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BOUNDS FOR KAKEYA-TYPE MAXIMAL OPERATORS ASSOCIATED WITH k-PLANES

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Abstract. A (d,k) set is a subset of $\mathbb{R}^d$ containing a translate of every k-dimensional plane. Bourgain showed that for $k \geq k_{cr}(d)$, where $k_{cr}(d)$ solves $2^{k_{cr}-1} + k_{cr} = d$, every (d,k) set has positive Lebesgue measure. We give a short proof of this result which allows for an improved L^p estimate of the corresponding maximal operator, and which demonstrates that a lower value of k_{cr} could be obtained if improved mixed-norm estimates for the x-ray transform were known.

1. Introduction

A measurable set $E \subset \mathbb{R}^d$ is said to be a (d,k) set if it contains a translate of every k-dimensional plane in $\mathbb{R}^d$. Once the definition is given, the question of the minimum size of a (d,k) set arises. This question has been extensively studied for the case $k = 1$, the Kakeya sets. It is known that there exist Kakeya sets of measure zero, and these are called Besicovitch sets. It is conjectured that all Besicovitch sets have Hausdorff dimension d. For $k \geq 2$, it is conjectured that (d,k) sets must have positive measure, i.e. that there are no (d,k) Besicovitch sets. These size estimates are related to L^p bounds on two maximal operators which we define below.

Let $G(d,k)$ denote the Grassmannian manifold of k-dimensional linear subspaces of $\mathbb{R}^d$. For $L \in G(d,k)$ we define

$$\mathcal{N}^k[f](L) = \sup_{x\in\mathbb{R}^d} \int_{x+L} f(y)dy$$

where we will only consider functions f supported on the unit ball $B(0,1) \subset \mathbb{R}^d$.

A limiting and rescaling argument shows that if $\mathcal{N}^k$ is bounded for some $p < \infty$ from $L^p(\mathbb{R}^d)$ to $L^1(G(d,k))$, then (d,k) sets must have positive measure. By testing $\mathcal{N}^k$ on the characteristic function of $B(0,\delta)$, $\chi_{B(0,\delta)}$, one sees that such a bound may only hold for $p \geq \frac{d}{k}$. For L in $G(d,k)$ and $a \in \mathbb{R}^d$ define the δ plate centered at a, $L_\delta(a)$, to be the δ neighborhood in $\mathbb{R}^d$ of the intersection of $B(a,\frac{1}{2})$ with $L + a$. Fixing L, considering $\mathcal{N}^k\chi_{L_\delta(0)}$, and using the fact that the dimension of $G(d,k)$ is $k(d-k)$, we see that a bound into $L^q(G(d,k))$ can only hold for $q \leq kp$. This leads to the following conjecture, where the case $k = 1$ is excluded due to the existence of Besicovitch sets.

Conjecture 1.1. *For* $2 \leq k < d, p > \frac{d}{k}, 1 \leq q \leq kp$

$$\|\mathcal{N}^k f\|_{L^q(G(d,k))} \lesssim \|f\|_{L^p(\mathbb{R}^d)}.$$

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It is also useful to consider a generalization of the Kakeya maximal operator, defined for $L \in G(d,k)$ by

$$\mathcal{M}_\delta^k[f](L) = \sup_{a\in\mathbb{R}^d} \frac{1}{\mathcal{L}^d(L_\delta(a))} \int_{L_\delta(a)} f(y)dy$$

where $\mathcal{L}^d$ denotes Lebesgue measure on $\mathbb{R}^d$. Using an argument analogous to that in Lemma 2.15 of [2], one may see that a bound

$$\|\mathcal{M}_\delta^k f\|_{L^1(G(d,k))} \lesssim \delta^{\frac{-\alpha}{p}} \|f\|_{L^p(\mathbb{R}^d)} \tag{1}$$

where $\alpha > 0$ and $p < \infty$, implies that the Hausdorff dimension of any (d,k) set is at least $d-\alpha$. Considering $\mathcal{M}_\delta^k \chi_{B(0,\delta)}$ and $\mathcal{M}_\delta^k \chi_{L_\delta(0)}$, we formulate

Conjecture 1.2. *For $k \geq 1, p < \frac{d}{k}, q \leq (d-k)p'$*

$$\|\mathcal{M}_\delta^k f\|_{L^q(G(d,k))} \lesssim \delta^{k-\frac{d}{p}} \|f\|_{L^p(\mathbb{R}^d)}.$$

In [6] Falconer showed that, for any $\epsilon > 0$, $\mathcal{N}^k$ is bounded from $L^{\frac{d}{k}+\epsilon}(\mathbb{R}^d)$ to $L^1(G(d,k))$ when $k > \frac{d}{2}$. Later, in [2], Bourgain used a Kakeya maximal operator bound combined with an L^2 estimate of the x-ray transform to show that $\mathcal{N}^k$ is bounded from $L^p(\mathbb{R}^d)$ to $L^p(G(d,k))$ for $(d,k,p) = (4,2,2+\epsilon)$ and $(d,k,p) = (7,3,3+\epsilon)$. He then showed, using a recursive metric entropy estimate, that for $d \leq 2^{k-1}+k$, $\mathcal{N}^k$ is bounded for a large unspecified p. Substituting in the proof Katz and Tao's more recent bound for the Kakeya maximal operator from [7]

$$\|\mathcal{M}_\delta^1 f\|_{L^{n+\frac{3}{4}}(G(n,1))} \lesssim \delta^{-\left(\frac{3(n-1)}{4n+3}+\epsilon\right)} \|f\|_{L^{\frac{4n+3}{7}}(\mathbb{R}^n)} \tag{2}$$

one now sees that this holds for $k > k_{cr}(d)$ where

$$k_{cr}(d) \text{ solves } d = \frac{7}{3} 2^{k_{cr}-2} + k_{cr}. \tag{3}$$

By Hölder's inequality, the following is true for any k-plate L_δ and positive f

$$\int_{L_\delta} f\ dx \lesssim \delta^{\frac{d-k}{r'}} \left(\int_{L^\perp} \left(\int_{L+y} f(x)\ d\mathcal{L}^k(x) \right)^r d\mathcal{L}^{d-k}(y) \right)^{\frac{1}{r}}.$$

Combining this with the $L^p \to L^q(L^r)$ bounds for the k-plane transform which were proven by Christ in Theorem A of [4], we see that Conjecture 1.2 holds when $p \leq \frac{d+1}{k+1}$. Except for a factor of $\delta^{-\epsilon}$, the same bound for $\mathcal{M}_\delta^k$ was proven with $k=2$ by Alvarez in [1] using a geometric-combinatorial "bush"-type argument. Alvarez also used a "hairbrush" argument to show that $(d,2)$ sets have Minkowski dimension at least $\frac{2d+3}{3}$. More recently, Mitsis proved a similar maximal operator bound in [11] and showed that $(d,2)$ sets have Hausdorff dimension at least $\frac{2d+3}{3}$ in [10]. In [3], Bueti used a hairbush type argument to show that, in the setting of vector spaces over finite fields, Conjecture 1.2 holds when $p \leq \frac{d+1+\frac{1}{k}}{k+1}$ and $k < d-1$. In [13], Rogers gave estimates for the Hausdorff dimension of sets which contain planes in directions corresponding to certain curved submanifolds of $G(4,2)$.

Our main result is the following.

Theorem 1.1. *Suppose $4 \leq k < d$ and $k > k_{cr}(d)$, where $k_{cr}(d)$ is defined in (3). Then*

$$\|\mathcal{N}^k f\|_{L^p(G(d,k))} \lesssim \|f\|_{L^p(\mathbb{R}^d)} \tag{4}$$

for f supported on the unit ball and $p \geq \frac{d-1}{2}$. If, additionally, we have $k-j > k_{cr}(d-j)$ for some integer j in $[1, k-4]$, then we may take $p \geq \frac{d-1}{2+j}$.

For $k < k_{cr}(d)$, we do not have a bound for $\mathcal{N}^k$, however our technique yields certain bounds for $\mathcal{M}_\delta^k$.

Theorem 1.2.

$$\|\mathcal{M}_\delta^k f\|_{L^q(G(d,k))} \lesssim \delta^{-\frac{\alpha}{p}} \|f\|_{L^p(\mathbb{R}^d)}$$

when

$$k \geq 2,\ \alpha = d - kp + \epsilon,\ p = \frac{d}{k + \frac{3}{4}},\ q \leq (d-k)\left(\frac{4(d-(k-1))}{7}\right)' \tag{5}$$

or

$$k \geq 2,\ \alpha = \frac{3(d-k)}{7(2^{k-1})} + \epsilon,\ p = \frac{d+1}{2},\ q = d+1 \tag{6}$$

or

$$3 \leq k \leq k_{cr}(d),\ \alpha = \frac{3(d-k)}{7(2^{k-2})} - 1 + \epsilon,\ p = q = \frac{d}{2} \tag{7}$$

where $\epsilon > 0$ may be taken arbitrarily small.

In (5) we have an optimal value for p relative to α, but a non-optimal value for q. In (6) and (7) we have improved values of α at the cost of a non-optimal p. For the "non-borderline" k, specifically when $k+1 < k_{cr}(d+1)$, (6) gives a smaller value of α than (7).

The number $p = \frac{d-1}{2+j}$ in Theorem 1.1 and the number $p = \frac{d}{k+\frac{3}{4}}$ in Theorem 1.2 are approximate and may be slightly improved through careful numerology. Also, in (7) we may take $k = 2$, but a slightly higher value of p and q is then required.

We prove (5) and (6) in Section 2 through a recursive maximal operator bound which is derived using Drury and Christ's bounds for the x-ray transform and which is inspired by Bourgain's recursive metric entropy estimates. This recursive maximal operator bound is a slight improvement of the result in [12], which will remain unpublished, and the new bound comes with a vastly simplified proof afforded by the explicit use of the x-ray transform. Additionally our argument reveals that with certain adjustments of p and q, the number 2 in the definition of $k_{cr}(d)$ and in the definition of α in (6) and (7) may be replaced by the ratio $\frac{\tilde{r}}{\tilde{p}}$ if the x-ray transform is known to be bounded, for certain values of n, from $L^{p_n}(\mathbb{R}^n)$ to $L^{q_n}_{\mathbb{S}^{n-1}}(L^{r_n}_{\mathbb{R}^{n-1}})$ for any r_n, p_n, q_n satisfying $\frac{r_n}{p_n} = \frac{\tilde{r}}{\tilde{p}}$.

We prove (7) and Theorem 1.1 in Section 3. There, we combine (5) and (6) with the L^2 method which Bourgain used to give bounds for $\mathcal{N}^k$ when $(d,k) = (4,2)$ or $(7,3)$.

From (6) and (7) we see that, for $k \geq 2$, the Hausdorff dimension of any (d,k) set is at least

$$\min\left(d, \max\left(d - \frac{3(d-k)}{7(2^{k-2})} + 1, d - \frac{3(d-k)}{7(2^{k-1})}\right)\right).$$

When $(d-k) < 7$, it is preferable to start with Wolff's $L^{\frac{n+2}{2}}$ bound for the Kakeya maximal operator from [15], instead of (2). A similar procedure then gives the lower bound

$$\min\left(d, \max\left(d - \frac{d-k-1}{2^{k-1}} + 1, d - \frac{d-k-1}{2^k}\right)\right)$$

for the Hausdorff dimension of a (d,k) set.

It should be noted that the dimension estimates provided by applying (6) and it's Wolff-variant are also a direct consequence of the metric entropy estimates in [2]. However, to the best of the author's knowledge they have not previously appeared in the literature, even without the improvement obtained from [16] and [7].

2. A recursive maximal operator bound

We start with the definition of the measure we will use on $G(d,k)$. Fix any $L \in G(d,k)$. For a Borel subset F of $G(d,k)$ let

$$\mathcal{G}^{(d,k)}(F) = \mathcal{O}(\{\theta \in O(d) : \theta(L) \in F\})$$

where $\mathcal{O}$ is normalized Haar measure of the orthogonal group on $\mathbb{R}^d$, $O(d)$. By the transitivity of the action of $O(d)$ on $G(d,k)$ and the invariance of $\mathcal{O}$, it is clear that the definition is independent of the choice of L. Also note that $\mathcal{G}^{(d,k)}$ is invariant under the action of $O(d)$. By the uniqueness of uniformly-distributed measures (see [9], pages 44-53), $\mathcal{G}^{(d,k)}$ is the unique normalized Radon measure on $G(d,k)$ invariant under $O(d)$.

It will be necessary to use an alternate formulation of $\mathcal{G}^{(d,k)}$. For each ξ in $\mathbb{S}^{d-1}$ let $T_\xi : \xi^\perp \to \mathbb{R}^{d-1}$ be an orthogonal linear transformation. Then T_ξ^{-1} identifies $G(d-1,k-1)$ with the $k-1$ dimensional subspaces of $\xi^\perp$. Now, define $T : \mathbb{S}^{d-1} \times G(d-1,k-1) \to G(d,k)$ by

$$T(\xi, M) = \operatorname{span}(\xi, T_\xi^{-1}(M)).$$

Choosing T_ξ continuously on the upper and lower hemispheres of $\mathbb{S}^{d-1}$, T^{-1} identifies the Borel subsets of $G(d,k)$ with the completion of the Borel subsets of $\mathbb{S}^{d-1} \times G(d-1,k-1)$. Under this identification, by uniqueness of rotation invariant measure, we have

$$\mathcal{G}^{(d,k)}(F) = \sigma^{d-1} \times \mathcal{G}^{(d-1,k-1)}(T^{-1}(F)) \tag{8}$$

where σ^{d-1} denotes normalized surface measure on the unit sphere.

For a function f on $\mathbb{R}^d$, $\xi \in \mathbb{S}^{d-1}$, and $y \in \xi^\perp$, the x-ray transform of f is defined

$$f_\xi(y) = \int_{\mathbb{R}} f(y + t\xi)\, dt.$$

It is conjectured that the x-ray transform is bounded from $L^p(\mathbb{R}^d)$ to $L^q_{\mathbb{S}^{d-1}}(L^r_{\mathbb{R}^{d-1}})$ when p, q, r satisfy

$$\begin{aligned} r &< \infty \\ p &= \frac{rd}{d+r-1} \\ q &\le r'd. \end{aligned} \tag{9}$$

This was shown to hold in [5] for $p < \frac{d+1}{2}$ and in [4] for $p = \frac{d+1}{2}$. Also, see [16] and [8] for certain improvements.

In the following proposition we exploit the fact that $r > p$ when $r \neq 1$ in (9), i.e. that the x-ray transform is L^p-improving.

Proposition 2.1. *Suppose that $p \leq d+1$ and $k \geq 2$. Then a bound*

$$\|\mathcal{M}_\delta^{k-1} f\|_{L^q(G(d-1,k-1))} \lesssim \delta^{-\frac{\alpha}{p}} \|f\|_{L^p(\mathbb{R}^{d-1})}$$

for all $f \in L^p(\mathbb{R}^{d-1})$ implies the bound

$$\|\mathcal{M}_\delta^{k} f\|_{L^{\tilde{q}}(G(d,k))} \lesssim \delta^{-\frac{\tilde{\alpha}}{\tilde{p}}} \|f\|_{L^{\tilde{p}}(\mathbb{R}^{d})}$$

for all $f \in L^{\tilde{p}}(\mathbb{R}^d)$ with

$$\tilde{p} = p\frac{d}{d+p-1}, \quad \tilde{\alpha} = \alpha\ \frac{\tilde{p}}{p} = \alpha\ \frac{d}{d+p-1}, \quad \textit{and} \quad \tilde{q} = \min(q, dp').$$

Proof. Without loss of generality, we assume that f is positive. Let $L \in G(d,k)$ and suppose that $L = \operatorname{span}(\xi, T_\xi^{-1}(M))$ where $M \in G(d-1,k-1)$. Let $a_L \in \mathbb{R}^d$ and let $a_M = T_\xi(\operatorname{proj}_{\xi^\perp}(a_L))$, where proj denotes orthogonal projection. Then

$$\begin{aligned}\int_{L_\delta(a_L)} f(y)\ dy &\leq \int_{M_\delta(a_M)} \int_{\mathbb{R}} f(T_\xi^{-1}(x) + t\xi)\ dt\ dx \\ &= \int_{M_\delta(a_M)} f_\xi(T_\xi^{-1}(x))\ dx\end{aligned}$$

where $L_\delta(a_L)$ and $M_\delta(a_M)$ are k and $k-1$ plates respectively. Noting that $d-k = (d-1)-(k-1)$, it follows that

$$\mathcal{M}_\delta^k[f](L) \lesssim \mathcal{M}_\delta^{k-1}[f_\xi \circ T_\xi^{-1}](M).$$

By (8), Hölder's inequality, and our hypothesized bound, we now have

$$\begin{aligned}\|\mathcal{M}_\delta^k[f]\|_{L^{\tilde{q}}(G(d,k))} &\lesssim \left(\int_{\mathbb{S}^{d-1}} \int_{G(d-1,k-1)} \mathcal{M}_\delta^{k-1}[f_\xi \circ T_\xi^{-1}](M)^{\tilde{q}}\ dM\ d\xi\right)^{\frac{1}{\tilde{q}}} \\ &\lesssim \left(\int_{\mathbb{S}^{d-1}} \left(\int_{G(d-1,k-1)} \mathcal{M}_\delta^{k-1}[f_\xi \circ T_\xi^{-1}](M)^{q}\ dM\right)^{\frac{\tilde{q}}{q}} d\xi\right)^{\frac{1}{\tilde{q}}} \\ &\lesssim \delta^{-\frac{\alpha}{p}} \left(\int_{\mathbb{S}^{d-1}} \left(\int_{\mathbb{R}^{d-1}} (f_\xi \circ T_\xi^{-1}(x))^p\ dx\right)^{\frac{\tilde{q}}{p}} d\xi\right)^{\frac{1}{\tilde{q}}} \\ &= \delta^{-\frac{\alpha}{p}} \left(\int_{\mathbb{S}^{d-1}} \left(\int_{\xi^\perp} f_\xi(x)^p\ dx\right)^{\frac{\tilde{q}}{p}} d\xi\right)^{\frac{1}{\tilde{q}}}.\end{aligned}$$

Finally, by our restrictions on p and $\tilde{q}$, we may apply Drury and Christ's bound for the x-ray transform, obtaining

$$\left(\int_{\mathbb{S}^{d-1}} \left(\int_{\xi^\perp} f_\xi(x)^p\ dx\right)^{\frac{\tilde{q}}{p}} d\xi\right)^{\frac{1}{\tilde{q}}} \lesssim \|f\|_{L^{\tilde{p}}(\mathbb{R}^d)}$$

when $\tilde{p} = \frac{pd}{d+p-1}$.

□

One should note that if $\alpha = (d-1) - (k-1)p$, then $\tilde{\alpha} = d - k\tilde{p}$. Hence, except for a non-optimal $\tilde{q}$, Proposition 2.1 yields the conjectured bound on $L^{\tilde{p}}(\mathbb{R}^d)$ when applied to the conjectured bound on $L^p(\mathbb{R}^{d-1})$.

Proof of (5). Observing that if

$$p = \frac{d-1}{m} \quad \text{then} \quad \tilde{p} = \frac{(d+1)-1}{m+1}, \tag{10}$$

we start from the bound

$$\|\mathcal{M}^1_\delta f\|_{L^{(n-1)\left(\frac{4n}{7}\right)'}} \lesssim \delta^{-\left(\frac{3}{4}+\epsilon\right)} \|f\|_{L^{\frac{4n}{7}}(\mathbb{R}^n)} \tag{11}$$

with $n = d-(k-1)$, which is weaker but more convenient for numerology than (2). Since (11) satisfies the left side of (10) with $m = \frac{7}{4}$ and $d = n+1$, we obtain (5) after $k-1$ iterations of Proposition 2.1. □

For a larger improvement in α, one may interpolate the known L^p bound for $\mathcal{M}^{k-1}_\delta$ with the trivial L^∞ bound and apply Proposition 2.1 to the resulting L^{d+1} bound. This allows us to use the maximum value, 2, of $\frac{r}{p}$ permitted by Drury and Christ's bound, and yields the following corollary.

Corollary 2.1. *Under the assumptions of Proposition 2.1, we may also take* $\tilde{p} = \frac{d+1}{2}$*,* $\tilde{\alpha} = \frac{\alpha}{2}$*, and* $\tilde{q} = \min(\frac{(d+1)q}{p}, (d+1))$*.*

Due to the interpolation, Corollary 2.1 cannot yield a bound for which α is sharp with respect to p as in Conjecture 1.2.

Proof of (6). Starting from (2) with $n = d-(k-1)$, we iteratively apply Corollary 2.1 $(k-1)$ times to obtain (6). □

We would like to point out that the proof of Proposition 2.1 and Corollary 2.1 is similar in spirit to Bourgain's recursive metric entropy estimate in the sense that a more efficient version of his technique, namely the proof of Proposition 3.1 in [12], could be used to derive the localized non-endpoint version of the $L^{\frac{d+1}{2}} \to L^{d+1}$ x-ray transform bound. The idea of expressing an average over a k-plane as the average over a $k-1$-plane of the x-ray transform and then "unraveling" the integration over $G(d,k)$ into a product integral over $\mathbb{S}^{d-1}$ and $G(d-1,k-1)$ is also due to Bourgain, as he used it in Propositions 3.3 and 3.20 of [2]. There, he gave bounds for $\mathcal{N}^k$ with $(d,k) = (4,2)$ and $(d,k) = (7,3)$. We state a generalization of these results below.

3. The L^2 method

Reducing α by a factor of two, as in Corollary 2.1, is not a substantial gain for small α. By using an L^2 estimate of the x-ray transform which takes advantage of cancellation, instead of the $L^{\frac{d+1}{2}}$ bound, we may take $\tilde{\alpha} = \alpha - 1$ when $\alpha \geq 1$ and obtain a bound for $\mathcal{N}^k$ when $\alpha < 1$.

Proposition 3.1. *Suppose $k, p \geq 2$ and that a bound for $\mathcal{M}_\delta^{k-1}$ on $L^p(\mathbb{R}^{d-1})$ of the form*

$$\|\mathcal{M}_\delta^{k-1} f\|_{L^p(G(d-1,k-1))} \lesssim \delta^{-\frac{\alpha}{p}} \|f\|_{L^p(\mathbb{R}^{d-1})} \tag{12}$$

is known. Then if $\alpha \geq 1$ we have the bound

$$\|\mathcal{M}_\delta^k f\|_{L^p(G(d,k))} \lesssim \delta^{-\frac{\alpha-1}{p}} \|f\|_{L^p(\mathbb{R}^d)} \tag{13}$$

for $f \in L^p(\mathbb{R}^d)$. If $\alpha < 1$ we have the bound

$$\|\mathcal{N}^k f\|_{L^p(G(d,k))} \lesssim \|f\|_{L^p(\mathbb{R}^d)} \tag{14}$$

for $f \in L^p(\mathbb{R}^d)$ supported on $B(0,1)$.

Before proving the proposition, we give its applications.

Proof of Theorem 1.1. We start from the bound (6) with $d_0 = d - 2 - j$ and $k_0 = k - 2 - j$. This gives

$$\alpha_0 = \frac{3(d-k)}{7 \cdot 2^{k-3-j}} + \epsilon, \; p_0 = \frac{d_0+1}{2}, \quad \text{and } q_0 = d_0 + 1. \tag{15}$$

The condition $k - j > k_{cr}(d - j)$ ensures that $\alpha_0 < 2$, and so no further improvement in α is necessary. Thus, we use our j "spare" iterations to improve p. We note that, in Proposition 2.1, when $m \leq d$,

$$p \leq \frac{d}{m} \quad \text{implies that} \quad \tilde{p} \leq \frac{d+1}{m+1}. \tag{16}$$

Since p_0 satisfies the left inequality in (16) with $m = 2$ and $d = d_0 + 1$, we see that we may take

$$p_1 = \frac{d_1+1}{3}, \; q_1 = d_0 + 1, \quad \text{and } \alpha_1 = \alpha_0,$$

where $d_1 = d_0+1 = d-2-(j-1)$ and $k_1 = k_0-1 = k-2-(j-1)$. Above, we ignore the improvement in α and, through interpolation, we ignore some slight additional improvement in p. After $j-1$ further iterations, we have

$$p_j = \frac{d_j+1}{2+j}, \; q_j = d_0 + 1, \quad \text{and } \alpha_j = \alpha_0, \tag{17}$$

where $d_j = d - 2$ and $k_j = k - 2$. Applying (13) to (17), and then applying (14) to the result, we obtain (4). □

Proof of (7). We obtain (7) by starting from (6) with $d_0 = d - 1$, and $k_0 = k - 1$ (In the case $k = 2$, we would simply start from (2)). We then apply (13) once. □

The main estimate needed to derive Proposition 3.1 was proven by Smith and Solmon in [14].

Lemma 3.1. *For $d \geq 3$*

$$\|f_\xi(y)\|_{L^2_{\xi,y}(\mathbb{S}^{d-1}\times\mathbb{R}^{d-1})} = C_d \|f\|_{\dot{H}^{-\frac{1}{2}}(\mathbb{R}^d)}$$

where C_d is a fixed constant depending only on d and $\dot{H}$ denotes the homogeneous L^2 Sobolev space.

It immediately follows that if the Fourier transform $\hat{g}$ of a function g is identically 0 on $B(0,R)$ then

$$\|g_\xi(y)\|_{L^2_{\xi,y}(\mathbb{S}^{d-1}\times\mathbb{R}^{d-1})} \lesssim R^{-\frac{1}{2}}\|g\|_{L^2(\mathbb{R}^d)}. \tag{18}$$

To effectively apply (18), we use a Littlewood-Paley decomposition. Let ϕ_0 be a Schwartz function with $\hat{\phi}_0 \equiv 1$ on $B(0,1)$ and with $\hat{\phi}_0$ supported on $B(0,2)$. For $j>0$, define $\phi_j = 2^{jd}\phi_0(2^j\cdot) - 2^{(j-1)d}\phi_0(2^{j-1}\cdot)$ so that $\hat{\phi}_j$ is supported on $B(0,2^{j+1})\setminus B(0,2^{j-1})$. Functions are decomposed

$$f = \sum_{j=0}^{\infty} f_j$$

where $f_j = f * \phi_j$.

Our last ingredients are two Schwartz-tail estimates needed to reconcile the localization properties of the space and frequency variables.

Lemma 3.2. *Suppose $g \geq 0$ and $\hat{\tilde{g}} = \hat{g}$ on $B(0,\frac{1}{\delta})$. Then*

$$\mathcal{M}_\delta^{k-1}[g] \lesssim \mathcal{M}_\delta^{k-1}[|\tilde{g}|]. \tag{19}$$

Proof. For $1 \leq n \leq d$ let Φ^n be a nonnegative Schwartz function on $\mathbb{R}^n$ such that $\Phi^n \geq 1$ on $B(0,1)$ and $\hat{\Phi}^n$ is supported on $B\left(0,\frac{1}{\sqrt{2}}\right)$. For $L \in G(d,k)$ let

$$\pi_{L,\delta}(x) = \Phi^k(\mathrm{proj}_L(x))\delta^{-(d-k)}\Phi^{d-k}\left(\mathrm{proj}_{L^\perp}\left(\frac{x}{\delta}\right)\right).$$

Now, define

$$\widetilde{\mathcal{M}}_\delta^k[f](L) = \sup_{a\in\mathbb{R}^d}\int_{\mathbb{R}^d}\pi_{L,\delta}(x-a)f(x)dx.$$

By construction, $\pi_{L,\delta}(\cdot - a) \gtrsim \frac{\chi_{L_\delta(a)}}{\mathcal{L}^d(L_\delta(a))}$ and $\hat{\pi}_{L,\delta}$ is supported on $B(0,\frac{1}{\delta})$. Thus

$$\mathcal{M}_\delta^k[g] \lesssim \widetilde{\mathcal{M}}_\delta^k[g] = \widetilde{\mathcal{M}}_\delta^k[\tilde{g}].$$

Since Φ^k and Φ^{d-k} are Schwartz functions, we have

$$\Phi^k \leq \sum_{j=1}^{\infty} c_j \chi_{B(y_j,\frac{1}{2})}$$

and

$$\Phi^{d-k} \leq \sum_{j=1}^{\infty} d_j \chi_{B(z_j,\frac{1}{2})}$$

for some $\{c_j\},\{d_j\} \in l^1(\mathbb{N})$, $\{y_j\}\subset\mathbb{R}^k$, and $\{z_j\}\subset\mathbb{R}^{d-k}$. Then, for an appropriately chosen $\{a_{j,l}\}$

$$\begin{aligned}\pi_{L,\delta}(x) &\leq \sum_{j,l=1}^{\infty} c_j d_l \chi_{B(y_j,\frac{1}{2})}(\mathrm{proj}_L(x))\delta^{-(d-k)}\chi_{B(z_l,\frac{1}{2})}\left(\mathrm{proj}_{L^\perp}\left(\frac{x}{\delta}\right)\right)\\ &\lesssim \sum_{j,l=1}^{\infty} c_j d_l \frac{\chi_{L_\delta(a_{j,l})}}{\mathcal{L}^d(L_\delta(a_{j,l}))}.\end{aligned}$$

Thus,

$$\widetilde{\mathcal{M}}_\delta^k[\tilde g] \lesssim \sum_{j,l=1}^\infty c_j, d_l \mathcal{M}_\delta^k[|\tilde g|] \lesssim \mathcal{M}_\delta^k[|\tilde g|].$$

□

Lemma 3.3. *Define*

$$\mathcal{N}_{\text{loc}}^{k-1}[g](L) = \sup_{a\in\mathbb{R}^d} \int_{a+(L\cap B(0,\frac12))} g(x)dx.$$

Suppose $\hat g$ is supported on $B(0, \frac{1}{\delta})$. Then

$$\mathcal{N}_{\text{loc}}^{k-1}[|g|] \lesssim \mathcal{M}_\delta^{k-1}[|g|].$$

Proof. Since $\hat g$ is supported on $B(0, \frac{1}{\delta})$,

$$g = g * \delta^{-d}\phi_0\left(\frac{\cdot}{\delta}\right)$$

and so

$$\int_{a+(L\cap B(0,\frac12)} |g(x)|dx \le \int_{\mathbb{R}^d} \left|\delta^{-d}\phi_0\left(\frac{y}{\delta}\right)\right| \int_{a+y+(L\cap B(0,\frac12))} |g(x)|dx\ dy.$$

Since ϕ_0 is a Schwartz function,

$$|\phi_0| \le \sum_{j=1}^\infty c_j \chi_{B(y_j,\frac12)}$$

for some $\{c_j\} \in l^1(\mathbb{N})$ and $\{y_j\} \subset \mathbb{R}^d$. Thus

$$\begin{aligned}
&\int_{\mathbb{R}^d} \left|\delta^{-d}\phi_0\left(\frac{y}{\delta}\right)\right| \int_{a+y+(L\cap B(0,\frac12))} |g(x)|dx\ dy \\
&\le \sum_{j=1}^\infty c_j \delta^{-d} \int_{B(\delta y_j,\frac{\delta}{2})} \int_{a+y+(L\cap B(0,\frac12))} |g(x)|dx\ dy \\
&\lesssim \sum_{j=1}^\infty c_j \mathcal{M}_\delta^{k-1}[|g|](L) \\
&\lesssim \mathcal{M}_\delta^{k-1}[|g|](L).
\end{aligned}$$

□

Proof of Proposition 3.1. We begin by proving (13). Averaging over each $L_\delta(a)$ is local and we are proving an $L^p \to L^q(L^r)$ bound where $p \le q \le r$, so we may assume that f is supported on the unit ball. Additionally, assume that f is nonnegative.

Following the proof of Proposition 2.1, we observe that for $L = \text{span}(\xi, T_\xi^{-1}(M))$ we have

$$\mathcal{M}_\delta^k[f](L) \lesssim \mathcal{M}_\delta^{k-1}[f_\xi \circ T_\xi^{-1}](M). \tag{20}$$

Since f is supported on the unit ball, we may switch the order of integration between convolution and the x-ray transform to obtain

$$\|(f_j)_\xi(y)\|_{L^\infty_{\xi,y}} \lesssim \|f\|_{L^\infty(\mathbb{R}^d)}$$

uniformly in j. Hence, interpolation with (18) gives

$$\|(f_j)_\xi(y)\|_{L^p_{\xi,y}} \lesssim (2^{-j})^{\frac{1}{p}} \|f\|_{L^p(\mathbb{R}^d)} \tag{21}$$

for any $p \geq 2$.

From Lemma 3.2, we obtain

$$\mathcal{M}_\delta^{k-1}[f_\xi \circ T_\xi^{-1}](M) \lesssim \sum_{j=0}^{|\log(\delta)|+1} \mathcal{M}_\delta^{k-1}[|(f_j)_\xi \circ T_\xi^{-1}|](M). \tag{22}$$

Averaging Lemma 3.3 gives, for each j,

$$\mathcal{M}_\delta^{k-1}[|(f_j)_\xi \circ T_\xi^{-1}|](M) \lesssim \mathcal{M}_{2^{-j}}^{k-1}[|(f_j)_\xi \circ T_\xi^{-1}|](M). \tag{23}$$

Integrating over $G(d,k)$ and combining the bounds (12) and (21) as in the proof of Proposition 2.1, we obtain

$$\|\mathcal{M}_\delta^k f\|_{L^p(G(d,k))} \lesssim \sum_{j=0}^{|\log\delta|+1} (2^j)^{\frac{\alpha-1}{p}} \|f\|_{L^p(\mathbb{R}^d)} \lesssim \delta^{-\frac{\alpha-1}{p}} \|f\|_{L^p(\mathbb{R}^d)}$$

from (20), (22), and (23), when $\alpha \geq 1$.

Proceeding to the proof of (14), we have f supported on the unit ball and we assume that f is nonnegative, giving

$$\mathcal{N}^k[f] \lesssim \mathcal{N}_{\text{loc}}^k[f].$$

As before,

$$\mathcal{N}_{\text{loc}}^k[f](L) \lesssim \mathcal{N}_{\text{loc}}^{k-1}[f_\xi \circ T_\xi^{-1}](M)$$

and

$$\mathcal{N}_{\text{loc}}^{k-1}[|(f_j)_\xi \circ T_\xi^{-1}|](M) \lesssim \mathcal{M}_{2^{-j}}^{k-1}[|(f_j)_\xi \circ T_\xi^{-1}|](M),$$

giving

$$\|\mathcal{N}^k f\|_{L^p(G(d,k))} \lesssim \sum_{j=0}^{\infty} (2^j)^{\frac{\alpha-1}{p}} \|f\|_{L^p(\mathbb{R}^d)} \lesssim \|f\|_{L^p(\mathbb{R}^d)}$$

when $\alpha < 1$. □

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