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ATKIN-SERRE TYPE CONJECTURES FOR AUTOMORPHIC REPRESENTATIONS ON $GL(2)$

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Abstract. Let $H(z)$ be a newform of weight $k \geq 4$ without complex multiplication on $\Gamma_0(N)$ with normalized L-function $L(H,s) = \prod_p (1-\alpha_p p^{-s})^{-1}(1-\beta_p p^{-s})^{-1}$. A conjecture of Atkin and Serre states that for sufficiently large primes p,

(1) $$|\alpha_p + \beta_p| \gg p^{-1-\epsilon}$$

for all $\epsilon > 0$. Let π a genuine cuspidal automorphic representation on $GL_2(\mathbb{A}_F)$, where F is a totally real number field. Assuming GRH for the symmetric power L-functions associated to π, we prove that

$$|\alpha_v + \beta_v| \geq q_v^{-\delta}$$

for all but $O(x^{1-\delta}/\log x)$ places v with $q_v \leq x$ provided $\delta \leq 1/8$. This implies a strong form of (1) for almost all primes p.

1. Introduction and Statement of Results

Let $H(z)$ be a normalized cuspidal newform of even weight k on $\Gamma_0(N)$ with trivial Nebentypus, and denote its Fourier expansion by

$$H(z) = \sum_{n=1}^{\infty} a(n) q^n, \quad q = e^{2\pi i z}.$$

It is known that there is a totally real number field K with ring of integers $\mathcal{O}_K$ so that $a(n) \in \mathcal{O}_K$ for all n. The distribution of the Fourier coefficients of such forms is an important and classical problem. For example, see many applications of such results in [16].

As a consequence of Deligne's proof of the Weil conjectures, it is known that if p is prime, then

$$|a(p)| \leq 2p^{\frac{k-1}{2}}.$$

In light of this result it is natural to ask how the numbers $|a(p)|$ are distributed in $[-2p^{\frac{k-1}{2}}, 2p^{\frac{k-1}{2}}]$. Define $\theta_p \in [0,\pi]$ by $a(p) = 2p^{\frac{k-1}{2}} \cos(\theta_p)$.

Conjecture. *Suppose that $H(z)$ does not have complex multiplication and let $f : [0,\pi] \to \mathbb{C}$ be a continuous function. Then,*

$$\lim_{x\to\infty} \frac{1}{\pi(x)} \sum_{p \leq x} f(\theta_p) = \int_0^{\pi} f(\theta)\, d\mu,$$

where $d\mu$ is the Sato-Tate measure $\frac{2}{\pi}\sin^2\theta\, d\theta$.

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Sato and Tate conjectured this result for elliptic curves without complex multiplication. Later, others formulated this conjecture in more general contexts (see Serre's article [19], Section 5.2 as well as Shahidi [22], section 3).

It is well-known that this conjecture follows from the analytic properties of the symmetric power L-functions associated to $H(z)$ predicted by Langlands functoriality (see for example [22], ([7], pg. 493)).

Remark. Richard Taylor [23] has recently proven the Sato-Tate conjecture for a wide class of elliptic curves over totally real number fields.

Another interesting question regarding the Fourier coefficients $a(p)$ is how small they can be. D. H. Lehmer conjectured that if

$$\Delta(z) = q\prod_{n=1}^{\infty}(1-q^n)^{24} = \sum_{n=1}^{\infty}\tau(n)q^n$$

is the unique normalized cusp form of weight 12 for $SL_2(\mathbb{Z})$, then $\tau(n) \neq 0$ for all n. It is well-known that $\tau(n) = 0$ only if $\tau(p) = 0$ for some prime $p|n$.

Despite the fact that Lehmer's conjecture is still open, much more is believed to be true. For a generic newform $H(z)$ of weight $k \geq 4$ without complex multiplication, the well-known conjectures of Lang and Trotter [12] predict that if c is fixed then $\#\{p : a(p) = c\}$ is finite. Moreover, in ([20], pg. 244) Atkin and Serre conjecture the following.

Conjecture (Atkin-Serre)**.** *If $H(z)$ has weight at least 4 and does not have complex multiplication, then for sufficiently large primes p we have*

$$|a(p)| \gg p^{\frac{k-3}{2}-\epsilon}$$

for all $\epsilon > 0$.

If $H(z)$ has complex multiplication by an imaginary quadratic field K, it is known ([20], Exercise 6.13, pg. 255) that if $\epsilon > 0$ and p is a sufficiently large prime that splits in K, then $|a(p)| \gg p^{\frac{k-3}{2}-\epsilon}$.

One approach to the conjectures of Atkin-Serre and Lang-Trotter is via the ℓ-adic Galois representations of Deligne and Serre. In this direction, Serre proved [21], assuming the Generalized Riemann Hypothesis (GRH), that

$$\#\{p \leq x \text{ prime} : a(p) = 0\} = O(x^{3/4})$$

provided the weight of $H(z)$ is at least 2. Moreover, for a fixed c, Serre's result states that

$$\#\{p \leq x \text{ prime} : a(p) = c\} = O\left(\frac{x^{7/8}}{\sqrt{\log x}}\right).$$

This latter result was improved to

$$\#\{p \leq x \text{ prime} : a(p) = c\} = O\left(\frac{x^{4/5}}{\log^{1/5} x}\right)$$

by M. Ram Murty, V. Kumar Murty, and N. Saradha in [14]. Also, they prove on GRH that for any $\epsilon > 0$

$$\#\{p \leq x \text{ prime} : |a(p)| < p^{1/4-\epsilon}\} = o\left(\frac{x}{\log x}\right).$$

V. Kumar Murty proved in [15] that if $H(z)$ corresponds to an elliptic curve E then

$$\#\{p \leq x \text{ prime} : a(p) = c\} = O(x^{3/4}\sqrt{\log x}),$$

assuming the GRH holds for the symmetric power L-functions corresponding to E.

In light of Taylor's work on the Sato-Tate conjecture and the symmetric power L-functions associated to elliptic curves, we wish to explore some of the implications of the conjectured properties of the symmetric power L-functions for higher weight modular forms. We consider implications for the Atkin-Serre conjecture.

Theorem 1.1. *Assume that $H(z)$ is a cuspidal newform of even weight $k \geq 4$ on $\Gamma_0(N)$ with trivial character and without complex multiplication. Assume also that the symmetric power L-functions of H are automorphic and satisfy the Riemann hypothesis. If $0 \leq \alpha \leq 1/8$, then*

$$\#\{p \leq x \textit{ prime} : |a(p)| \leq p^{\left(\frac{k-1}{2}-\alpha\right)}\} \asymp \frac{x^{1-\alpha}}{\log x}.$$

Remark. An improved bound on the conductors of the symmetric power L-functions would yield the same result for $\alpha < 1/4$, and for $\alpha = 1/4$ gives

$$\#\{p \leq x \text{ prime} : |a(p)| \leq p^{\frac{k-1}{2}-\frac{1}{4}} \log^{1/2} p\} \asymp \frac{x^{3/4}}{\sqrt{\log x}}.$$

In particular, if c is fixed then

$$\#\{p \leq x \text{ prime} : a(p) = c\} = O\left(\frac{x^{3/4}}{\sqrt{\log x}}\right).$$

Remark. One may view Theorem 1.1 as a strong form of the predicted equidistribution. In particular, if $F : [0, \pi] \to \mathbb{C}$ is a continuous function with $F(0) = F(\pi)$, one may ask for precise bounds on

$$\frac{1}{\pi(x)} \sum_{p \leq x} F(\theta_p) - \int_0^\pi F(\theta)\, d\mu.$$

Similar arguments to those used in the proof of Theorem 1.1 can address this question.

At present, it is not possible to extend Theorem 1.1 to say that the set of numbers n for which $a(n)$ is "too small" has density zero. Indeed, if p is a prime for which $a(p) = 0$ and $\gcd(m, p) = 1$, then by the multiplicativity of the coefficients, $a(mp) = 0$. In particular, the density of n for which $a(n) = 0$ is positive in this case. However, it is possible to demonstrate that, subject to $a(n) \neq 0$, $a(n)$ is not small very often. In this direction, Murty, Murty and Saradha proved [14] under GRH that if $H(z)$ has (rational) integral Fourier coefficients, then there is a constant $c > 0$ so that

$$\{n : a(n) = 0 \text{ or } |a(n)| > n^c\}$$

has density 1. Under the same assumptions of Theorem 1.1, this may be improved.

Corollary 1.2. *Suppose K is a number field with $a(n) \in O_K$ for all n, and let $d = [K : \mathbb{Q}]$. If the assumptions of Theorem 1.1 are satisfied, $0 \leq \alpha \leq 1/8$ and $0 < \lambda < 1$, then we have*

$$\#\{n \leq x : a(n) \neq 0 \textit{ and } |N_{K/\mathbb{Q}}(a(n))| \leq n^{d(1-\lambda)\left(\frac{k-1}{2}-\alpha\right)}\} = O(x^{1-\alpha\lambda} \log^C x).$$

Here, C is some constant depending only on α and F.

Theorem 1.1 will follow as a consequence of the following general result about cuspidal automorphic representations on $GL_2(\mathbb{A}_F)$. To state this result, we first fix some notation. Let $F/\mathbb{Q}$ be a totally real number field. Let π be an irreducible cuspidal automorphic representation of $GL_2(\mathbb{A}_F)$ (see [1] for definitions and basic facts). We assume that π has trivial central character and that π is genuine (in the sense of Shahidi [22], see section 2 for a precise definition). Let

$$L(\pi, s) = \prod_v (1 - \alpha_v q_v^{-s})^{-1}(1 - \beta_v q_v^{-s})^{-1}$$

be the L-function associated to π, where the product is over all (finite) places v of F and q_v is the cardinality of the residue field. Then, we have the following result.

Theorem 1.3. *Assume the notation above. If the symmetric power L-functions of π are automorphic and satisfy the Riemann hypothesis, then for $0 \leq \alpha \leq 1/8$ we have*

$$\#\{v : q_v \leq x, |\alpha_v + \beta_v| \leq q_v^{-\alpha}\} \asymp \frac{x^{1-\alpha}}{\log x}.$$

2. Symmetric Power L-Functions

Let $F/\mathbb{Q}$ be a totally real number field and let π be a cuspidal automorphic representation of $GL_2(\mathbb{A}_F)$ with trivial central character. Then, π has a factorization

$$\pi = \otimes_v \pi_v$$

over all places of F. We must require that the representation π is genuine in the sense of Shahidi ([22], pg. 162). This means that π is not a twist by an idèle class character of a monomial representation or a representation of Galois type. A monomial representation ρ is one for which $\rho \otimes \eta \cong \rho$ for some nontrivial character η of $F^\times/\mathbb{A}_F^\times$. A representation ρ is of Galois type if for every archimedean place v of F, ρ_v factors through the Galois group of $\overline{F}_v/F_v$.

Let S denote the set of places v for which π_v is ramified. The representation π has an L-function

$$L(\pi, s) = \prod_{v \notin S} (1 - \alpha_v q_v^{-s})^{-1}(1 - \beta_v q_v^{-s})^{-1} \prod_{v \in S} L_v(\pi, s),$$

where the product is over the finite places, and q_v is the cardinality of the residue field of the local ring $\mathcal{O}_v \subseteq F_v$. The $L_v(\pi, s)$ are appropriate local factors at the ramified places. It is known that $L(\pi, s)$ converges absolutely for $\mathrm{Re}(s) > 1$ and hence $|\alpha_v| < q_v$. Also, $\alpha_v \beta_v = 1$ since π has trivial central character.

Langlands functoriality predicts, for all $m \geq 2$, that there is an automorphic representation denoted $\mathrm{Sym}^m \pi$ on $GL_{m+1}(\mathbb{A}_F)$ with an L-function

$$L(\mathrm{Sym}^m \pi, s) = \prod_v L_v(\mathrm{Sym}^m \pi, s).$$

For $v \notin S$ the local factor is given by

$$L_v(\mathrm{Sym}^m \pi, s) = \prod_{j=0}^{m} (1 - \alpha_v^j \beta_v^{m-j} q_v^{-s})^{-1}.$$

In particular, $L(\mathrm{Sym}^m\pi, s)$ is a degree $m+1$ L-function over F, and hence a degree $(m+1)[F:\mathbb{Q}]$ L-function over $\mathbb{Q}$.

There are other appropriate local factors $L_v(\mathrm{Sym}^m\pi, s)$ for $v \in S$. These local factors, together with the local factor at infinity, are predicted by the local Langlands correspondence. The existence of such a $\mathrm{Sym}^m\pi$ has been established unconditionally for $m \leq 4$ by work of Gelbart, Jacquet, Kim and Shahidi ([5], [10], [11], [9]). (If $m = 1$, then $\mathrm{Sym}^m\pi = \pi$ and if $m = 0$, we take $L(\mathrm{Sym}^m\pi, s) = \zeta_F(s)$, the Dedekind zeta function of F).

When π is genuine, $L(\mathrm{Sym}^m\pi, s)$ is predicted to have an analytic continuation to all of $\mathbb{C}$ and a functional equation of the usual type. If we let

$$\Lambda(\mathrm{Sym}^m\pi, s) = q_m^{s/2}\gamma(\mathrm{Sym}^m\pi, s)L(\mathrm{Sym}^m\pi, s),$$

where q_m is the conductor, $\gamma(\mathrm{Sym}^m\pi, s)$ is the prescribed gamma factor, and ϵ_m is the prescribed root number, then

$$\Lambda(\mathrm{Sym}^m\pi, s) = \epsilon_m\Lambda(\mathrm{Sym}^m\pi, 1-s).$$

The distribution of the zeroes of $L(\mathrm{Sym}^m\pi, s)$ is governed by $\gamma(\mathrm{Sym}^m\pi, s)$ and q_m.

In [13], Moreno and Shahidi work out the predictions of Langlands functoriality for the symmetric powers of a representation π_∞ of $PGL(2, \mathbb{R})$. An explicit description of the gamma factors $\gamma(\mathrm{Sym}^m\pi, s)$ follows from this. It follows that

$$\gamma(\mathrm{Sym}^m\pi, s) = \pi^{-(m+1)[F:\mathbb{Q}]s/2} \prod_{j=1}^{(m+1)[F:\mathbb{Q}]} \Gamma\left(\frac{s+\kappa_{j,m}}{2}\right),$$

where $\kappa_{j,m}$ is a complex number with $\mathrm{Re}(\kappa_{j,m}) \leq 0$ and $|\kappa_{j,m}| \leq (m+1)\max_j\{|\kappa_{j,1}|\}$.

Remark. Serre ([19], equation (32)) gives the form of the gamma factors of the symmetric power L-functions associated to $\Delta(z) = q\prod_{n=1}^{\infty}(1-q^n)^{24}$, the unique newform of weight 12 for $SL_2(\mathbb{Z})$.

Now, we will bound the conductor q_m by considering certain Rankin-Selberg convolutions.

Lemma 2.1. *Assume that for all $m \geq 1$, $L(\mathrm{Sym}^m\pi, s)$ is the L-function of a cuspidal automorphic representation of $GL_{m+1}(\mathbb{A}_F)$. Then, $q_m = O(q_1^{am^3})$ for some constant a.*

Proof. Jacquet, Piatetski-Shapiro and Shalika have shown [8] that if f and g are cuspidal automorphic representations on $GL_m(\mathbb{A}_F)$ and $GL_n(\mathbb{A}_F)$ with conductors $q(f)$ and $q(g)$ respectively, then they admit a Rankin-Selberg convolution $L(f \otimes g, s)$ where if

$$L_v(f, s) = \prod_{i=1}^{m}(1 - \alpha_{i,v}q_v^{-s})^{-1},$$

and

$$L_v(g, s) = \prod_{j=1}^{n}(1 - \beta_{j,v}q_v^{-s})^{-1},$$

then for a place v at which neither f nor g is ramified then

$$L_v(f \otimes g, s) = \prod_{i=1}^{m} \prod_{j=1}^{n} (1 - \alpha_{i,v} \beta_{j,v} q_v^{-s})^{-1}.$$

For other v, appropriate local factors exist, but they are not necessarily given by the above equation.

For v with $q_v \nmid q(f)q(g)$, one can easily see that

$$L_v(\pi \otimes \mathrm{Sym}^m \pi, s) = L_v(\mathrm{Sym}^{m+1} \pi, s) L_v(\mathrm{Sym}^{m-1} \pi, s).$$

It follows that the L-functions

$$F(s) = L(\pi \otimes \mathrm{Sym}^m \pi, s)$$

and

$$G(s) = L(\mathrm{Sym}^{m+1} \pi, s) L(\mathrm{Sym}^{m-1} \pi, s)$$

are equal up to the factors at the ramified places. Then, $G(s)/F(s)$ is a finite product of ratios of local factors at the ramified places. Let $N(T, F)$ and $N(T, G)$ denote the number of zeroes of F and G, respectively, in the set $\{\beta + i\tau : 0 \leq \beta \leq 1, -T \leq \tau \leq T\}$. Let S denote the set of ramified places and for $v \in S$ let $k(v)$ denote the number of local factors occurring in $F(s)$ that do not occur in $G(s)$. Then,

$$N(T, G) = N(T, F) + T \sum_{p \in S} \frac{k(v) \log q_v}{\pi} + O(1).$$

From Iwaniec and Kowalski Theorem 5.8 ([7], pg. 104), it follows that for an L-function $L(f, s)$ of degree d and conductor $q(f)$,

$$N(T, f) = \frac{T}{\pi} \log \frac{q(f) T^d}{(2\pi e)^d} + O(\log T).$$

In particular,

$$\lim_{T \to \infty} \exp\left(\frac{\pi}{T} N(T, f) - d \log \frac{T}{2\pi e} \right) = q(f).$$

It follows from this that if p is a prime, then we have

$$\mathrm{ord}_p(q(G)) \leq \mathrm{ord}_p(q(F)) + \sum_{v \mid p} k(v) \leq \mathrm{ord}_p(q(F)) + 2m[F : \mathbb{Q}].$$

Bushnell and Henniart [3] derive a bound on the conductor of a Rankin-Selberg convolution. In this situation, it gives that $q(\pi \otimes \mathrm{Sym}^m \pi)$ divides $q(\pi)^{m+1} q(\mathrm{Sym}^m \pi)^2$. Hence

$$\mathrm{ord}_p(q_{m+1} q_{m-1}) \leq 2\mathrm{ord}_p(q_m) + (m+1)\mathrm{ord}_p(q_1) + 2m[F : \mathbb{Q}].$$

Rewriting this, we obtain

$$\mathrm{ord}_p(q_m) \leq 2\mathrm{ord}_p(q_{m-1}) - \mathrm{ord}_p(q_{m-2}) + m\left(\mathrm{ord}_p(q_1) + 2[F : \mathbb{Q}]\right) - 2[F : \mathbb{Q}].$$

It follows easily by induction that

$$\mathrm{ord}_p(q_m) \leq (\mathrm{ord}_p(q_1) + 2[F : \mathbb{Q}]) \binom{m+2}{3} - 2[F : \mathbb{Q}] \binom{m+1}{2},$$

and the result follows from this. □

Remark. The result of Lemma 2.1 can probably be improved to $q_m = O(q_1^{am})$ using the local Langlands correspondence to compute the local factors of $L(\mathrm{Sym}^m\pi, s)$ at all the ramified places. See Section 5 for improvements on this Lemma in certain special cases.

Define the numbers $\Lambda_m(n)$ by

$$-\frac{L'(\mathrm{Sym}^m\pi, s)}{L(\mathrm{Sym}^m\pi, s)} = \sum_{n=1}^{\infty} \frac{\Lambda_m(n)}{n^s}.$$

Then,

$$\Lambda_m(n) = \sum_{\substack{v \\ q_v^k = n}} \sum_{i=0}^{m} \alpha_{i,v}^k \log(q_v). \tag{2}$$

Here, $\alpha_{1,v}, \ldots, \alpha_{m,v}$ are the appropriate local roots for a place q_v. By the existence of a Rankin-Selberg convolution of $\mathrm{Sym}^m\pi$ with itself, it follows that $|\alpha_v| < q_v^{1/2}$, which will be used later. In particular, if n is not a power of q_v for any $v \in S$ then (2) can be written as

$$\Lambda_m(n) = \sum_{\substack{v \\ q_v^k = n}} \sum_{i=0}^{m} \alpha_v^{ki} \beta_v^{k(m-i)} \log(q_v).$$

By the absolute convergence of $L(\mathrm{Sym}^m\pi, s)$ for $\mathrm{Re}(s) > 1$ it follows that $|\alpha_v^m| < q_v$. Since this is true for all m, it follows that $|\alpha_v| \le 1$. Since $\alpha_v\beta_v = 1$ and $|\beta_v| \le 1$, it follows that $|\alpha_v| = |\beta_v| = 1$, so the Ramanujan-Petersson conjecture holds for π (at least at the unramified places).

Since $\alpha_v\beta_v = 1$ and $|\alpha_v| = |\beta_v| = 1$, it follows that there is a number $\theta_v \in [0, \pi]$ so that $\alpha_v = e^{i\theta_v}$ and $\beta_v = e^{-i\theta_v}$. If p is prime and π_v is unramified for all v with $q_v = p$, then

$$\Lambda_m(p) - \Lambda_{m-2}(p) = \sum_{\substack{v \\ q_v = p}} 2\cos(m\theta_v). \tag{3}$$

Here, the $m-2$ term is omitted if $m = 0, 1$.

3. Preliminary Lemmas

Throughout, we assume that π is an irreducible cuspidal automorphic representation of $GL_2(\mathbb{A}_F)$, where F is a totally real number field. We assume that π is genuine and has trivial central character. We assume that the symmetric power L-functions associated to π are automorphic.

To analyze the distribution of places v for which $|\alpha_v + \beta_v|$ is small we wish to study the sum

$$\sum_{\substack{x \le q_v \le 2x \\ |\theta_v - \frac{\pi}{2}| < z}} \log q_v.$$

Here, $0 < z \le 1/10$ and will be chosen later as a function of x.

Following Sarnak [17], we smooth this sum by introducing certain test functions. Let $f(y)$ be an infinitely-differentiable, even function supported on $[0,1]$. Let

$$f_z(y) = f\left(\frac{y+\pi/2}{z}\right) + f\left(\frac{y-\pi/2}{z}\right).$$

Hence $f_z(y)$ is even and $f_z(y) \neq 0$ only on $(-\pi/2 - z, -\pi/2 + z) \cup (\pi/2 - z, \pi/2 + z)$.

Let $g(y)$ be an infinitely differentiable, non-negative function supported in $[1/2, 5/2]$ with $g(y) = 1$ on $[1,2]$. Let $g_x(y) = g(y/x)$. Then, $g_x(y)$ is supported on $[x, 2x]$. We study the smoothed sum

$$\sum_v f_z(\theta_v) g_x(q_v) \log q_v. \tag{4}$$

We write f_z as an absolutely convergent Fourier series, and switching the order of summation we obtain

$$\sum_{m=0}^{\infty} a_m(z) \sum_v \cos(m\theta_v) g_x(q_v) \log q_v,$$

where the $a_m(z)$ are the Fourier coefficients of $f_z(y)$. The relation (3) provides a relationship with the symmetric power L-functions.

The first step is to derive bounds on the Fourier coefficients of $f_z(y)$. This is equivalent to bounding the Fourier transform of f. Recall that the Fourier transform of a function ϕ is defined by

$$\hat{\phi}(y) = \int_{-\infty}^{\infty} \phi(x) e^{-2\pi i x y}\, dx.$$

Lemma 3.1. *Assume the notation above and write*

$$f_z(y) = \sum_{n=0}^{\infty} a_n(z) \cos(ny).$$

If α and β are non-negative integers, then for $0 < z \leq 1/10$ we have

$$\sum_{n=0}^{\infty} |a_n(z)| n^{\alpha} \log^{\beta} n = O\left(\frac{1}{z^{\alpha}} \log^{\beta}(1/z)\right).$$

Proof. The Fourier coefficients of $f_z(y)$ are given by

$$a_n(z) = \begin{cases} \frac{1}{2\pi} \int_{-\pi}^{\pi} f_z(y)\, dy & n = 0 \\ \frac{1}{\pi} \int_{-\pi}^{\pi} f_z(y) \cos(ny)\, dy & n > 0. \end{cases}$$

Plugging in the definition of $f_z(y)$ and using that $z \leq 1/10$ gives

$$a_n(z) = \frac{1}{\pi(1+\delta_{n,0})} \int_{-\infty}^{\infty} f(y/z) \cos(ny) \cos\left(\frac{n\pi}{2}\right) dy,$$

and hence

$$|a_n(z)| \leq \frac{1}{\pi} \left| \int_{-\infty}^{\infty} f(y/z) e^{iny}\, dy \right|.$$

Setting $u = y/z$, $du = \frac{1}{z} dy$ we get

$$|a_n(z)| \leq \frac{z}{\pi} \left| \int_{-\infty}^{\infty} f(u) e^{inuz}\, dz \right| = \frac{z}{\pi} \left| \hat{f}\left(\frac{-nz}{2\pi}\right) \right|.$$

Now, taking the equation

$$\hat{f}(y) = \int_{-\infty}^{\infty} f(x)e^{-2\pi ixy}\,dx,$$

and integrating by parts r times gives

$$|\hat{f}(y)| \leq \frac{1}{(2\pi)^r y^r}\int_{-\infty}^{\infty} |f^{(r)}(x)|\,dx.$$

Taking $r = \alpha + 2$ shows that there is a constant C so that

$$|a_n(z)|n^{\alpha}\log^{\beta} n \leq \frac{C\log^{\beta} n}{n^2\pi z^{\alpha+1}}.$$

Now, $\sum_{n\geq(1/z)} |a_n(z)|n^{\alpha}\log^{\beta} n$ is bounded by

$$\frac{C}{\pi z^{\alpha+1}}\sum_{n\geq(1/z)}\frac{\log^{\beta} n}{n^2} = O\left(\frac{1}{z^{\alpha+1}}\left(z(\log(1/z)^{\beta}+1)\right)\right) = O\left(\frac{1}{z^{\alpha}}\log(1/z)^{\beta}\right).$$

Now, $\sum_{n\leq(1/z)} |a_n(z)|n^{\alpha}\log^{\beta} n$ is bounded by

$$\left(\frac{z}{\pi}\int_{-\infty}^{\infty}|f(x)|\,dx\right)\sum_{n\leq(1/z)} n^{\alpha}\log^{\beta} n = O\left(\frac{1}{z^{\alpha}}\log^{\beta}(1/z)\right),$$

and the desired result holds. □

Next, we bound the error in changing from a sum over all places v to the sum in equation (2).

Lemma 3.2. *Assume the notation above. Then,*

$$\sum_{n=1}^{\infty}\Lambda_m(n)g_x(n) - \sum_{v}\left(\cos(m\theta_v) - \cos((m-2)\theta_v)\right)g_x(v)\log q_v = O(m\sqrt{x}).$$

Again, if $m = 0, 1$ the $m-2$ term is omitted.

Proof. From equation (2), the difference is at most

$$\begin{aligned}&\sum_{\substack{p^{\alpha}\leq(5/2)x\\ q_v|p^{\alpha}\text{ with }\pi_v\text{ ramified}}}[F:\mathbb{Q}](m+1)p^{\alpha/2}\log p + \sum_{\substack{p^{\alpha}\leq(5/2)x\\ \alpha\geq 2}}[F:\mathbb{Q}](m+1)\log p\\ &\leq O((m+1)[F:\mathbb{Q}]\sqrt{x}) + [F:\mathbb{Q}](m+1)\psi(\sqrt{5x/2}) = O([F:\mathbb{Q}]m\sqrt{x}).\end{aligned}$$

Here $\psi(x) = \sum_{n\leq x}\Lambda(n)$, and $\psi(x) = O(x)$. □

Now, we will prove a version of the explicit formula.

Lemma 3.3. *Assume the notation above. Let $G_x(s) = \int_0^{\infty} g_x(y)y^s\frac{dy}{y}$. For $m \geq 0$ and $x > 1$ we have*

$$\sum_{n=1}^{\infty}\Lambda_m(n)g_x(n) = \delta_{m,0}G_x(1) - \sum_{\rho}G_x(\rho), \tag{5}$$

where the sum is over the zeroes of $L(\mathrm{Sym}^m\pi, s)$.

Proof. First, note that the Mellin transform is simply a change of variables of the Fourier transform, and hence we have the Mellin inversion formula

$$g_x(y) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} G_x(s) y^{-s} \, ds,$$

(see equation (4.106) on pg. 90 of [7]). Since $-\frac{L'(\mathrm{Sym}^m \pi, s)}{L(\mathrm{Sym}^m \pi, s)}$ converges absolutely for $\mathrm{Re}(s) \geq 2$, the left hand side of (5) equals

$$\sum_{n=1}^{\infty} \frac{\Lambda_m(n)}{n^s} \int_{2-i\infty}^{2+i\infty} G_x(s) \, ds = \int_{2-i\infty}^{2+i\infty} \frac{-L'(\mathrm{Sym}^m \pi, s)}{L(\mathrm{Sym}^m \pi, s)} G_x(s) \, ds. \tag{6}$$

Now, it is easy to see from the definition that $G_x(s)$ is entire and that $|G_x(\sigma + it)| \leq x^{\sigma}$ for all x. Also, making the change of variables $u = \log y$ in the definition of $G_x(s)$, we have for $s = \sigma + it$ that

$$G_x(s) = \int_{-\infty}^{\infty} g_x(e^u) e^{u\sigma} (\cos(tu) + i \sin(tu)) \, du.$$

Integrating by parts n times shows that $|G_x(s)| \leq C_{n,x} t^{-n}$ for some constant $C_{n,x}$, depending on n and x. Theorem 5.8 of [7] implies that if T is sufficiently large, there are $O(\log T)$ zeroes of $L(\mathrm{Sym}^m \pi, s)$ with imaginary part between T and $T + 1$. Also, the zeroes of $L(\mathrm{Sym}^m \pi, s)$ with $\mathrm{Re}(s) < 0$ arise from the gamma factor. We choose a large real number T so that there are no zeroes with imaginary part between $T - \frac{C}{\log T}$ and $T + \frac{C}{\log T}$ (and no zeroes with real part between $-T - C$ and $-T + C$). We add the additional contours $2 + iT$ to $-T + iT$, $-T + iT$ to $-T - iT$ and $-T - iT$ to $-2 - iT$ to the integral in (6). From Proposition 5.7 (part 2) of [7] (pg. 103), it follows that

$$\left| -\frac{L'(\mathrm{Sym}^m, s)}{L(\mathrm{Sym}^m \pi, s)} \right| = O(\log^2 T)$$

on $\mathrm{Im}(s) = \pm T$ for $-1/2 \leq \mathrm{Re}(s) \leq 2$. It follows from $|G_x(s)| \leq C_{n,x} T^{-n}$ that the integral along this portion of the contour tends to zero as $T \to \infty$.

Now, $L'(\mathrm{Sym}^m \pi, s)/L(\mathrm{Sym}^m \pi, s)$ is bounded in $\mathrm{Re}(s) \geq 3/2$ and by the functional equation, it follows that

$$|L'(\mathrm{Sym}^m \pi, s)/L(\mathrm{Sym}^m \pi, s)| = O(\log T)$$

for $\mathrm{Re}(s) \leq -1/2$, the only substantial contribution coming from the Γ'/Γ terms. Again, it follows that the integral along this contour tends to zero as $T \to \infty$, since $|G_x(s)| \leq C_{n,x} T^{-n}$.

The last piece is $\mathrm{Re}(s) = -T$. Here, we use that $|G_x(s)| \leq x^{\sigma} = x^{-T}$. Since $x > 1$, the integral along this piece tends to zero as $T \to \infty$. Thus, we have that

$$\int_{2-i\infty}^{2+i\infty} -\frac{L'(\mathrm{Sym}^m \pi, s)}{L(\mathrm{Sym}^m \pi, s)} G_x(s) \, ds = \lim_{T \to \infty} \int_{C_T} -\frac{L'(\mathrm{Sym}^m \pi, s)}{L(\mathrm{Sym}^m \pi, s)} G_x(s),$$

where C_T is the box with vertices $-T \pm iT$ and $2 \pm iT$. By Cauchy's theorem, the integral over this box is the sum of the residues of the poles. The poles come from poles or zeroes of $\frac{L'(\mathrm{Sym}^m \pi, s)}{L(\mathrm{Sym}^m \pi, s)}$, and hence the limit is

$$\delta_{m,0} G_x(1) - \sum_{\substack{\rho \\ L(\mathrm{Sym}^m \pi, \rho) = 0}} G_x(\rho).$$

From this, the result follows. □

Now, we estimate the errors arising from the sum over the zeroes in the explicit formula (5).

Lemma 3.4. *Assume the notation above and the Riemann hypothesis for $L(\mathrm{Sym}^m\pi, s)$. For z sufficiently small and $x \geq 2$ we have*

$$\sum_{m=0}^{\infty} |a_m(z)| \sum_{\substack{\rho \\ L(\mathrm{Sym}^m\pi,\rho)=0}} |G_x(\rho)| = O\left(\frac{\sqrt{x}}{z^3}\right).$$

Proof. First, we consider the trivial zeroes. These arise as poles of the gamma factor. Since the $\kappa_{j,m}$ all have real part less than or equal to zero, one can easily see that if ρ is a trivial zero then $|G_x(\rho)| \leq 2x^{\mathrm{Re}(\rho)}$. It follows then that the contribution of the trivial zeroes is at most

$$\sum_{n=0}^{\infty} 2[F:\mathbb{Q}]mx^{-n/2} = O(m).$$

Now, we will bound the contribution of the non-trivial zeroes. Theorem 5.8 of [7] states that if $N(T,f)$ is the number of zeroes $\rho = \beta + i\gamma$ of an L-function $L(f,s)$ with $0 \leq \beta \leq 1$ and $|\gamma| \leq T$, then

$$N(T,f) = \frac{T}{\pi} \log \frac{qT^d}{(2\pi e)^d} + O(\log \mathfrak{q}(f, iT)). \tag{7}$$

Here, d is the degree of the L-function, and $\mathfrak{q}(f,s) = q(f)\prod_{j=1}^{d}(|s+\kappa_j|+3)$ is the analytic conductor. The $q(f)$ is the conductor and the κ_j arise in the gamma factor

$$\gamma(f,s) = \pi^{-ds/2} \prod_{j=1}^{d} \Gamma\left(\frac{s+\kappa_j}{2}\right).$$

From the form of the γ factor for $L(\mathrm{Sym}^m\pi, s)$ and Lemma 2.1, it follows that

$$|\mathfrak{q}(\mathrm{Sym}^m\pi, iT)| = O(N^{a(d[F:\mathbb{Q}])^3}(T + m[F:\mathbb{Q}])^{m+1}),$$

and hence $\log|\mathfrak{q}(\mathrm{Sym}^m\pi, iT)| = O(m^3 + m(\log T + \log m))$. It follows from (7) that $N(n+1, \mathrm{Sym}^m\pi) - N(n, \mathrm{Sym}^m\pi) = O(m^3 + m\log m + m\log n)$.

Suppose that $\rho = \frac{1}{2} + i\gamma$ is a non-trivial zero of $L(\mathrm{Sym}^m\pi, s)$. Making a few changes of variables, it follows that

$$G_x(\rho) = \int_0^{\infty} g(y/x)y^{1/2+i\gamma}\frac{dy}{y} = x^{1/2+i\gamma}\hat{h}(-\gamma).$$

Here, $h(y) = 2\pi g(e^{2\pi y})e^{\pi y}$. In particular,

$$|G_x(\rho)| = \sqrt{x}|\hat{h}(-\gamma)|.$$

Now, $h(y)$ is a compactly supported infinitely differentiable function, and hence $|\hat{h}(y)| \leq C_n y^{-n}$ for all $n \geq 0$ by the same argument as in Lemma 3.1. Hence, we have that the error from the non-trivial zeroes is bounded by

$$\sqrt{x} \sum_{m=0}^{\infty} |a_m(z)| \sum_{n=0}^{\infty} \#\{n \leq |\gamma| \leq n+1 : L(\mathrm{Sym}^m \pi, 1/2 + i\gamma) = 0\} \cdot |\hat{h}(-n)|.$$

For $n = 0$ we use the bound $|\hat{h}(0)| \leq C_0$. For $n > 0$ we use the bound $|\hat{h}(-n)| \leq C_2/n^2$. Hence, the contribution is

$$O\left(\sqrt{x} \sum_{m=0}^{\infty} |a_m(z)| m^3\right).$$

By Lemma 3.1, this is $O\left(\sqrt{x}(1/z)^3\right)$.

□

Remark. If we have the bound $q_m = O(q_1^{am})$, then the log of the analytic conductor is bounded by $O(m \log m + m \log T)$. This gives the bound $O\left(\frac{\sqrt{x}}{z} \log(1/z)\right)$.

We will now turn to the proof of Theorem 1.3.

4. Proof of Theorem 1.3

In this section, we combine the results of Section 3 and prove Theorem 1.3.

Proof. For x sufficiently large,

$$\sum_{\substack{x \leq q_v \leq 2x \\ |\theta_v - \frac{\pi}{2}| < z}} \log q_v \leq \sum_v f_z(\theta_v) g_x(q_v) \log q_v.$$

Expanding f_z as a Fourier series, we obtain

$$\sum_{m=0}^{\infty} a_m(z) \sum_v \cos(m\theta_v) g_x(q_v) \log q_v.$$

Expressing this in terms of $\Lambda_m(n)$ (as in Lemma 3.2), we have

$$\sum_{m=0}^{\infty} a_m(z) \sum_n \left(\Lambda_m(n) - \Lambda_{m-2}(n)\right) g_x(n) + O(m\sqrt{x}).$$

Plugging in the explicit formula yields

$$\sum_{m=0}^{\infty} a_m(z) \left(\delta_{m,0} G_x(1) - \delta_{m,2} G_x(1) - \sum_{L(\mathrm{Sym}^m \pi, \rho) = 0} G_x(\rho) + \sum_{L(\mathrm{Sym}^{m-2} \pi, \rho) = 0} G_x(\rho) \right).$$

From Lemma 3.4, this quantity is

$$(a_0(z) - a_2(z)) G_x(1) + O(\sqrt{x} z^{-3}). \tag{8}$$

Now,

$$G_x(1) = \int_0^{\infty} g_x(y)\, dy = x \int_0^{\infty} g(u)\, du.$$

Using the bounds $|a_m(z)| = O(z)$, the first term in (8) is $O(xz)$, giving

$$\sum_{\substack{x \le q_v \le 2x \\ |\theta_v - \frac{\pi}{2}| < z}} \log q_v \le \sum_v f_z(\theta_v) g_x(q_v) \log q_v = O(xz + \sqrt{x} z^{-3}).$$

We choose $z = \frac{1}{2} x^{-\alpha}$. The error is then $O(x^{1-\alpha})$ if $\alpha \le 1/8$. Now, if $|\alpha_v + \beta_v| < q_v^{-\alpha}$ then $2|\cos(\theta_v)| \le q_v^{-\alpha}$. This implies that $|\cos(\theta_v)| \le \frac{1}{2} q_v^{-\alpha}$ and hence $|\theta_v - \frac{\pi}{2}| < z$. Hence,

$$\sum_{\substack{x \le q_v \le 2x \\ |\alpha_v + \beta_v| \le q_v^{-\alpha}}} \log q_v = O(x^{1-\alpha}).$$

It follows easily then that for x sufficiently large,

$$\sum_{\substack{\sqrt{x} \le q_v \le 2x \\ |\alpha_v + \beta_v| \le q_v^{-\alpha}}} \log q_v = O(x^{1-\alpha}).$$

For $\sqrt{x} \le q_v \le x$, $\log q_v \ge \frac{1}{2} \log x$ and hence

$$\#\{\sqrt{x} \le q_v \le x : |\alpha_v + \beta_v| \le q_v^{-\alpha}\} = O\left(\frac{x^{1-\alpha}}{\log x}\right).$$

There are $O(x^{1/2})$ places v with $q_v \le \sqrt{x}$ and from this, the upper bound follows. A proof of the lower bound is analogous, but we choose g to be supported on $[1, 2]$ with $g(y) \le 1$. $\square$

5. Special Cases

In this section, we consider the special case of $F = \mathbb{Q}$ and π corresponding to a classical newform $H(z)$. In particular, we will prove Theorem 1.1 and Corollary 1.2.

See Chapter 7 of [2] for a discussion of the correspondence between classical newforms and automorphic representations. To assure that the representation π is genuine with trivial central character, we assume that $H(z)$ has even weight k with trivial character and does not have complex multiplication.

When $N > 1$, the conductor bound from Lemma 2.1 is quite crude. One approach to computing the conductor and determining the local factors at the ramified primes is to use the local Langlands correspondence to predict them. This is done by Cogdell and Michel for a classical newform with trivial character and squarefree level in [4]. In their case $q_m = q_1^m$.

Another approach to computing the conductor was indicated to me by Professor Serre. Let K be a number field with the property that the $a(n) \in \mathcal{O}_K$ for all n. If $\mathfrak{p}$ is a prime ideal of $\mathcal{O}_K$ then there is a continuous, semisimple Galois representation $\rho_{\mathfrak{p}}$ unramified outside $N \cdot N_{K/\mathbb{Q}}(\mathfrak{p})$ that provides $\mathfrak{p}$-adic information about the coefficients of $H(z)$. One may then compute the conductors of the symmetric power L-functions attached to $H(z)$ by determining the conductors of the symmetric powers of $\rho_{\mathfrak{p}}$. A precise recipe for computing the conductor of $\mathfrak{p}$-adic Galois representations is given in Section 2 of [18]. In this way one derives the bound $\mathrm{ord}_p q_m \le m C_p \mathrm{ord}_p q_1$, where C_p doesn't depend on m. It follows then that $q_m = O(q_1^{am})$ for some a.

Now, we will derive Theorem 1.1 from Theorem 1.3.

Proof. In light of the correspondence between classical newforms and automorphic representations, Theorem 1.3 gives that

$$\#\{p \leq x : |\alpha_p + \beta_p| \leq p^{-\alpha}\} \asymp O\left(\frac{x^{1-\alpha}}{\log x}\right).$$

Now, $|a(p)| = p^{\frac{k-1}{2}}|\alpha_p + \beta_p|$ and the desired result follows immediately. $\square$

Now, we will show how Corollary 1.2 follows from Theorem 1.1.

Proof. Suppose that $F(z) = \sum_{n=1}^{\infty} a(n)q^n$ is a newform with coefficients in $\mathcal{O}_K$, where K is a degree d totally real extension of $\mathbb{Q}$. Let $\sigma_1, \ldots, \sigma_d$ be the embeddings of K into $\mathbb{R}$.

Let $P_\alpha = \{p^r : |\sigma_i(a(p^r))| < p^{r\left(\frac{k-1}{2}-\alpha\right)}$ for some r and some $i, 1 \leq i \leq d\}$. It is well-known that $\sigma_i(F(z)) = \sum_{n=1}^{\infty} \sigma_i(a(n))q^n$ is a newform in the same space as $F(z)$ for all i. Applying Theorem 1.1 to $\sigma_i(F(z))$ for all i gives that

$$\#\{q \in P_\alpha : q \leq x\} = O\left(\frac{x^{1-\alpha}}{\log x}\right).$$

Let $B_\alpha = \{b :$ if $p^e \| b$ then $p^e \in P_\alpha\}$. For $n \leq x$, let

$$b_\alpha(n) = \prod_{\substack{p^e \| n \\ p^e \in P_\alpha}} p^e,$$

and let $c = n/b_\alpha(n)$. Then, since the coefficients $a(n)$ are multiplicative, it follows that

$$|\sigma_i(a(c))| \geq c^{\left(\frac{k-1}{2}-\alpha\right)},$$

for all i and hence

$$|N_{K/\mathbb{Q}}(a(c))| = \prod_{i=1}^{d} |\sigma_i(a(c))| \geq c^{d\left(\frac{k-1}{2}-\alpha\right)}.$$

If $a(b_\alpha(n)) \neq 0$, it follows that $|N_{K/\mathbb{Q}}(a(b_\alpha(n)))| \geq 1$ and hence

$$|N_{K/\mathbb{Q}}(a(n))| = |N_{K/\mathbb{Q}}(a(c))||N_{K/\mathbb{Q}}(a(b))| \geq c^{d\left(\frac{k-1}{2}-\alpha\right)}.$$

If $b_\alpha(n) \leq n^\lambda$, then we have

$$|N_{K/\mathbb{Q}}(a(n))| \geq c^{d\left(\frac{k-1}{2}-\alpha\right)} \geq n^{d(1-\lambda)\left(\frac{k-1}{2}-\alpha\right)}.$$

The bound $\#\{q \leq x : q \in P_\alpha\} = O\left(\frac{x^{1-\alpha}}{\log x}\right)$ implies that for $s - \alpha$ positive and small,

$$\sum_{q \in P_\alpha} \frac{1}{q^s} = O\left(\log \frac{1}{s-\alpha}\right).$$

If D is the implied constant, we have that

$$\sum_{b \in B_\alpha} \frac{1}{b^s} = \exp\left(\sum_{q \in P_\alpha} \frac{1}{q^s}\right) + O(1) = O\left(\frac{1}{(s-\alpha)^D}\right).$$

The usual inversion formulas for Dirichlet series (see for example equation A.10 on pg. 486 of [6]) imply that

$$B(x) := \#\{b \le x : b \in B_\alpha\} = O(x^{1-\alpha} \log^{D+1} x).$$

Now, using partial summation, we have

$$\begin{aligned}\sum_{\substack{b \in B_\alpha \\ x^\lambda \le b}} \frac{1}{b} &= \frac{B(x)}{x} - \frac{B(x^\lambda)}{x^\lambda} + \int_{x^\lambda}^{x} B(t) \frac{1}{t^2}\, dt \\ &= O\left(\frac{\log^{D+1} x}{x^{\alpha\lambda}} \right).\end{aligned}$$

Hence, the number of $x \le n \le 2x$ with $b_\alpha(n) \ge n^\lambda$ is at most

$$\sum_{\substack{b \in B_\alpha \\ x^\lambda \le b}} \left\lfloor \frac{2x}{b} \right\rfloor \le 2x \sum_{x^\lambda \le b} \frac{1}{b} = O(x^{1-\alpha\lambda} \log^{D+1} x).$$

It follows that the number of $n \le x$ with $b_\alpha(n) \ge n^\lambda$ is $O(x^{1-\alpha\lambda} \log^{D+1} x)$, and the result follows. □

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