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# ABELIAN VARIETIES WITHOUT HOMOTHETIES

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Abstract. A celebrated theorem of Bogomolov asserts that the $\ell$-adic Lie algebra attached to the Galois action on the Tate module of an abelian variety over a number field contains all homotheties. This is not the case in characteristic $p$: a "counterexample" is provided by an ordinary elliptic curve defined over a finite field. In this note we discuss (and explicitly construct) more interesting examples of "non-constant" absolutely simple abelian varieties (without homotheties) over global fields in characteristic $p$.

## 1. Introduction

Let $K$ be a field, $K_a$ its algebraic closure and $\mathrm{Gal}(K) = \mathrm{Aut}(K_a/K)$ the absolute Galois group. If $X$ is an abelian variety over $K$ then we write $\mathrm{End}_K(X)$ for the ring of $K$-endomorphisms of $X$ and $\mathrm{End}^0_K(X)$ for the corresponding $\mathbb{Q}$-algebra $\mathrm{End}_K(X)\otimes\mathbb{Q}$. We write $\mathrm{End}(X)$ for the ring of $K_a$-endomorphisms of $X$ and $\mathrm{End}^0(X)$ for the corresponding $\mathbb{Q}$-algebra $\mathrm{End}(X)\otimes\mathbb{Q}$. The notation $1_X$ stands for the identity automorphism of $X$. It is well-known [5] that $\mathrm{End}^0(X)$ is a finite-dimensional semisimple $\mathbb{Q}$-algebra and its center $\mathfrak{C}(X)$ is a product of number fields; in addition, either of those fields is either totally real or a CM-field.

Let $E$ be a number field. Suppose we are given an embedding

$$i : E \hookrightarrow \mathrm{End}^0_K(X),\ i(1) = 1_X.$$

Then $[E:\mathbb{Q}]$ divides $2\dim(X)$ [15, Ch. 2, Sect. 5, Prop. 2]; let us put

$$r(X,E) = \frac{2\dim(X)}{[E:\mathbb{Q}]}.$$

Let $\ell$ be a prime different from $\mathrm{char}(K)$. We write $T_\ell(X)$ for the corresponding Tate $\mathbb{Z}_\ell$-module of $X$ and $V_\ell(X)$ for the corresponding $\mathbb{Q}_\ell$-vector space $T_\ell(X)\otimes_{\mathbb{Z}_\ell}\mathbb{Q}_\ell$. It is well-known that $T_\ell(X)$ is a free $\mathbb{Z}_\ell$-module of rank $2\dim(X)$ and $V_\ell(X)$ is a $2\dim(X)$-dimensional $\mathbb{Q}_\ell$-vector space. We write Id for the identity automorphism of $V_\ell(X)$. It is well-known that $\mathrm{Aut}_{\mathbb{Z}_\ell}(T_\ell(X)) \cong \mathrm{GL}(2\dim(X),\mathbb{Z}_\ell)$ is a compact $\ell$-adic Lie group with Lie algebra $\mathrm{End}_{\mathbb{Q}_\ell}(V_\ell(X))$. Let

$$\det : \mathrm{Aut}_{\mathbb{Q}_\ell}(V_\ell(X)) \to \mathbb{Q}_\ell^*$$

be the determinant map. As usual, we write $\mathrm{SL}(V_\ell(X))$ for its kernel. It is well-known that $\mathrm{SL}(V_\ell(X))$ is a Lie subgroup in $\mathrm{Aut}_{\mathbb{Q}_\ell}(V_\ell(X))$ and its Lie algebra coincides with

$$\mathfrak{sl}(V_\ell(X)) := \{u \in \mathrm{End}_{\mathbb{Q}_\ell}(V_\ell(X)) \mid \mathrm{tr}(u) = 0\}$$

where

$$\mathrm{tr} : \mathrm{End}_{\mathbb{Q}_\ell}(V_\ell(X)) \to \mathbb{Q}_\ell$$

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is the trace map.

On the other hand, $T_\ell(X)$ carries a natural structure of $\mathrm{End}_K(X) \otimes \mathbb{Z}_\ell$-module and $V_\ell(X)$ carries a natural structure of $\mathrm{End}^0_K(X) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$-module.

Let us put

$$E_\ell = E \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \subset \mathrm{End}^0_K(X) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell.$$

The embedding $i$ provides $V_\ell(X)$ with a natural structure of $E_\ell$-module: it is known [14, 9] that this module is free of rank $r(X, E)$.

One may view $E_\ell^*$ as a commutative $\ell$-adic Lie (sub)group with (commutative) Lie algebra $E_\ell$. We have

$$\mathbb{Z}_\ell^* \mathrm{Id} \subset E_\ell^* \subset \mathrm{Aut}_{\mathbb{Q}_\ell}(V_\ell(X));$$

clearly, $\mathbb{Z}_\ell^* \mathrm{Id}$ is a compact $\ell$-adic Lie subgroup whose Lie algebra coincides with $\mathbb{Q}_\ell \mathrm{Id}$.

**Remark 1.1.** Let $G \subset \mathrm{Aut}_{\mathbb{Q}_\ell}(V_\ell(X))$ be a compact subgroup. Then the ($\ell$-adic variant of) Cartan's theorem [12, Part 2, Ch. 5, Sect. 9] tells us that $G$ is a Lie subgroup. Clearly, the intersection $G \bigcap \mathbb{Z}_\ell^* \mathrm{Id}$ is infinite if and only if the Lie algebra $\mathrm{Lie}(G)$ of $G$ contains $\mathbb{Q}_\ell \mathrm{Id}$.

Let us consider the centralizer $\mathrm{End}_{E_\ell}(V_\ell(X))$ of $E_\ell$ in $\mathrm{End}_{Q_\ell}(V_\ell(X))$ and its group of invertible elements $\mathrm{Aut}_{E_\ell}(V_\ell(X))$. One may view $\mathrm{Aut}_{E_\ell}(V_\ell(X))$ as an $\ell$-adic Lie group with Lie algebra $\mathrm{End}_{E_\ell}(V_\ell(X))$.

Since $V_\ell(X)$ is a free $E_\ell$-module of finite rank, there are the natural $E_\ell$-determinant homomorphism of $\ell$-adic Lie groups

$$\det{}_{E_\ell} : \mathrm{Aut}_{E_\ell}(V_\ell(X)) \to E_\ell^*$$

and the $E_\ell$-trace map

$$\mathrm{tr}_{E_\ell} : \mathrm{End}_{E_\ell}(V_\ell(X)) \to E_\ell.$$

Clearly, $\mathrm{tr}_{E_\ell}$ is the *tangent map* of Lie algebras attached to $\det{}_{E_\ell}$.

**Remark 1.2.** Let $G$ be a (closed) compact subgroup in $\mathrm{Aut}_{E_\ell}(V_\ell(X))$. Then $G$ is an $\ell$-adic Lie (sub)group and its Lie algebra $\mathrm{Lie}(G)$ is a $\mathbb{Q}_\ell$-Lie subalgebra of $\mathrm{End}_{E_\ell}(V_\ell(X))$. In addition, if $\mathrm{Lie}(G)$ is a semisimple Lie algebra then $\det{}_{E_\ell}(G)$ is a finite subgroup in $E_\ell^*$. Indeed, the semisimplicity of $\mathrm{Lie}(G)$ implies that $\mathrm{tr}_{E_\ell}(\mathrm{Lie}(G)) = \{0\}$ and therefore $\det{}_{E_\ell} = 1$ on an open subgroup of $G$. One has only to recall that every open subgroup in a compact $\ell$-adic Lie group has finite index.

There is a natural continuous homomorphism ($\ell$-adic representation) [10]

$$\rho_{\ell,X} : \mathrm{Gal}(K) \to \mathrm{Aut}_{\mathbb{Z}_\ell}(T_\ell(X)) \subset \mathrm{Aut}_{\mathbb{Q}_\ell}(V_\ell(X));$$

its image $G_{\ell,X}$ is a compact $\ell$-adic Lie subgroup of $\mathrm{Aut}_{\mathbb{Q}_\ell}(V_\ell(X))$. We write $\mathfrak{g}_{\ell,X}$ for the Lie algebra of $G_{\ell,X}$; one may view $\mathfrak{g}_{\ell,X}$ as a Lie $\mathbb{Q}_\ell$-subalgebra in $\mathrm{End}_{\mathbb{Q}_\ell}(V_\ell(X))$ [10].

The following assertion is proven in [18].

**Theorem 1.3.** *Suppose that $K$ is a global field of characteristic $p > 2$ and $X$ is an abelian variety of positive dimension over $K$. Then:*

(I) *$\mathfrak{g}_{\ell,X}$ is a reductive $\mathbb{Q}_\ell$ algebra, i.e. $\mathfrak{g}_{\ell,X} \cong \mathfrak{g}^{ss} \oplus \mathfrak{c}$ where $\mathfrak{g}^{ss}$ is a semisimple $\mathbb{Q}_\ell$-Lie algera and $\mathfrak{c}$ is the center of $\mathfrak{g}_{\ell,X}$.*

(II) $\dim_{\mathbb{Q}_\ell}(\mathfrak{c}) = 1$.

(III) *If $\mathfrak{C}(X)$ is a product of totally real number fields then $\mathfrak{c} = \mathbb{Q}_\ell \cdot \mathrm{Id}$.*

When $K$ is a number field, a theorem of Bogomolov [1, 2] asserts that $\mathfrak{g}_{\ell,X}$ always contains homotheties $\mathbb{Q}_\ell \cdot \mathrm{Id}$.

However, one may easily check that this is not the case if $K$ is a global field of characteristic $p$. For example, if $X$ is an ordinary elliptic curve that is defined over a finite field then $\mathfrak{g}_{\ell,X}$ is a one-dimensional $\mathbb{Q}_\ell$-Lie algebra that is generated by the $\ell$-adic logarithm of the corresponding Frobenius endomorphism, which is not a scalar. The aim of this note is to prove the existence of an absolutely simple abelian variety $X$ over a global field of characteristic $p$ such that $\mathfrak{g}_{\ell,X}$ does *not* contain homotheties and $X$ is *not* isogenous over $K_a$ to an abelian variety over a finite field. Recall [6] that the latter condition means that $X$ is not an abelian variety of CM-type over $K_a$. Our main result is described by the following two statements.

**Theorem 1.4.** *Suppose that $K$ is a global field of characteristic $p > 2$. Suppose that $X$ is an ordinary abelian variety of positive dimension over $K$. Let $E \subset \mathrm{End}^0(X)$ be a subfield that contains $1_X$. Assume that $r(X, E)$ is an odd integer.*

*Then $\mathfrak{g}_{\ell,X} \bigcap \mathbb{Q}_\ell \cdot \mathrm{Id} = \{0\}$, i.e., $\mathfrak{g}_{\ell,X}$ does not contain homotheties except zero and $G_{\ell,X} \bigcap \mathbb{Z}_\ell^* \mathrm{Id}$ is finite.*

We prove Theorem 1.4 in Section 3.

**Theorem 1.5.** *Let $Z$ be an ordinary elliptic curve over a finite field $k$ of characteristic $p > 2$ and $E = \mathrm{End}^0(Z)$ the corresponding imaginary quadratic field.*

*Then for every odd $g > 1$ there exist a global field $K$ of characteristic $p$ and an ordinary $g$-dimensional abelian variety $X$ over $K$ that enjoys the following properties:*

(i) *All endomorphisms of $X$ are defined over $K$ and $\mathrm{End}^0(X) = E$. In particular, $X$ is absolutely simple.*

(ii) *$X$ is not isogenous over $K_a$ to an abelian variety that is defined over a finite field.*

(ii) *$\mathfrak{g}_{\ell,X} \bigcap \mathbb{Q}_\ell \cdot \mathrm{Id} = \{0\}$, i.e., $\mathfrak{g}_{\ell,X}$ does not contain homotheties except zero and $G_{\ell,X} \bigcap \mathbb{Z}_\ell^* \mathrm{Id}$ is finite.*

**Remark 1.6.** (i) In light of Theorem 2(b) of [17], the second assertion of Theorem 1.5 follows readily from the first one, because in this case

$$\dim_{\mathbb{Q}}(\mathrm{End}^0(X)) = \dim_{\mathbb{Q}}(E) = 2 < 2g = 2\dim(X).$$

(ii) In light of Theorem 1.4, the third assertion of Theorem 1.5 follows readily from the first one, because in this case $r(X, E) = g$ is odd.

We prove Theorem 1.5(i) in Section 2. In Section 4 we discuss an explicit example of an abelian variety that satisfies the conditions and conclusions of Theorem 1.4.

## 2. Abelian varieties and imaginary quadratic fields

*Proof of Theorem 1.5(i).* Notice that all endomorphisms of $Z$ are defined over $k$. (This well-known result goes back to Deuring [3]; it follows easily from Main Theorem of [17].) Since $Z$ is ordinary and $g-1$ is a multiple of $2 = \dim_{\mathbb{Q}}(E)$, a theorem of Oort -van der Put [7, Th. 1.1] implies the existence of an ordinary $g$-dimensional abelian variety $Y$ over $k((t))$ with all endomorphisms defined over $k((t))$ and $\mathrm{End}^0(Y) = E$. Clearly, $Y$ and all its endomorphisms are defined over a field $K$ that is finitely generated over $k$. Now, Mori's specialization arguments [4, Cor. 5.4] allow us to assume that $K$ has transcendence degree 1, i.e., is global. □

## 3. Ordinary abelian varieties

**Lemma 3.1.** *Let $k$ be a finite field that consists of $q$ elements, $A$ an ordinary abelian variety over $k$ and $d$ a positive odd integer. If $\{\alpha_1, \dots, \alpha_d\}$ are $d$ eigenvalues of the Frobenius endomorphism $Fr$ of $A$ then $q^{-d}(\prod_{i=1}^d \alpha_i)^2$ is not a root of unity.*

*Proof of Lemma 3.1.* If $p = \mathrm{char}(k)$ then $q$ is a power of $p$. Let us choose a $p$-adic valuation map $\mathrm{ord}_p : \bar{\mathbb{Q}}^* \to \mathbb{Q}$ normalized by the condition $\mathrm{ord}_p(q) = 1$. Since $A$ is ordinary, the Honda-Tate theory [16] tells us that $\mathrm{ord}_p(\alpha) = 0$ or 1 for every eigenvalue of the Frobenius endomorphism of $A$. This implies that

$$\mathrm{ord}_p(q^{-d}(\prod_{i=1}^d \alpha_i)^2) = -d + 2\sum_{i=1}^d \mathrm{ord}_p(\alpha_i) \in -d + 2\mathbb{Z}$$

is an *odd* integer and therefore does not vanish. It follows that $q^{-d}(\prod_{i=1}^d \alpha_i)^2$ is not a root of unity. □

*Proof of Theorem 1.4.* Replacing (if necessary) $K$ by its finite separable algebraic extension, we may and will assume that all endomorphisms of $X$ are defined over $K$; in particular, $E \subset \mathrm{End}_K^0(X) = \mathrm{End}^0(X)$. Let us assume that $\mathfrak{g}_{\ell,X} \bigcap \mathbb{Q}_\ell \cdot \mathrm{Id} \neq \{0\}$. This means that $\mathfrak{g}_{\ell,X}$ contains $\mathbb{Q}_\ell \cdot \mathrm{Id}$ and therefore $\mathfrak{c} = \mathbb{Q}_\ell \cdot \mathrm{Id}$.

Let us put $G^0 = G_{\ell,X} \bigcap SL(V_\ell(X)$. Clearly, $G^0$ is a closed (compact) Lie subgroup of $G_{\ell,X}$ and $\mathrm{Lie}(G^0)$ has codimension 1 in $\mathrm{Lie}(G_{\ell,X}) = \mathfrak{g}^{ss} \oplus \mathbb{Q}_\ell \cdot \mathrm{Id}$. The semisimplicity of $\mathfrak{g}^{ss}$ implies that $\mathrm{Lie}(G^0) = \mathfrak{g}^{ss}$.

Let us put

$$S = G_{\ell,X} \bigcap (1 + \ell^2 \mathbb{Z}_\ell)\mathrm{Id} \subset \mathbb{Z}_\ell^* \mathrm{Id}.$$

Clearly, $S$ is compact. Since $\mathfrak{g}_{\ell,X} = \mathrm{Lie}(G_{\ell,X})$ contains $\mathbb{Q}_\ell \cdot \mathrm{Id}$, the group $G_{\ell,X}$ contains an open subgroup of $\mathbb{Z}_\ell^* \mathrm{Id}$. It follows that $S$ is an open subgroup of finite index in $\mathbb{Z}_\ell^* \mathrm{Id}$. Since $1 + \ell^2 \mathbb{Z}_\ell$ does not contain nontrivial roots of unity, $S$ does not contain elements of finite order (except Id) and therefore $G^0 \bigcap S = \{\mathrm{Id}\}$. Recall that both $G^0$ and $S$ are subgroups of $G_{\ell,X}$. Let us consider the homomorphism of compact $\ell$-adic Lie groups

$$\pi : G^0 \times S \to G_{\ell,X}, \ (u, c) \mapsto uc = cu.$$

Clearly, $\pi$ is injective and the corresponding tangent map of Lie algebras is an isomorphism. It follows that $G^1 := \pi(G^0 \times S)$ is an open compact subgroup in $G_{\ell,X}$ and $\pi$ induces an isomorphism of $\ell$-adic Lie groups $G^0 \times S$ and $G^1$.

**Lemma 3.2.** *There exists a positive integer $m$ such that*

$$\det_{E_\ell}(g)^m \in \mathbb{Q}_\ell^* \mathrm{Id} \ \forall g \in G_{\ell,X}.$$

*Proof of Lemma 3.2.* Since $\mathrm{Lie}(G^0)$ is semisimple, it follows from Remark 1.2 that $\det_{E_\ell}(G^0)$ is a finite group. If $m_0$ is its order then $\det_{E_\ell}(g_0)^{m_0} = 1$ for all $g_0 \in G^0$. Notice that

$$\det_{E_\ell}(c) = c^{r(X,E)} \ \forall c \in \mathbb{Z}_\ell^* \mathrm{Id},$$

because $\mathbb{Z}_\ell^* \mathrm{Id} \subset E_\ell^*$. It follows that $\det_{E_\ell}(g)^{m_0} \in \mathbb{Q}_\ell^* \mathrm{Id} \ \forall g \in G^1$. In order to finish the proof, one has only to recall that $G^1$ is a subgroup of finite index in $G_{\ell,X}$ and put $m := m_0 \cdot [G_{\ell,X} : G^1]$. □

There exists a place $v$ of $K$ such that the abelian variety $X$ has ordinary good reduction. (In fact, this condition is fulfilled for all but finitely many places of $K$.) Let $k(v)$ be the residue field at $v$, let $q(v)$ be the cardinality of $k(v)$ and $X(v)$ the reduction of $X$ at $v$, which is an ordinary abelian variety over $k(v)$ whose dimension coincides with $\dim(X)$. Let $\mathbb{P}_v(t) \in \mathbb{Z}[t]$ be the (degree $2\dim(X)$) characteristic polynomial of the Frobenius endomorphism Fr of $X(v)$. One may view the roots of $\mathbb{P}_v$ as eigenvalues of the Frobenius endomorphism with respect to its natural action on $V_\ell(X(v))$.

Let us choose a place $\bar{v}$ of $K_a$ that lies above $v$. Such a choice gives rise to natural isomorphisms [13, 10]

$$T_\ell(X) \cong T_\ell(X(v)),\ V_\ell(X) \cong V_\ell(X(v))$$

in such a way that $\mathrm{Fr} \in \mathrm{Aut}_{\mathbb{Z}_\ell}(T_\ell(X(v))$ corresponds to a certain element of $G_{\ell,X}$: this element is called the *Frobenius element* attached to $\bar{v}$ and denoted by $F_{\bar{v}}$. It is known [14, Chap. 7, proof of Prop. 7.23] (see also [19, p. 167]) that

$$b_v := \det_{E_\ell}(F_{\bar{v}}) \in E^* \subset E_\ell^*$$

and $b_v$ is a product of $r(X,E)$ eigenvalues of Fr.

In other words, let $L$ be the splitting field of $\mathbb{P}_v(t)$ over $E$: it is a finite Galois extension of $E$. Then there exist roots $\alpha_1, \dots, \alpha_{r(X,E)}$ of $\mathbb{P}_v(t)$ such that their product coincides with $b_v$. On the other hand, it follows from a famous theorem of A. Weil (the Riemann hypothesis) [5, Sect. 21] that if we fix a field embedding $L \subset \mathbb{C}$ then

$$| b_v^2 |_\infty = q(v)^{r(X,E)}$$

where $|\ |_\infty$ is the standard (archimedean) absolute value on the field of complex numbers. On the other hand, by Lemma 3.2, there exists a positive integer $m$ such that $b_v^m \in \mathbb{Q}_\ell$. Since the intersection of $E = E \otimes 1$ and $\mathbb{Q}_\ell = 1 \otimes \mathbb{Q}_\ell$ in $E_\ell = E \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ coincides with $\mathbb{Q}$, we conclude that $b_v^m$ is a rational number. This implies that $b_v^{2m}$ is a positive rational number and, by Weil's theorem, coincides with $q(v)^{mr(X,E)}$. This implies that

$$1 = \left(q(v)^{-r(X,E)} \cdot b_v^2\right)^m.$$

However, by Lemma 3.1, $q(v)^{-r(X,E)} b_v^2$ is *not* a root of unity. (Here we use the *oddity* of $r(X,E)$.) We get a contradiction, which proves the Theorem. □

## 4. Superelliptic jacobians

**Proposition 4.1.** *Let $K$ be a number field with the ring of integers $\mathcal{O}_K$. Let $Y$ be an abelian variety of positive dimension over $K$, let $L$ be a CM-field of degree $2\dim(X)$ and $i : L \hookrightarrow \mathrm{End}^0(Y)$ an embedding that sends $1$ to $1_Y$. Let $p$ be a prime that splits completely in $L$, i.e. $L \otimes_{\mathbb{Q}} \mathbb{Q}_p$ splits into a product of $[L:\mathbb{Q}]$ copies of $\mathbb{Q}_p$. Let $\mathfrak{p}$ be maximal ideal in $\mathcal{O}_K$ with residual characteristic $p$.*

*If $Y$ has good reduction at $\mathfrak{p}$ then this reduction is ordinary.*

*Proof.* Let $\bar{\mathbb{Q}}_p$ be an algebraic closure of $\mathbb{Q}_p$. Let $L_\mathfrak{p}$ be the $\mathfrak{p}$-adic completion of $L$. By assumption, $L_\mathfrak{p} = \mathbb{Q}_p$ and therefore the set $H_\mathfrak{p}$ of $\mathbb{Q}_p$-linear field embeddings $L_\mathfrak{p} \hookrightarrow \bar{\mathbb{Q}}_p$ is a singleton that consists of the inclusion map $\mathbb{Q}_p \subset \bar{\mathbb{Q}}_p$; in particular, $\#(H_\mathfrak{p}) = 1$. Now the assertion follows readily from Lemma 5 in Sect. 4 of [16]. □

**Lemma 4.2.** *Let us consider the curve $C_0 : y^3 = x^9 - x$ and its jacobian $J(C_0)$ over $\mathbb{Q}$.*

*Then:*

(i) *If $p$ is a prime such that $p-1$ is divisible by 24 then $J(C_0)$ has ordinary good reduction at $p$.*

(ii) *$J(C_0)$ is a (non-simple) abelian variety of CM-type over $\bar{\mathbb{Q}}$.*

*Proof.* Clearly, both $C_0$ and $J(C_0)$ have good reduction at $p$, because $x^9-x = x(x^8-1)$ has 9 distinct roots in $\mathbb{F}_p$ and therefore has no multiple roots in characteristic $p$. In order to check that $J(C_0)$ has ordinary reduction, pick a number field $F$ such that $F$ contains $\mathbb{Q}(\zeta_{24})$, all endomorphisms of $J(C_0)$ are defined over $F$ and all homomorphisms between $J(C_0)$ and the elliptic curve $y^2 = x^3 - x$ are defined over $F$. Let us consider both $C_0$ and $J(C_0)$ over $F$, and let $\mathfrak{p}$ be a place of $F$ that lies above $p$. For our purposes, it suffices to check that $J(C_0)$ has ordinary reduction at $\mathfrak{p}$.

Pick a primitive cubic root of unity $\zeta_3 \in F$. Then the map

$$(x, y) \mapsto (x, \zeta_3 y)$$

induces an automorphism $\delta_3 : C_0 \to C_0$, which, in turn, induces by Albanese functoriality an automorphism $J(C_0) \to J(C_0)$, which we still denote by $\delta_3$. It is known [8, p. 149] that $\delta_3^2 + \delta_3 + 1 = 0$ in $\mathrm{End}(J(C_0))$, which leads to the embedding

$$\mathbb{Z}[\zeta_3] \hookrightarrow \mathrm{End}(J(C_0)),\ \zeta_3 \mapsto \delta_3, 1 \mapsto 1_{J(C_0)}.$$

Extending it by $\mathbb{Q}$-linearity, we get an embedding

$$\mathbb{Q}(\sqrt{-3}) = \mathbb{Q}(\zeta_3) \hookrightarrow \mathrm{End}^0(J(C_0)),\ \zeta_3 \mapsto \delta_3, 1 \mapsto 1_{J(C_0)}.$$

On the other hand, pick a primitive 8th root of unity $\zeta_8 \in F$. Then the map

$$(x, y) \mapsto (\zeta_8^{-1} x, \zeta_8^{-3} y)$$

induces an automorphism $\delta_8 : C_0 \to C_0$, which commutes with $\delta_3$. Again $\delta_8$ induces by Albanese functoriality an automorphism of $J(C_0)$, which we still denote by $\delta_8$; clearly $\delta_8$ and $\delta_3$ do commute in $\mathrm{End}(J(C_0)$. In order to understand $\delta_8$ better, let us divide both sides of the equation for $C_0$ by $x^9 = (x^3)^3$: we get $(y/x^3)^3 = 1 - (1/x)^8$. It follows that $C$ is $F$-birationally isomorphic to the curve

$$C' : w^8 = -u^3 + 1;\ w = 1/x, u = y/x^3$$

and $\delta_8$ is induced by

$$(u, w) \mapsto (u, \zeta_8 w).$$

This implies that the jacobian $J(C')$ of $C'$ and $J(C)$ are isomorphic over $F$. Let us put $f(w) = -w^3 + 1$. Then in notations of [21], $C' = C_{f,8}$ and the structure of its jacobian $J(C') = J(C_{f,8})$ is described as follows [21, Sect. 5, Cor. 5.12, Rem. 5.14, Th. 5.17 ]. First, $J(C_{f,8})$ contains a $\delta_8$-invariant abelian fourfold

$$J^{(f,8)} = (\delta_8^3 + \delta_8^2 + \delta_8 + 1)(J(C_{f,8})) \subset J(C_{f,8})$$

provided with an embedding

$$\mathbb{Z}[\zeta_8] \hookrightarrow \mathrm{End}(J^{(f,8)}),\ \zeta_8 \mapsto \delta_8, 1 \mapsto 1_{J(C_{f,8})}.$$

Clearly, $J^{(f,8)}$ is $\delta_3$-invariant. This gives rise to an embedding

$$\mathbb{Q}(\zeta_{24}) = \mathbb{Q}(\zeta_3) \otimes \mathbb{Q}(\zeta_8) \hookrightarrow \mathrm{End}^0(J^{(f,8)}),\ \zeta_3 \mapsto \delta_3, \zeta_8 \mapsto \delta_8$$

and 1 goes to the identity map. This implies that $J^{(f,8)}$ is an abelian fourfold of CM-type. Since $p$ splits in $\mathbb{Q}(\zeta_{24})$, it follows from Proposition 4.1 that $J^{(f,8)}$ has ordinary reduction at all places of $F$ over $p$. Second, $J(C_{f,8})$ is isogenous (over $F$) to a product of $J^{(f,8)}$, two copies of the elliptic curve $y^2 = x^3 - x$ and the elliptic curve $w^2 = -v^3 + 1$. Since 24 divides $p-1$, the prime $p$ splits in the imaginary quadratic fields $\mathbb{Q}(\sqrt{-1})$ and $\mathbb{Q}(\sqrt{-3})$. Therefore the CM-elliptic curves $y^2 = x^3 - x$ with multiplication by $\mathbb{Q}(\sqrt{-1})$ and $w^2 = -v^3 + 1$ with multiplication by $\mathbb{Q}(\sqrt{-3})$ have ordinary reduction at $p$. It follows that $J(C_{f,8})$ has ordinary reduction at $\mathfrak{p}$. □

**Example 4.3.** Fix a prime $p$ with $p-1$ divisible by 24, let $K = \mathbb{F}_p(t)$ and let $X$ be the 7-dimensional jacobian of the $K$-curve $C : y^3 = x^9 - x - t$. Since $p$ divides neither 9 nor 8, $x^9 - x \in \mathbb{F}_p[x]$ is a *Morse polynomial* [11, p. 39], i.e., its derivative $9x^8 - 1$ has 8 distinct roots $\beta_1, \dots, \beta_8$ and all eight critical values $\beta_i^9 - \beta_i = -\frac{8}{9}\beta_i$ are distinct. It follows that the Galois group of $x^9 - x - t$ over $\mathbb{F}_p(t)$ is the full symmetric group $\mathbf{S}_9$ [11, p. 41]. On the other hand, if $\zeta \in \mathbb{F}_p$ is a primitive cubic root of unity then

$$(x, y) \mapsto (x, \zeta y)$$

gives rise to a non-trivial automorphism of $C$ (of period 3), which, in turn, allows us to define the embedding

$$\mathbb{Q}(\zeta_3) = \mathbb{Q}(\sqrt{-3}) \hookrightarrow \mathrm{End}^0(J(C)),\ 1 \mapsto 1_{J(C)}.$$

By Theorem 0.1 of [20], $E = \mathbb{Q}(\sqrt{-3})$ coincides with its own centralizer in $\mathrm{End}^0(J(C))$ and therefore contains $\mathfrak{C}(J(C))$. This means that $\mathfrak{C}(J(C)) = E$ or $\mathbb{Q}$. On the other hand, the reduction of $J(C)$ at $t = 0$ is the jacobian of the $\mathbb{F}_p$-curve $y^3 = x^9 - x$, which is ordinary, by Lemma 4.2. Applying Theorem 1.4, we obtain that $\mathfrak{g}_{\ell,J(C)}$ does not contain non-zero homotheties. On the other hand, if $\mathfrak{C}(J(C)) = \mathbb{Q}$ then, by Theorem 1.3(III), $\mathfrak{g}_{\ell,J(C)}$ does contain all the homotheties. This contradiction proves that $\mathfrak{C}(J(C)) = E$ and therefore the centralizer of $E$ coincides with the whole $\mathrm{End}^0(J(C))$. This implies that $\mathrm{End}^0(J(C)) = E$ and therefore $J(C)$ is absolutely simple and is not of CM-type. It follows that $J(C)$ is not isogenous to an abelian variety that is defined over a finite field.

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