z})$ contains G_k as a subgroup of finite index.)

Lemma 3.2. *Let $Y \subseteq F$. Then the following hold:*

- (1) *If $\rho_A(Y)$ exists then $\sigma_A(Y)$ and $\lim_{n \rightarrow \infty} \frac{\gamma_A(n, Y)}{\gamma_A(n, F)}$ exist and*

$$\rho_A(Y) = \sigma_A(Y) = \lim_{n \rightarrow \infty} \frac{\gamma_A(n, Y)}{\gamma_A(n, F)}.$$

- (2) *If $\lim_{n \rightarrow \infty} \frac{\gamma_A(n, Y)}{\gamma_A(n, F)}$ exists then $\rho_A(Y)$ and $\sigma_A(Y)$ exist (and hence, by part (1), all three of these quantities are equal).*

Proof. For two sequences x_n, y_n of real numbers we will write $x_n \sim' y_n$ if

$$\lim_{n \rightarrow \infty} (x_n - y_n) = 0.$$

Denote $\eta := 2k - 1$. Then $\gamma_A(n, F) = \frac{\eta+1}{\eta} \eta^n$ and $\rho_A(n, F) = \frac{\eta+1}{\eta-1} \eta^n - \frac{\eta+1}{\eta-1}$. Hence $\frac{\gamma_A(n, F)}{\rho_A(n, F)} \sim' \frac{\eta-1}{\eta}$. Now,

$$\begin{aligned} \frac{\gamma_A(n, Y)}{\gamma_A(n, F)} &= \frac{\rho_A(n, Y) - \rho_A(n-1, Y)}{\gamma_A(n, F)} = \frac{\rho_A(n, Y)}{\gamma_A(n, F)} - \frac{\rho_A(n-1, Y)}{\gamma_A(n, F)} = \\ &= \frac{\rho_A(n, Y)}{\gamma_A(n, F)} - \frac{\rho_A(n-1, Y)}{\eta \gamma_A(n-1, F)} \sim' \frac{\rho_A(n, Y)}{\frac{\eta-1}{\eta} \rho_A(n, F)} - \frac{\rho_A(n-1, Y)}{\eta \frac{\eta-1}{\eta} \rho_A(n-1, F)} \\ &= \frac{\eta}{\eta-1} \frac{\rho_A(n, Y)}{\rho_A(n, F)} - \frac{1}{\eta-1} \frac{\rho_A(n-1, Y)}{\rho_A(n-1, F)}. \end{aligned}$$

Therefore if the strict asymptotic density $\delta := \rho_A(Y) = \lim_{n \rightarrow \infty} \frac{\rho_A(n, Y)}{\rho_A(n, F)}$ exists then

$$\lim_{n \rightarrow \infty} \frac{\gamma_A(n, Y)}{\gamma_A(n, F)} = \frac{\eta}{\eta-1} \delta - \frac{1}{\eta-1} \delta = \frac{\eta-1}{\eta-1} \delta = \delta.$$

By definition of $\sigma_A(Y)$ it follows that $\sigma_A(Y)$ exists and $\sigma_A(Y) = \frac{1}{2}(\delta + \delta) = \delta$. This proves part (1) of the lemma.

For (2) suppose that the limit $\lim_{n \rightarrow \infty} \frac{\gamma_A(n, Y)}{\gamma_A(n, F)}$ exists. Recall that by Stolz' Theorem (see, for example [44]), if x_n, y_n are sequences of real numbers with $y_n < y_{n+1}$ and $\lim_{n \rightarrow \infty} y_n = \infty$ and if the limit

$$\lim_{n \rightarrow \infty} \frac{x_n - x_{n-1}}{y_n - y_{n-1}}$$

exists and is finite then $\lim_{n \rightarrow \infty} \frac{x_n}{y_n}$ also exists and

$$\lim_{n \rightarrow \infty} \frac{x_n}{y_n} = \lim_{n \rightarrow \infty} \frac{x_n - x_{n-1}}{y_n - y_{n-1}}.$$

Put $y_n = \rho_A(n, F)$ and $x_n = \rho_A(n, Y)$. Then $y_n - y_{n-1} = \gamma_A(n, F)$ and $x_n - x_{n-1} = \gamma_A(n, Y)$ and Stolz' Theorem implies part (2) of the lemma. \square

Proposition 3.3. *Let $S \subseteq \mathbb{Z}^k$ be a subset such that $\delta = \rho_\infty(S)$ exists and such that S is G_k -invariant. Then for every bounded nice open set $\Omega \subseteq \mathbb{R}^k$ we have*

$$\lim_{r \rightarrow \infty} \mu_{r,S}(\Omega) = \delta \lambda(\Omega).$$

Proof. The proof is the same as for Proposition 2.3. The only difference is that instead of ergodicity of the $SL(k, \mathbb{Z})$ -action on \mathbb{R}^k we use ergodicity of the G_k -action on \mathbb{R}^k with respect to the Lebesgue measure (see [49] for the proof of this ergodicity). \square

Let $p_n(x)$ be defined exactly as in Notation 2.6.

Theorem 3.4. *Let $\Omega \subseteq \mathbb{R}^k$ be a nice bounded open set. Let $S \subseteq \mathbb{Z}^k$ be a G_k -invariant subset such that $\delta := \rho_\infty(S)$ exists.*

Then

$$\lim_{n \rightarrow \infty} \sum_{x \in S \cap \sqrt{n}\Omega} p_n(x/\sqrt{n}) = \delta \mathfrak{N}(\Omega).$$

Proof. The proof is exactly the same as that of Theorem 2.8, with the only change that instead of Proposition 2.3 we use Proposition 3.3. \square

Convention 3.5. From now and until the end of this section we assume that $k = 2$.

Proposition 3.6. *We have*

$$\rho_\infty(U_1^{ev}) = \frac{1}{3} \rho_\infty(U_1) = \frac{1}{3\zeta(2)}.$$

Proof. Let $r, s \geq 1$ be real numbers. For $X, Y \in \{A, O, E\}$ we denote by $XY(r, s)$ the number of all $z = (z_1, z_2) \in U_1$ such that $0 \leq z_1 < r, 0 \leq z_2 < s$ and such that the parity of z_1 is X and the parity of z_2 is Y . Here A stands for “any”, E stands for “even” and O stands for “odd”.

Let $n \gg 1, m \gg 1$ be integers. We will also use $='$ to signify the equality up to an additive error term that is $o(nm)$. Note that $EE(n, m) = 0$. Then we have

$$\begin{aligned} EO(n, m) &= AO(n/2, m) = AA(n/2, m) - AE(n/2, m) = \\ &AA(n/2, m) - OE(n/2, m) = ' AA(n/2, m) - EO(n/2, m) = ' \\ &\frac{1}{2}AA(n, m) - \frac{1}{2}EO(n, m). \end{aligned}$$

Therefore

$$\frac{3}{2}EO(n, m) = ' \frac{1}{2}AA(n, m) \quad \Rightarrow \quad EO(n, m) = ' OE(n, m) = ' \frac{1}{3}AA(n, m).$$

Hence $EO(n, m) + OE(n, m) = ' \frac{2}{3}AA(n, m)$ which implies

$$OO(n, m) = OO(n, m) + EE(n, m) = ' \frac{1}{3}AA(n, m).$$

Since $\rho_\infty(U_1) = \frac{1}{\zeta(2)}$, we have

$$\lim_{n \rightarrow \infty} \frac{AA(n, n)}{n^2} = \frac{1}{\zeta(2)}$$

Therefore

$$\lim_{n \rightarrow \infty} \frac{EE(n, n) + OO(n, n)}{n^2} = \frac{1}{3\zeta(2)},$$

which implies $\rho_\infty(U_1^{ev}) = \frac{1}{3\zeta(2)}$, as required. □

We can now compute the limits for the spherical densities of the set of visible points for even and odd n tending to infinity for the case $k = 2$.

Theorem 3.7. *Let $k = 2$. We have*

$$\lim_{m \rightarrow \infty} \frac{\gamma_A(2m, V_1)}{\gamma_A(2m, F)} = \frac{2}{3\zeta(2)} = \frac{4}{\pi^2}$$

and

$$\lim_{m \rightarrow \infty} \frac{\gamma_A(2m - 1, V_1)}{\gamma_A(2m - 1, F)} = \frac{8}{\pi^2}.$$

Moreover, $\rho_A(V_1)$ does not exist.

Proof. The proof is essentially the same as that of Theorem A. We present the details for completeness.

For $c > 0$ denote $\Omega_c := \{x \in \mathbb{R}^2 : \|x\| < c\}$. Then $\lim_{c \rightarrow \infty} \mathfrak{N}(\Omega_c) = 1$. Let $\epsilon > 0$ be arbitrary. Choose $c > 0$ such that

$$|\mathfrak{N}(\Omega_c) - 1| \leq \epsilon/3$$

and such that

$$\lim_{n \rightarrow \infty} \sum \{p_n(x) : x \in \mathbb{Z}^2/\sqrt{n} \text{ and } \|x\| \geq c\} \leq \epsilon/6.$$

By Theorem 3.4, Proposition 3.6 and the above formula there is some $n_0 \geq 1$ such that for all $n \geq n_0$ we have

$$\left| \sum_{x \in \mathbb{Z}^2/\sqrt{n}} p_n(x/\sqrt{n}) - \frac{1}{3\zeta(2)} \mathfrak{N}(\Omega_c) \right| \leq \epsilon/3$$

and

$$\sum \{p_n(x) : x \in \mathbb{Z}^k/\sqrt{n} \text{ and } \|x\| \geq c\} \leq \epsilon/3.$$

For an even $n \geq 2$ let

$$Q(n) := \frac{\gamma_A(n-1, \{w \in F : \bar{w} \in U_1^{ev}\})}{2\gamma_A(n-1, F)} + \frac{\gamma_A(n, \{w \in F : \bar{w} \in U_1^{ev}\})}{2\gamma_A(n, F)} = \frac{\gamma_A(n, \{w \in F : \bar{w} \in U_1^{ev}\})}{2\gamma_A(n, F)}.$$

In the above equality we use the fact that n is even and all the points of U_1^{ev} are even.

For an even $n \geq n_0$ we have

$$\begin{aligned}
 Q(n) = & \frac{\#\{w \in F : \bar{w} \in U_1^{ev}, |w|_A = n - 1 \text{ and } \|\bar{w}\| < c\sqrt{n}\}}{2\gamma_A(n - 1, F)} + \\
 & \frac{\#\{w \in F : \bar{w} \in U_1^{ev}, |w|_A = n \text{ and } \|\bar{w}\| < c\sqrt{n}\}}{2\gamma_A(n, F)} + \\
 & \frac{\#\{w \in F : \bar{w} \in U_1^{ev}, |w|_A = n - 1 \text{ and } \|\bar{w}\| \geq c\sqrt{n}\}}{2\gamma_A(n - 1, F)} + \\
 & \frac{\#\{w \in F : \bar{w} \in U_1^{ev}, |w|_A = n \text{ and } \|\bar{w}\| \geq c\sqrt{n}\}}{2\gamma_A(n, F)} = \\
 & \sum_{x \in U_1^{ev} \cap \sqrt{n}\Omega_c} p_n(x/\sqrt{n}) + \sum_{x \in U_1^{ev} \cap (\mathbb{R}^k - \sqrt{n}\Omega_c)} p_n(x/\sqrt{n}).
 \end{aligned}$$

In the last line of the above equation, the first sum differs from $\frac{1}{3\zeta(2)}\mathfrak{N}(\Omega_c)$ by at most $\epsilon/3$ since $n \geq n_0$, and the second sum is $\leq \epsilon/3$ by the choice of c and n_0 . Therefore, again by the choice of c , we have $|Q(n) - \frac{1}{3\zeta(2)}| \leq \epsilon$. Since $\epsilon > 0$ was arbitrary, this implies that $\lim_{m \rightarrow \infty} Q(2m) = \frac{1}{3\zeta(2)}$. Therefore

$$\lim_{m \rightarrow \infty} \frac{\gamma_A(2m, V_1)}{\gamma_A(2m, F)} = 2 \lim_{m \rightarrow \infty} Q(2m) = \frac{2}{3\zeta(2)} = \frac{4}{\pi^2}.$$

Together with the conclusion of Theorem B this implies that

$$\lim_{m \rightarrow \infty} \frac{\gamma_A(2m - 1, V_1)}{\gamma_A(2m - 1, F)} = \frac{8}{\pi^2},$$

as claimed.

Since $\frac{4}{\pi^2} \neq \frac{8}{\pi^2}$, it follows that the limit $\lim_{n \rightarrow \infty} \frac{\gamma_A(n, V_1)}{\gamma_A(n, F)}$ does not exist. Hence by Lemma 3.2 the strict asymptotic density $\rho_A(V_1)$ does not exist. \square

4. Test elements in the free group of rank two

A subgroup H of a group G is called a *retract* of G if there exists a *retraction* from G to H , that is, an endomorphism $\phi : G \rightarrow G$ such that $H = \phi(G)$ and that $\phi|_H = Id_H$. A retract $H \leq G$ is *proper* if $H \neq G$ and $H \neq 1$.

The following result is due to Turner [47]:

Proposition 4.1. *Let F be a free group of finite rank $k \geq 2$ and let $w \in F$. Then w is a test element in F if and only if w does not belong to a proper retract of F .*

If F is a free group of rank two, then a proper retract of F is necessarily cyclic. The following explicit characterization of retracts in this case is actually Exercise 25 on page 103 of Magnus, Karrass, Solitar [29]. We present a proof here for completeness.

Lemma 4.2. *Let F be a free group of rank two and let $H = \langle h \rangle \leq F$ be an infinite cyclic subgroup of F .*

Then H is a retract of F if and only if there is a free basis $\{a, b\}$ of F such that h can be represented as $h = ac$, where c belongs to the normal closure of b in F . In particular, if H is a retract of F then H is a maximal cyclic subgroup of F .

Proof. Suppose first that H is a retract of F and that $\phi : F \rightarrow F$ is a retraction with $\phi(F) = H$. Choose a free basis x, y of F . Since $H = \langle h \rangle = \langle \phi(x), \phi(y) \rangle \leq F$ is infinite cyclic, the pair $(\phi(x), \phi(y))$ is Nielsen equivalent to the pair $(h, 1)$. Applying the same sequence of Nielsen transformations to (x, y) we obtain a free basis (a, b) of F such that $\phi(a) = h$ and $\phi(b) = 1$. Then the kernel of ϕ is the normal closure of b in F .

Since ϕ is a retraction onto H , we have $\phi(h) = h = \phi(a)$. Hence $a^{-1}h \in \ker(\phi)$ and therefore $h = ac$, where c belongs to the kernel of ϕ , that is, to the normal closure of b , as required.

Suppose now that for some free basis a, b of F we have $h = ac$ where c belongs to the normal closure of b in F . Consider the endomorphism $\psi : F \rightarrow F$ defined by $\psi(a) = h$, $\psi(b) = 1$. Then, clearly, $\psi(h) = h$ and ψ is a retraction from F to H . \square

We can now obtain an explicit characterization of test elements in the free group $F = F(a, b)$ of rank two with free basis $A = \{a, b\}$. We identify the abelianization of F with \mathbb{Z}^2 so that $\bar{a} = (1, 0)$ and $\bar{b} = (0, 1)$. If $x \in A$ and $w \in F$, then w_x denotes the exponent sum on x in w when w is written as a freely reduced word in A and \bar{w} denotes the image of w in the abelianization of F . Thus $\bar{w} = (w_a, w_b)$.

Note that if $w \in F = F(a, b)$ then \bar{w} is an n -th power in \mathbb{Z}^2 for some $n \geq 2$ if and only if $\gcd(w_a, w_b) > 1$. By convention we set $\gcd(0, 0) = \infty$.

Proposition 4.3. *Let F be a free group of rank two. Let $w \in F$ be a nontrivial element that is not a proper power in F .*

Then w is a test element in F if and only if there exists an integer $n \geq 2$ such that \bar{w} is an n -th power in \mathbb{Z}^2 . That is, w is not a test element if and only if w is visible in F .

Proof. Suppose first that w is a test element but that \bar{w} cannot be represented as an n -th power in \mathbb{Z}^2 for $n \geq 2$. Then $\gcd(w_a, w_b) = 1$. Hence there exist integers p and q such that $pw_a + qw_b = 1$. Consider an endomorphism $\phi : F \rightarrow F$ defined by $\phi(a) = w^p$ and $\phi(b) = w^q$. Then $\phi(w) = w$ and ϕ is not an automorphism of F since $\phi(F)$ is cyclic. Hence, by definition, w is not a test element in F , yielding a contradiction.

Suppose now that \bar{w} is an n -th power in \mathbb{Z}^2 for some $n \geq 2$ but that w is not a test element. Then by Proposition 4.1 w belongs to an infinite cyclic proper retract H of F . Since by assumption w is not a proper power in F , it follows that w generates H . Lemma 4.2 implies that for some free basis (a_1, b_1) of F we have $w = a_1c$ where c belongs to the normal closure of b_1 in F . Hence when w is expressed as a word in a_1, b_1 , the exponent sum on a_1 in w is equal to 1, which contradicts the assumption that \bar{w} is an n -th power in the abelianization of F . \square

It is well-known and easy to prove that the set of proper powers in a free group is negligible [2]:

Proposition 4.4. *Let $F = F(A)$ be a free group of finite rank $k \geq 2$ with free basis $A = \{a_1, \dots, a_k\}$. Let P be the set of all nontrivial elements of F that are proper powers.*

Then

$$\lim_{n \rightarrow \infty} \frac{\gamma_A(n, P)}{\gamma_A(n, F)} = \lim_{n \rightarrow \infty} \frac{\rho_A(n, P)}{\rho_A(n, F)} = 0,$$

and the convergence in both limits is exponentially fast.

Proof of Theorem D. Since $\zeta(2) = \frac{\pi^2}{6}$, Theorem D now follows directly from Theorem B, Proposition 4.3 and Proposition 4.4. \square

5. Open Problems

As before, let $F = F(A)$ be a free group of rank $k \geq 2$ with free basis $A = \{a_1, \dots, a_k\}$ and let $\alpha : F \rightarrow \mathbb{Z}^k$ be the abelianization homomorphism.

Problem 5.1. Let $k \geq 3$. Is the set of test elements negligible in F ? In view of Proposition 4.1, this is equivalent to asking if the union of all proper retracts of F is generic in F .

The proof of Proposition 4.3 shows that a visible element in F is never a test element and therefore by Theorem B the asymptotic density of the set of test elements in F is at most $1 - \frac{1}{\zeta(k)}$. For $k \geq 2$ we have $0 < 1 - \frac{1}{\zeta(k)} < 1$ and $\lim_{k \rightarrow \infty} 1 - \frac{1}{\zeta(k)} = 0$. Thus the asymptotic density of the set of test elements of F tends to zero as the rank k of F tends to infinity.

Note that every free factor of F is a retract, but the converse is not true. As mentioned in the Introduction, the union of all proper free factors is negligible in F , whereas the union of all proper retracts is not since every visible element of F belongs to a proper retract.

Problem 5.2. For $k \geq 2$ find a subset $S \subseteq \mathbb{Z}^k$ such that $\bar{\rho}_{\|\cdot\|_\infty}(S) \neq \bar{\sigma}_A(\alpha^{-1}(S))$.

Note that if such a set S exists then it is *not* invariant under the action of $SL(k, \mathbb{Z})$ in view of Theorem A,

Problem 5.3. For $w \in F$ define $T(w) = 0$ if $\alpha(w) = 0$ and define $T(w)$ to be the greatest common divisor of the coordinates of $\alpha(w)$ if $\alpha(w) \neq 0$. Let T'_n be the expected value of T over the sphere of radius n in F with respect to the uniform distribution on that sphere and let $T_n = (T'_{n-1} + T'_n)/2$. What can one say about the behavior of T_n as $n \rightarrow \infty$?

Using the results of this paper we can show that $\lim_{n \rightarrow \infty} T_n = \infty$ for the case $k = 2$. It also seems plausible that for each $k \geq 3$ we have $\limsup_{n \rightarrow \infty} T_n < \infty$ and heuristic considerations allow us to conjecture that in fact $\lim_{n \rightarrow \infty} T_n = \frac{\zeta(k-1)}{\zeta(k)}$. A similar question for \mathbb{Z}^2 has been studied in detail by Diaconis and Erdős [12], who computed the precise asymptotics, as $n \rightarrow \infty$, of the expected value for the greatest common divisor of the coordinates, computed for the uniform distribution on the $n \times n$ -square in \mathbb{Z}^2 .

As we have seen, the set of proper powers is negligible in free groups of rank $k \geq 2$ but has positive asymptotic density in free abelian groups of finite rank. This raises the corresponding question about free groups in other varieties. It is possible to show using Lemma A.1 of [5] that if G is a finitely generated nilpotent group and $t \geq 2$ then the set of t -th powers has positive asymptotic density in G . The referee asked if it is true that a finitely generated group is amenable if and only if the set of all proper powers has positive asymptotic density. The answer to that question is negative. In a cyclic group of a finite odd order every element is a square and hence the same

is true for any group where all elements have finite odd orders. For example, let $G = B(2, 665)$ be the free Burnside group of rank two and of exponent 665. Then every element of G is a square, but G is non-amenable by a theorem of Adian [1]. It also seems plausible that there exist finitely generated amenable groups (e.g. perhaps free metabelian groups) where the set of all proper powers has asymptotic density zero.

Problem 5.4. Let G be a finitely generated nonabelian free solvable group. What can be said about the asymptotic density of the set of all proper powers in G ?

Acknowledgements

The authors are grateful to Laurent Bartholdi, John D'Angelo, Iwan Duursma, Kevin Ford, Steve Lalley, Alexander Ol'shanskii, Yuval Peres, Yannis Petridis, Alexandru Zaharescu and Andrzej Zuk for very helpful conversations. The authors thank the referee for the careful reading of the paper and for many helpful suggestions. The second author acknowledges the support of the Princeton University, the Hebrew University at Jerusalem and of the Lady Davis Fellowship Trust.

The first author was supported by NSF grants DMS-0404991 and DMS-0603921. The third author was supported by the NSF grant DMS-0404991 and the fourth author was supported by the NSF grant DMS-0405105.

References

- [1] S. I. Adian, *Random walks on free periodic groups*. (Russian) *Izv. Akad. Nauk SSSR Ser. Mat.* **46** (1982), no. 6, 1139–1149.
- [2] G. Arzhantseva and A. Ol'shanskii, *Generality of the class of groups in which subgroups with a lesser number of generators are free*, (Russian) *Mat. Zametki* **59** (1996), no. 4, 489–496; translation in: *Math. Notes* **59** (1996), no. 3–4, 350–355.
- [3] G. Arzhantseva, *On groups in which subgroups with a fixed number of generators are free*, (Russian) *Fundam. Prikl. Mat.* **3** (1997), no. 3, 675–683.
- [4] ———, *Generic properties of finitely presented groups and Howson's theorem*, *Comm. Algebra* **26** (1998), no. 11, 3783–3792.
- [5] I. Belegradek and V. Kapovitch, *Classification of negatively pinched manifolds with amenable fundamental groups*, *Acta Math.*, to appear; <http://www.arxiv.org/math.DG/0402268>
- [6] A. Borovik, A. G. Myasnikov and V. Shpilrain, *Measuring sets in infinite groups*, *Computational and Statistical Group Theory* (R. Gilman et al, Editors), *Contemp. Math.* **298** Amer. Math. Soc. (2002), 21–42.
- [7] J. Burillo, E. Ventura, *Counting primitive elements in free groups*, *Geom. Dedicata* **93** (2002), 143–162.
- [8] C. Champetier, *Propriétés statistiques des groupes de présentation finie*, *Adv. Math.* **116** (1995), no. 2, 197–262.
- [9] ———, *The space of finitely generated groups*, *Topology* **39** (2000), no. 4, 657–680.
- [10] P.-A. Cherix and G. Schaeffer, *An asymptotic Freiheitssatz for finitely generated groups*, *Enseign. Math. (2)* **44** (1998), no. 1–2, 9–22.
- [11] J. Christopher, *The asymptotic density of some k -dimensional sets*. *Amer. Math. Monthly* **63** (1956), 399–401.
- [12] P. Diaconis and P. Erdős, *On the distribution of the greatest common divisor*. A festschrift for Herman Rubin, 56–61, *IMS Lecture Notes Monogr. Ser.*, 45, Inst. Math. Statist., Beachwood, OH, 2004.
- [13] N. Dunfield and D. Thurston, *A random tunnel number one 3-manifold does not fiber over the circle*, *Geom. Topol.* **10** (2006) 2431–2499.

- [14] W. Feller, *An introduction to probability theory and its applications. Vol. II.* Second edition. John Wiley & Sons, Inc., New York-London-Sydney, 1971.
- [15] B. Fine, G. Rosenberger, D. Spellman and M. Stille, *Test words, generic elements and almost primitivity.* Pacific J. Math. **190** (1999), no. 2, 277–297.
- [16] R. Z. Goldstein, *The density of small words in a free group is 0,* Group theory, statistics, and cryptography, 47–50, Contemp. Math. **360** Amer. Math. Soc., Providence, RI, 2004.
- [17] M. Gromov, *Hyperbolic Groups,* in "Essays in Group Theory (G.M.Gersten, editor)", MSRI publ. **8** (1987), 75–263
- [18] ———, *Asymptotic invariants of infinite groups.* Geometric group theory, Vol. 2 (Sussex, 1991), 1–295, London Math. Soc. Lecture Note Ser. **182** Cambridge Univ. Press, Cambridge, 1993.
- [19] ———, *Random walks in random groups,* Geom. Funct. Anal. **13** (2003), no. 1, 73–146.
- [20] G. Hardy, E. Wright, *The Theory of Numbers,* 4th ed., Oxford University Press, 1965.
- [21] S. Ivanov, *On certain elements of free groups.* J. Algebra **204** (1998), no. 2, 394–405.
- [22] I. Kapovich, A. Myasnikov, P. Schupp and V. Shpilrain, *Generic-case complexity, decision problems in group theory and random walks,* J. Algebra **264** (2003), no. 2, 665–694.
- [23] ———, *Average-case complexity for the word and membership problems in group theory,* Advances in Mathematics **190** (2005), no. 2, 343–359
- [24] I. Kapovich, P. Schupp and V. Shpilrain, *Generic properties of Whitehead's Algorithm and isomorphism rigidity of random one-relator groups,* Pacific J. Math. **223** (2006), no. 1, 113–140.
- [25] I. Kapovich and P. Schupp, *Genericity, the Arzhantseva-Ol'shanskii method and the isomorphism problem for one-relator groups,* Math. Ann. **331** (2005), no. 1, 1–19.
- [26] ———, *Delzant's T-invariant, Kolmogorov complexity and one-relator groups,* Comment. Math. Helv. **80** (2005), no. 4, 911–933.
- [27] D. Lee, *On certain C-test words for free groups.* J. Algebra **247** (2002), no. 2, 509–540.
- [28] R. Lyndon and P. Schupp, *Combinatorial Group Theory,* Springer-Verlag, 1977. Reprinted in the "Classics in mathematics" series, 2000.
- [29] W. Magnus, A. Karrass and D. Solitar, *Combinatorial Group Theory. Presentations of groups in terms of generators and relations.* Second revised edition. Dover Publications, Inc., New York, 1976.
- [30] F. Mertens, *Ueber einige asymptotische Gesetze der Zahlentheorie,* J. Reine Angew. Math. **77** (1874), 289–338.
- [31] A. Mikhalev, V. Shpilrain and J.-T. Yu, *Combinatorial methods. Free groups, polynomials, and free algebras.* CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, 19. Springer-Verlag, New York, 2004.
- [32] A. Mikhalev, U. Umirbaev and J.-T. Yu, *Generic, almost primitive and test elements of free Lie algebras.* Proc. Amer. Math. Soc. **130** (2002), no. 5, 1303–1310.
- [33] Y. Ollivier, *Critical densities for random quotients of hyperbolic groups.* C. R. Math. Acad. Sci. Paris **336** (2003), no. 5, 391–394.
- [34] A. Yu. Ol'shanskii, *Almost every group is hyperbolic,* Internat. J. Algebra Comput. **2** (1992), no. 1, 1–17.
- [35] J. O'Neill and E. Turner, *Test elements and the retract theorem in hyperbolic groups,* New York J. Math. **6** (2000), 107–117.
- [36] Y. Petridis and M. Risager, *Equidistribution of geodesics on homology classes and analogues for free groups,* Forum Mathematicum, to appear.
- [37] H. Rumsey, *Sets of visible points.* Duke Math. J. **33** (1966), no. 2, 263–274.
- [38] I. Rivin, *Growth in free groups (and other stories),* preprint, 1999,
<http://www.arxiv.org/abs/math.CO/9911076>
- [39] ———, *Simple curves on surfaces,* Geom. Dedicata **87** (2001), no. 1-3, 345–360.
- [40] V. Roman'kov, *On test elements of free solvable groups of rank 2.* Algebra Logika **40** (2001), no. 2, 192–201.
- [41] R. Sharp, *Local limit theorems for free groups.* Math. Ann. **321** (2001), no. 4, 889–904.
- [42] V. Shpilrain, *Recognizing automorphisms of the free groups.* Arch. Math. (Basel) **62** (1994), no. 5, 385–392.

- [43] ———, *Test elements for endomorphisms of free groups and algebras*. Israel J. Math. **92** (1995), no. 1-3, 307–316.
- [44] R. Spigler and M. Vianello, *Cesàro's theorems for complex sequences*, J. Math. Anal. Appl. **180** (1993), no. 2, 317–324.
- [45] J. R. Stallings, *Whitehead graphs on handlebodies*. Geometric group theory down under (Canberra, 1996), 317–330, de Gruyter, Berlin, 1999.
- [46] E. Timoshenko, *Test elements and test rank of a free metabelian group*. Sibirsk. Mat. Zh. **41** (2000), no. 6, 1451–1456.
- [47] E. Turner, *Test words for automorphisms of free groups*. Bull. London Math. Soc. **28** (1996), no. 3, 255–263.
- [48] W. Woess, *Cogrowth of groups and simple random walks*, Arch. Math. (Basel) **41** (1983), no. 4, 363–370.
- [49] R. J. Zimmer. *Ergodic theory and semisimple groups*, Monographs in Mathematics, **81** Birkhäuser Verlag, Basel, 1984.
- [50] A. Zuk, *On property (T) for discrete groups*. Rigidity in dynamics and geometry (Cambridge, 2000), 473–482, Springer, Berlin, 2002.

DEPARTMENT OF MATHEMATICS,, UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN,, 1409 WEST GREEN STREET, URBANA, IL 61801, USA

E-mail address: kapovich@math.uiuc.edu

URL: <http://www.math.uiuc.edu/~kapovich/>

DEPARTMENT OF MATHEMATICS,, TEMPLE UNIVERSITY,, PHILADELPHIA, PA 19122, USA

E-mail address: rivin@euclid.math.temple.edu

URL: <http://www.math.temple.edu/~rivin/>

DEPARTMENT OF MATHEMATICS,, UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN,, 1409 WEST GREEN STREET, URBANA, IL 61801, USA

URL: <http://www.math.uiuc.edu/People/schupp.html>

E-mail address: schupp@math.uiuc.edu

DEPARTMENT OF MATHEMATICS,, THE CITY COLLEGE OF THE CITY UNIVERSITY OF NEW YORK,, NEW YORK, NY 10031, USA

E-mail address: shpil@groups.sci.ccny.cuny.edu

URL: <http://www.sci.ccny.cuny.edu/~shpil>