

## SESHADRI CONSTANTS ON SYMMETRIC PRODUCTS OF CURVES

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ABSTRACT. Let  $X_g = C_g^{(2)}$  be the second symmetric product of a very general curve of genus  $g$ . We reduce the problem of describing the ample cone on  $X_g$  to a problem involving the Seshadri constant of a point on  $X_{g-1}$ . Using this we recover a result of Ciliberto-Kouvidakis that reduces finding the ample cone of  $X_g$  to the Nagata conjecture when  $g \geq 9$ . We also give new bounds on the the ample cone of  $X_g$  when  $g = 5$ .

### 1. Introduction

Consider the second symmetric product  $C^{(2)}$  of a smooth curve  $C$  of genus  $g \geq 2$ . This smooth surface comes with some naturally defined divisors. Given a point  $p \in C$  there is the divisor  $x_p = \{p + q : q \in C\}$  whose numerical class is independent of  $p$  and will be denoted by  $x$ . Another divisor on  $C^{(2)}$  is given by the diagonal  $\Delta = \{p + p | p \in C\}$  whose numerical class is denoted by  $\delta$ .

We will be interested in describing the the intersection  $N$  of the ample cone with the plane in  $N^1(C^{(2)})_{\mathbb{R}}$  spanned by  $x$  and  $\delta$ . Note that when  $C$  is a very general curve the classes  $x$  and  $\delta/2$  generate  $N^1(C^{(2)})$  so in this case  $N$  is the entire ample cone of  $C^{(2)}$ . Since  $N$  is a two dimensional cone it is described by two boundary rays. The first boundary is easily given: since the diagonal  $\Delta$  is an irreducible curve of negative self-intersection it spans a boundary of the effective cone, so its dual ray is one boundary. The more interesting boundary of  $N$  is characterised by the quantity

$$\tau(C) = \inf \{s > 0 : (s + 1)x - (\delta/2) \text{ is ample}\}.$$

There is the obvious universal bound

$$\tau(C) \geq \sqrt{g},$$

coming from the fact that if  $(s + 1)x - (\delta/2)$  is ample then it has positive self-intersection. The following conjecture governs the ample cone of a  $C^{(2)}$  when  $C$  is very general.

**Conjecture 1.1.** *If  $C$  is a very general curve of genus  $g \geq 4$  then  $\tau(C) = \sqrt{g}$ .*

This conjecture asserts that for a very general curve  $C$  the other boundary of  $N$  has zero-self intersection, and thus the ample cone of  $C^{(2)}$  is as large as possible. It is only known to hold when  $g$  is a perfect square [8].

Our aim is to give a lower bound for  $\tau(C)$  in terms of the Seshadri constant of a point in  $D^{(2)}$  where  $D$  is a smooth curve of genus  $g - 1$ . If  $X$  is a smooth surface and

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$L$  is the (numerical class of) a nef  $\mathbb{R}$ -divisor on  $X$  the Seshadri constant at a collection of distinct points  $p_1, \dots, p_m \in X$  is defined to be

$$\epsilon(p_1, \dots, p_m; X, L) = \inf_C \left\{ \frac{L.C}{\sum_i \text{mult}_{p_i} C} \right\},$$

where the infimum is over all reduced irreducible curves  $C \subset X$  passing through at least one of the  $p_i$ . We will prove the following connecting Seshadri constants and the ample cone of second symmetric products of smooth curves.

**Theorem 1.2.** *Let  $D$  be a smooth curve of genus  $g - 1$ . Suppose  $a, b > 0$  are such that  $a/b > \tau(D)$  and for a very general point  $p \in D^{(2)}$*

$$\epsilon\left(p; D^{(2)}, (a+b)x - b(\delta/2)\right) \geq b.$$

Then for a very general curve  $C$  of genus  $g$ ,

$$\tau(C) \leq \frac{a}{b}.$$

Thus Conjecture 1.1 is implied by the following conjecture about Seshadri constants:

**Conjecture 1.3.** *Let  $D$  be a very general curve of genus  $g - 1$  with  $g \geq 5$  and  $p$  be a very general point in  $D^{(2)}$ . Then*

$$\epsilon\left(p; D^{(2)}, (\sqrt{g} + 1)x - (\delta/2)\right) = 1. \quad (1.4)$$

We note that this conjecture is not easy to prove as the class  $L = (\sqrt{g} + 1)x - (\delta/2)$  has degree  $L^2 = 1$  so the equality in (1.4) asserts that the Seshadri constant of this  $\mathbb{R}$ -divisor is maximal (see 2.5). There is a general lower bound due to Ein-Lazarsfeld [6] for the Seshadri constant of very general points in surfaces with respect to *integral* divisors but this does not extend to the case of  $\mathbb{R}$ -divisors. However when  $g$  is a perfect square we can apply [6] to deduce that  $\epsilon(p; D^{(2)}, L) = 1$  for a very general  $p \in D^{(2)}$  where  $D$  is a smooth curve of genus  $g - 1$ . Thus we get another proof that if  $g \geq 4$  is a perfect square then  $\tau(C) = \sqrt{g}$  for a very general curve  $C$  of genus  $g$ .

**Remark 1.5.** As pointed out by Lazarsfeld, it is not the case that the analogy of Conjecture 1.3 holds for all polarised surfaces  $(X, L)$  such that  $L$  is an ample  $\mathbb{R}$ -divisor with  $L^2 = 1$  and  $L.E \geq 1$  for all but finitely many curves  $E \subset X$ . For example let  $X$  be an abelian surface of type  $(1, d)$  with Picard number 1 generated by the ample line bundle  $L'$  and set  $L = L'/\sqrt{2d}$ . Then  $L^2 = 1$  and  $L.E \geq 1$  for all irreducible curves  $E \subset X$ . But whenever  $\sqrt{2d}$  is irrational it is known that  $\epsilon(p, X; L') < \sqrt{2d}$  (in fact it is rational [2]) so  $\epsilon(p; X, L) < 1$  for all  $p \in X$ .

The proof of Theorem 1.2 uses a degeneration of the symmetric product that arises from letting  $C$  degenerate to the nodal curve  $C_0$  obtained by gluing two points in  $D$ . The same degeneration allows us to compare the multipoint Seshadri constants of  $C^{(2)}$  and  $D^{(2)}$ . We define  $\epsilon_m(X, L)$  to be the Seshadri constant of a collection of  $m$  very general points in  $X$ .

**Theorem 1.6.** *Let  $D$  be a smooth curve of genus  $g - 1$  and fix an integer  $m \geq 1$ . Suppose that there are numbers  $a, b > 0$  with  $a/b > \tau(D)$  such that*

$$\epsilon_{m+1} \left( D^{(2)}, (a+b)x - b(\delta/2) \right) \geq b.$$

*Then for a very general curve  $C$  of genus  $g$  the class  $(a+b)x - b(\delta/2) \in N^1(C^{(2)})$  is nef and*

$$\epsilon_m \left( C^{(2)}, (a+b)x - b(\delta/2) \right) \geq b.$$

A more concise way to state this theorem is to let  $\epsilon_{m,g}(s)$  be the Seshadri constant of  $m$  very general points in  $C^{(2)}$  with respect to the class  $(s+1)x - \delta/2$ , where  $C$  is a very general curve of genus  $g \geq 0$ . Then

$$\epsilon_{m,g}(s) \geq \epsilon_{m+1,g-1}(s),$$

where this is to be interpreted as holding for all  $s, g$  such that the right hand makes sense.

As the second symmetric product of  $\mathbb{P}^1$  is  $\mathbb{P}^2$ , induction on  $g$  yields:

**Corollary 1.7** (Ciliberto-Kouvidakis [4]). *Let  $C$  be a very general curve of genus  $g \geq 1$ . Then*

$$\tau(C) \leq \frac{1}{\epsilon_g(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))}.$$

As is well known, one formulation of the Nagata conjecture states that if  $g \geq 9$  then the Seshadri constant of  $g \geq 9$  very general points in  $\mathbb{P}^2$  is maximal:

**Conjecture 1.8** (Nagata Conjecture). *If  $g \geq 9$  then*

$$\epsilon_g(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)) = \frac{1}{\sqrt{g}}.$$

Thus, as proved in [4], the Nagata conjecture yields the ample cone of  $C^{(2)}$  for a very general curve of genus  $g \geq 9$ . Currently the Nagata conjecture is only proved when  $g$  is a perfect square. For other  $g$  there are several bounds on  $\epsilon_g(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$  (e.g. [7, 13, 14, 15]). For instance using (1.7) and results from [13] we get that for a very general curve of genus  $g \geq 10$ ,

$$\tau(C) \leq \frac{\sqrt{g}}{\sqrt{1 - \frac{1}{8g}}}. \tag{1.9}$$

When  $g$  is not a perfect square this improves on the bound  $\tau(C) \leq \frac{g}{\lfloor \sqrt{g} \rfloor}$  from [8].

**Remark 1.10.** From the discussion above there are implications

$$\begin{array}{ccccc} \text{Conjecture 1.8} & \Rightarrow & \text{Conjecture 1.3} & \Rightarrow & \text{Conjecture 1.1} \\ (\text{Nagata conjecture}) & & (\text{Seshadri constants}) & & (\text{Ample cone of } C^{(2)}) \end{array}$$

so Conjecture 1.3 concerning Seshadri constants sits between the Nagata conjecture and the conjecture governing the ample cone of a general  $C^{(2)}$ . It is possible that Conjecture 1.3 is easier than the full Nagata conjecture but there is, of course, currently no proof of this.

**Remark 1.11.** The degeneration we use in the proofs of the above theorems is related to the Franchetta degeneration used in [4] which arises from degenerating  $C$  to a rational nodal curve. The main difference is that here we consider the case that  $C$  develops one node at a time. Moreover rather than using a degeneration of  $C^{(2)}$  we find it easier to use a degeneration of  $C \times C$  and only consider divisors that are invariant under permuting the factors. A related degeneration of  $C^{(2)}$  coming from letting  $C$  develop cusps is described in [12].

The values of  $\tau(C)$  are known for a very general curve of genus  $g \leq 4$  (see Section 2.2) and from the discussion above the Nagata conjecture governs the case that  $g \geq 9$ . There appears to be little known in the intermediate range  $5 \leq g \leq 8$  and one would imagine that one of the two extreme cases holds (namely that either  $\tau(C) = \epsilon_g(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))^{-1}$  or  $\tau(C) = \sqrt{g}$  for a very general curve  $C$  of genus  $5 \leq g \leq 8$ ). In Section 4 we apply Theorem 1.2 to show that the first case does not hold when  $g = 5$ ; more precisely we show that if  $C$  is a very general curve of genus 5 then  $\tau(C) \leq 16/7$  which gives

$$2.236 \simeq \sqrt{5} \leq \tau(C) \leq 16/7 \simeq 2.286 < \epsilon_5(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2})^{-1} = 2.5.$$

To achieve this we use the techniques of Ein-Lazarsfeld [6] to get lower bounds of Seshadri constants of points in  $D^{(2)}$  where  $D$  is a curve of genus 4. This bound is stronger than that obtained from Corollary 1.7 as  $\epsilon_5(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)) = 2/5$  [13]. The number  $16/7$  is not expected to be optimal, but it is, as far as I am aware, the best that is currently known.

**Notation and conventions:** We work throughout over  $\mathbb{C}$ . The Néron-Severi space of divisors (resp.  $\mathbb{R}$ -divisors) on a variety  $V$  modulo numerical equivalence is denoted  $N^1(V)$  (resp.  $N^1(V)_{\mathbb{R}}$ ). An  $\mathbb{R}$ -divisor  $L$  on a variety  $V$  is ample (resp. nef) if it is a formal sum  $\sum_{i=1}^r a_i D_i$  of ample (resp. nef) divisors where the  $a_i \in \mathbb{R}$  are positive (resp. non-negative). Equivalently  $D \in N^1(V)_{\mathbb{R}}$  is nef if and only if it has non-negative self intersection with every irreducible curve  $C \subset V$ .

We say that  $p \in V$  is a very general point if there is a countable collection of proper subvarieties  $(V_n)_{n \geq 1}$  of  $V$  such that  $p$  is not contained in the union  $\bigcup_{n \geq 1} V_n$ . A collection  $p_1, \dots, p_m$  of points in  $V$  is very general if  $(p_1, \dots, p_m) \in V^{\times m}$  is very general. By a very general curve we mean a smooth curve whose corresponding point in the moduli space  $M_g$  is very general.

## 2. Preliminaries

**2.1. Divisors the second symmetric product.** Let  $C$  be a smooth curve of genus  $g \geq 0$ . The product  $C \times C$  has a natural involution and the second symmetric product is the quotient  $\sigma_C: C \times C \rightarrow C^{(2)}$  which is a smooth surface. We denote the image of a point  $(p, q) \in C \times C$  by  $p + q$ . In  $N^1(C^{(2)})$  we have the classes  $x$  and  $\delta$  as defined in the introduction. (Of course these classes really depend on  $C$  but this will always be clear from context.) It is well known that when  $C$  is a very general curve then  $N^1(C^{(2)})$  is spanned by  $x$  and  $\delta/2$  ([1] p.359) and when  $g \geq 1$  they are independent. When  $g = 0$ ,  $C^{(2)} = \mathbb{P}^2$  and both  $x$  and  $(\delta/2)$  are the class of the hyperplane.

The intersection of these classes is given by  $x^2 = 1$ ,  $\delta^2 = 4 - 4g$ , and  $x \cdot \delta = 2$ , so

$$((n + \gamma)x - \gamma(\delta/2)) \cdot ((n' + \gamma')x - \gamma'(\delta/2)) = nn' - \gamma\gamma'g.$$

Notice that if  $(n + \gamma)x - \gamma(\delta/2)$  is effective then intersecting with the ample class  $x$  implies  $n > 0$ .

As in the introduction define

$$\begin{aligned} \tau(C) &= \inf \{s > 0 : (s + 1)x - (\delta/2) \text{ is ample} \} \\ &= \min\{s \geq 0 : (s + 1)x - (\delta/2) \text{ is nef} \}. \end{aligned}$$

Since  $(\tau(C) + 1)x - (\delta/2)$  is nef, it has non-negative self-intersection which yields

$$\tau(C) \geq \sqrt{g}.$$

Now as is standard in such situations, the function  $\tau(C)$  is semicontinuous with respect to  $C$ :

**Lemma 2.1.** *Let  $X \rightarrow T$  be a flat family of smooth curves over an irreducible base  $T$  and for  $t \in T$  denote the fibre by  $C_t$ . If  $t_0 \in T$  is fixed then*

$$\tau(C_t) \leq \tau(C_{t_0}) \text{ for very general } t \in T.$$

*Proof.* Let  $Y \rightarrow T$  be the relative second symmetric product of  $X$ , and denote the fibre of  $Y$  over  $t$  by  $Y_t$ . Let  $\mathcal{D} \subset Y$  be the diagonal and pick a divisor  $D$  on  $Y$  which has class  $x$  on a each fibre  $Y_t$ . If  $\tau = \tau(C_{t_0})$  then by hypothesis  $F = (\tau + 1)D - (\mathcal{D}/2)$  restricts to a nef divisor on  $Y_{t_0}$ . Hence for very general  $t$  the restriction of  $F$  to  $Y_t$  is nef ([10] 1.4.14) which implies  $\tau(C_t) \leq \tau$ .  $\square$

In particular by applying this to a complete family of smooth curves we see that if  $\tau(C) \leq \tau_0$  for some smooth curve  $C$  of genus  $g$ , then the same bound holds for a very general curve of genus  $g$ . Moreover if  $C$  is a very general curve then  $\tau(C)$  is independent of the actual curve chosen.

A geometric interpretation of Conjecture 1.1 can be given in terms of the existence of “exceptional” curves in  $C^{(2)}$ :

**Lemma 2.2.** *Let  $C$  be a smooth curve of genus  $g \geq 2$ .*

- (1) *If  $\tau(C) > \sqrt{g}$  then there exists a reduced irreducible curve  $D \subset C^{(2)}$  with numerical class  $(n + \gamma)x - \gamma(\delta/2) + \sigma$  where  $\sigma \cdot x = \sigma \cdot \delta = 0$  such that  $\tau(C) = \frac{\gamma g}{n}$ .*
- (2) *If  $C$  is a very general curve then  $\tau(C) = \sqrt{g}$  if and only if  $\Delta$  is the only reduced irreducible curve in  $C^{(2)}$  with negative self-intersection. Moreover if there does exist another such curve of negative self intersection then it is unique.*

*Proof.* Suppose  $\tau = \tau(C) > \sqrt{g}$ . Then the  $\mathbb{R}$ -divisor

$$F = (\tau + 1)x - (\delta/2)$$

has positive self-intersection and, by definition of  $\tau$ , is nef but not ample. Thus by the Nakai criterion for real divisors [3] there is a reduced irreducible curve  $D \subset C^{(2)}$  with  $D \cdot F = 0$ . We can write the numerical class of  $D$  as  $(n + \gamma)x - \gamma(\delta/2) + \sigma$  where  $\sigma$  is a class orthogonal to  $x$  and  $\delta$  so  $D \cdot F = 0$  implies  $\tau = \frac{\gamma g}{n}$ . We note that since  $n > 0$  this implies  $\gamma > 0$ .

Now suppose  $C$  is very general. Then the effective cone of  $C^{(2)}$  is spanned by  $x$  and  $\delta/2$ . Thus if  $\tau = \tau(C) > \sqrt{g}$  the curve  $D$  above has class  $(n + \gamma)x - \gamma(\delta/2)$  (i.e.

$\sigma = 0$ ). Hence  $D^2 = n^2 - g\gamma^2 = \frac{n^2}{g}(g - \tau^2) < 0$  so  $D$  has negative self intersection and clearly  $D \neq \Delta$  as  $\gamma > 0$ .

Before proving the converse we deal with uniqueness. To this end suppose that  $(n + \gamma)x - \gamma(\delta/2)$  and  $(n' + \gamma')x - \gamma'(\delta/2)$  are classes of distinct reduced irreducible curves of negative self intersection. Then  $n^2 - \gamma^2 g < 0$ ,  $n'^2 - \gamma'^2 g < 0$  and  $nn' - \gamma\gamma'g \geq 0$  which implies  $\gamma_1$  and  $\gamma_2$  have opposite sign. Hence any irreducible curve  $D \neq \Delta$  in  $C^{(2)}$  with  $D^2 < 0$  must have numerical class  $(n + \gamma)x - \gamma(\delta/2)$  with  $\gamma > 0$ , and if it exists it is unique. Thus if there exists a reduced irreducible curve  $D \neq \Delta$  with negative self intersection it has numerical class  $(n + \gamma)x - \gamma(\delta/2)$  with  $\gamma > 0$  and  $D^2 = n^2 - g\gamma^2 < 0$ . As  $F$  is nef we know  $0 \leq F.D = \tau n - g\gamma$  which implies  $\tau \geq \frac{g\gamma}{n} > \sqrt{g}$ .  $\square$

Before proceeding with the main results of this paper we digress to discuss a finiteness result concerning the possible values of  $\tau(C)$  as  $C$  ranges over all curves of fixed genus  $g \geq 2$ .

**Proposition 2.3.** *Fix a real number  $\alpha > \sqrt{g}$ . Then*

$$\{\tau \geq \alpha : \tau = \tau(C) \text{ for some smooth curve } C \text{ of genus } g\}$$

*is a finite set. Equivalently the only possible accumulation point of the set given by  $\{\tau(C) : C \text{ a smooth curve of genus } g\}$  is  $\sqrt{g}$ .*

*Proof.* Fix a number  $s \in (\sqrt{g}, \alpha) \cap \mathbb{Q}$ . We first prove that there exists an integer  $k$  such that for any smooth curve  $C$  of genus  $g$  the divisor  $k[(s + 1)x - (\delta/2)]$  on  $C^{(2)}$  is effective.

To this end let  $C$  be any smooth curve of genus  $g$  and fix a  $\mathbb{Q}$ -divisor  $F$  on  $C^{(2)}$  whose numerical class is  $(s + 1)x - (\delta/2)$ . We note that the canonical class of  $C^{(2)}$  is  $K = (2g - 2)x - (\delta/2)$  ([9] Prop. 2.6). Thus if  $k \in \mathbb{N}$  is sufficiently large (with  $ks \in \mathbb{N}$ ) then  $x.(K - kF) < 0$ . So by Serre duality  $h^2(\mathcal{O}(kF)) = h^0(\mathcal{O}(K - kF)) = 0$  as  $x$  is ample. Hence for such  $k$ ,

$$h^0(\mathcal{O}(kF)) \geq h^0(\mathcal{O}(kF)) - h^1(\mathcal{O}(kF)) = \chi(kF) = p(k)$$

where by the Riemann-Roch theorem  $p(k)$  is a polynomial whose coefficients depend only on  $s$  and  $g$  (and not on the specific curve  $C$  or choice of  $F$ ). The leading order coefficient of  $p(k)$  is  $F^2/2 = (s^2 - g)/2 > 0$  so there exists a  $k$  (independent of  $C$ ) such that  $h^0(\mathcal{O}(kF)) > 0$  as claimed.

Now suppose  $C$  is chosen so that  $\tau(C) \geq \alpha$ . By Lemma 2.2(a) there exists a reduced irreducible curve  $D \subset C^{(2)}$  with numerical class  $(n + \gamma)x - \gamma(\delta/2) + \sigma$  where  $\sigma.x = \sigma.\delta = 0$  and  $\tau(C) = \frac{g\gamma}{n}$ . With  $k, F$  as above there is a divisor  $E \subset |kF|$ . But as  $s < \alpha \leq \tau(C)$  we have  $D.F = (ns - \gamma g) < 0$  so  $D \subset E$ . Since  $D$  is reduced this implies that  $n = x.D \leq x.E = ks$ . Thus letting  $N := ks$  we have  $n \leq N$ . To complete the proof note that the divisor  $G = (g - 1)x + (\delta/2)$  is always nef as it is the pullback of the theta divisor under the Abel-Jacobi map  $u: C^{(2)} \rightarrow \text{Jac}(C)$ . Hence  $D \subset E$  also implies  $D.G \leq E.G$  which yields  $gn + \gamma g \leq k(gs + g)$  so  $\gamma \leq k(s + 1) =: M$ . Thus  $\tau(C)$  lies in the set

$$\{\tau : \tau = \frac{g\gamma}{n} \text{ with } n, \gamma \in \mathbb{N} \text{ and } n \leq N, \gamma \leq M\}$$

which is finite.  $\square$

**Remark 2.4.** A similar finiteness result for Seshadri constants in families of surfaces can be found in [11]. It would be interesting to know if the finiteness from Proposition 2.3 still holds when  $\alpha = \sqrt{g}$ .

**2.2. The case of low genus.** For low genus it is possible to describe the intersection of the ample cone with the plane spanned by  $x$  and  $\delta$  by finding explicit irreducible curves of negative self intersection. For details see [4, 8] (or [10] Section 1.5.B).

- $g = 0$ : Here  $C^{(2)} = \mathbb{P}^2$  and  $(s + 1)x - (\delta/2) = sh$  where  $h$  is the class of the hyperplane, so trivially  $\tau(\mathbb{P}^1) = 0$ .
- $g = 1$ : In this case it is well known that if  $C$  is a very general genus 1 curve then the closure of the effective cone of  $C^{(2)}$  is the nef cone. It is a closed circular cone described by the equations  $\alpha^2 \geq 0$ ,  $\alpha.h \geq 0$  where  $h$  is an ample class ([10] Lemma 1.5.4). Thus  $\tau(C) = 1$ .
- $g = 2$ : Any curve  $C$  of genus 2 is hyperelliptic. Using the  $g_1^2$  one can produce an irreducible curve in  $C^{(2)}$  of negative self intersection whose class is  $2x - (\delta/2)$ , and thus  $\tau(C) = 2$ .
- $g = 3$ : If  $C$  is a very general curve of genus 2 then it is possible to construct an irreducible curve in  $C^{(2)}$  whose class is  $16x - 6(\delta/2)$  and thus has self-intersection  $-8$  [4, 8]. Using this one deduces that  $\tau(C) = 9/5$ .
- $g = 4$ : If  $C$  is a very general curve of genus 4 then  $\tau(C) = 2$ . In fact any such curve admits two  $g_1^3$ , and the associated  $\Gamma_3$  is an irreducible curve whose class is  $3x - (\delta/2)$  spans one boundary of the effective cone. (This can also be obtained from Corollary 1.7).

**2.3. Seshadri constants.** We record some basic definitions and properties of Seshadri constant and refer the reader to [2, 10] for a comprehensive treatment. Let  $X$  be a smooth variety of dimension  $n$  and  $L$  be a nef numerical class in  $N^1(X)_{\mathbb{R}}$ . If  $p_1, \dots, p_m$  are points in  $X$  define

$$\epsilon(p_1, \dots, p_m; X, L) = \inf_C \left\{ \frac{L.C}{\sum_{i=1}^r \text{mult}_{p_i} C} \right\},$$

where the infimum is over all reduced irreducible curves  $C$  in  $X$  that pass through at least one of the  $p_i$ . Equivalently if  $\pi: B \rightarrow X$  is the blowup of  $X$  at these points with exceptional divisor  $E$  then

$$\epsilon(p_1, \dots, p_m; X, L) = \max\{s \geq 0 : \pi^*L - sE \text{ is nef}\}.$$

By a standard semicontinuity argument similar to (2.1) the Seshadri constant of  $m$  very general points does not depend on the actual points chosen. Thus we can set

$$\epsilon_m(X, L) = \epsilon(p_1, \dots, p_m; X, L) \text{ where } p_1, \dots, p_m \text{ are in very general position.}$$

Notice that if  $\pi^*L - cE$  is nef then  $(\pi^*L - cE)^n \geq 0$  which implies

$$\epsilon(p_1, \dots, p_m; X, L) \leq \sqrt[n]{\frac{L^n}{m}}. \tag{2.5}$$

It is an interesting and difficult problem to get non-trivial bounds for Seshadri constants in general [5, 6] or even to calculate them in examples (see [10] and the reference therein).

We will make use of the following simple lemma which says that Seshadri constants of a collection of points in  $C^{(2)}$  can be calculated by looking at their preimage in  $C^{\times 2}$ .

**Lemma 2.6.** *Let  $D$  be a smooth curve and  $\sigma_D: D^{\times 2} \rightarrow D^{(2)}$  be the quotient map. Let  $p_1, \dots, p_m$  be general points in  $D^{(2)}$  and suppose for each  $i$  that  $\sigma_D^{-1}(p_i) = \{q_i^1, q_i^2\}$ . Then for any nef class  $L \in N^1(D^{(2)})_{\mathbb{R}}$ ,*

$$\epsilon(p_1, \dots, p_m; D^{(2)}, L) = \epsilon(q_1^1, q_1^2, \dots, q_m^1, q_m^2; D^{\times 2}, \sigma_D^* L).$$

*Proof.* Let  $p_X: X \rightarrow D^{(2)}$  be the blowup of  $D^{(2)}$  at  $p_1, \dots, p_m$  with exceptional divisor  $E$ , and  $p_Y: Y \rightarrow D^{\times 2}$  be the blowup of  $D^{\times 2}$  at  $q_1^1, q_1^2, \dots, q_m^1, q_m^2$  with exceptional divisor  $F$ . Then under the induced map  $\tilde{\sigma}_D: Y \rightarrow X$  that lifts  $\sigma_D$  we have for any  $c > 0$  that  $\tilde{\sigma}_D^*(p_X^* L - cE) = p_Y^* L - cF$ . Since  $\tilde{\sigma}_D$  is surjective this implies  $p_X^* L - cE$  is nef if and only if  $p_Y^* L - cF$  is nef, which proves the lemma.  $\square$

### 3. Proofs

Let  $\mathcal{C} \rightarrow T$  be a family of smooth curves of genus  $g$  over a disc  $T$  that develops a node. By this we mean that the family is proper and flat, the fibre  $C_t$  over  $t \in T$  is a smooth curve for  $t \neq 0$  and that the fibre  $C_0$  over  $0 \in T$  is an irreducible curve that has a single node. We assume further that  $\mathcal{C}$  has a smooth total space. Taking the relative second symmetric product of  $\mathcal{C}$  gives a degeneration of  $C^{(2)}$  which when suitably blown up is essentially the Franchetta degeneration used in [4]. Instead of using this we find it easier to consider the fibred product  $\mathcal{C} \times_T \mathcal{C}$  (and thus a degeneration of  $C \times C$ ) and only deal with divisors that are invariant under permuting the factors.

To this end suppose  $p$  is the node in  $C_0$  and let  $\mathcal{Y} \rightarrow \mathcal{C} \times_T \mathcal{C}$  be the blowup of  $\mathcal{C} \times_T \mathcal{C}$  at  $(p, p)$  with exceptional divisor  $E$ . One can easily check by working in local analytic coordinates that  $\mathcal{Y}$  has smooth total space. Clearly the fibre of  $\mathcal{Y}$  over  $t \neq 0$  is  $\mathcal{Y}_t = C_t \times C_t$  and the next two lemmas describe the central fibre of  $\mathcal{Y}$ . Denote the normalisation of  $C_0$  by  $D$  and let  $q, r \in D$  be the preimage of the node  $p \in C_0$ .

**Lemma 3.1.**

- (1) *The central fibre  $\mathcal{Y}_0$  has two irreducible components namely the exceptional divisor  $E$  which is isomorphic to  $\mathbb{P}^1 \times \mathbb{P}^1$  and another we denote by  $F$ .*
- (2) *The normalisation  $\tilde{F}$  of  $F$  is the blowup  $\pi: \tilde{F} \rightarrow D \times D$  at the four points  $(q, q), (q, r), (r, q), (r, r)$  making the natural diagram*

$$\begin{array}{ccc} \tilde{F} & \longrightarrow & F \\ \pi \downarrow & & \downarrow \\ D \times D & \longrightarrow & C_0 \times C_0 \end{array}$$

*commute. We denote the exceptional curve in  $\tilde{F}$  that sits over the point  $(s, t)$  by  $\tilde{e}_{st}$  and the corresponding curve in  $F$  by  $e_{st}$ .*

- (3) *The two components of  $\mathcal{Y}_0$  are glued along the four rational curves  $\{e_{qq}, e_{rr}\}$  and  $\{e_{qr}, e_{rq}\}$  in  $F$ . Each set consists of a pair of lines in one of the two rulings of  $E$ .*

*Proof.* In local analytic coordinates around the node  $p \in C_0$  the family  $\mathcal{C}$  has the form  $xy = t$  in  $\mathbb{C}^2 \times T$  where  $t$  is the parameter on  $T$ . Hence locally  $\mathcal{C} \times_T \mathcal{C}$  is given by  $x_1 y_1 = x_2 y_2 = t$  in  $\mathbb{C}^4 \times T$ . Thus a local model for  $\mathcal{Y}$  is given by the proper



transform in the blowup  $B \rightarrow \mathbb{C}^4 \times T$  at the origin. On the exceptional  $\mathbb{P}^4$  in  $B$  we pick coordinates  $\lambda_1, \lambda_2, \mu_1, \mu_2, \sigma$  such that for  $t \neq 0$

$$\frac{\lambda_i}{x_i} = \frac{\mu_j}{y_j} = \frac{\sigma}{t} \quad i, j = 1, 2.$$

Then  $E$  is the intersection of  $\mathcal{Y}$  with  $\mathbb{P}^4$  and is given by

$$\lambda_1 \mu_1 = \lambda_2 \mu_2 \text{ and } \sigma = 0$$

which is a quadric hypersurface in  $\mathbb{P}^3$ , and thus  $E \simeq \mathbb{P}^1 \times \mathbb{P}^1$ .

Now let  $U$  and  $V$  be the two components of the normalisation of  $xy = t$  corresponding to  $x = 0$  and  $y = 0$  respectively. Then locally the other components of  $\mathcal{Y}$  are the proper transform of  $U \times U, U \times V, V \times U$  and  $V \times V$  given by  $x_1 = x_2 = 0, x_1 = y_2 = 0, y_1 = x_2 = 0$  and  $y_1 = y_2 = 0$  respectively. These are glued along normal crossing curves and the normalisation  $\tilde{F}$  is obtained by pulling them apart. Thus  $\tilde{F}$  is the blowup of  $D \times D$  in the four points as claimed.

Now the proper transform of  $U \times U$  is the blowup at the point  $(p, p)$  and meets  $E$  in the line given by  $\lambda_1 = \lambda_2 = 0$ . Similarly  $V \times V$  meets  $E$  in the line  $\mu_1 = \mu_2$  which is easily seen to be in the same ruling. A completely analogous analysis applies to  $U \times V$  and  $V \times U$ .  $\square$

For simplicity denote the numerical class of the curves  $\tilde{e}_{st}$  by the same letter. Then

$$N^1(\tilde{F}) = \pi^* N^1(D \times D) \oplus \mathbb{Z}[\tilde{e}_{qq}, \tilde{e}_{rr}, \tilde{e}_{qr}, \tilde{e}_{rq}].$$

Moreover  $N^1(E)$  is a free group of rank 2 with two generators  $\alpha$  and  $\beta$ . We declare that  $\alpha$  is the class of the curve  $e_{qq}$  (equivalently of  $e_{rr}$ ) inside  $E$  and  $\beta$  is the class of  $e_{qr}$  (equivalently of  $e_{rq}$ ).

Consider now the proper transform  $\mathcal{D} \subset \mathcal{Y}$  of the diagonal in  $\mathcal{C} \times_T \mathcal{C}$ .

**Lemma 3.2.** *The restriction of  $\mathcal{D}$  to  $E$  has class  $\alpha \in N^1(E)$ . The pullback of  $\mathcal{D}|_F$  to  $\tilde{F}$  is the proper transform of the diagonal  $\Delta_D \subset D \times D$  and thus has class  $\pi^* \Delta_D - \tilde{e}_{qq} - \tilde{e}_{rr} \in N^1(\tilde{F})$ .*

*Proof.* We continue to use the local coordinates introduced in the proof of Lemma 3.1. For  $t \neq 0$  the diagonal is given by  $x_1 = y_1$  and  $x_2 = y_2$  and thus meets  $E$  in the line  $\lambda_1 = \mu_1$  and  $\lambda_2 = \mu_2$ . As is easily checked this line has class  $\alpha$ .

Now clearly the proper transform of  $\mathcal{D}|_{U \times U}$  is, locally, the proper transform of the diagonal  $y_1 = y_2$  in  $U \times U$  and similarly for  $V \times V$ . Moreover for  $\Delta_D$  is disjoint from  $U \times V$  and  $V \times U$  for  $t \neq 0$  and thus the pullback of  $\mathcal{D}|_F$  to  $\tilde{F}$  has the numerical class as claimed.  $\square$

With these preliminaries we are ready to give the proofs of the theorems stated in the introduction.

*Proof of Theorem 1.2.* By hypothesis there is a smooth curve  $D$  of genus  $g - 1$  and distinct points  $q, r \in D$  such that

$$\epsilon(q + r; D^{(2)}, (a + b)x - b(\delta/2)) \geq b. \quad (3.3)$$

By gluing  $q$  and  $r$  we get a curve  $C_0$  with a single node  $p$  whose arithmetic genus is  $g$  and whose normalisation is  $D$ . Let  $\mathcal{C} \rightarrow T$  be a proper flat family of curves of genus  $g$  which has a smooth total space, smooth general fibre and whose central

fibre is  $C_0$ . Let  $\mathcal{Y} \rightarrow \mathcal{C} \times_T \mathcal{C}$  be the blowup at  $(p, p)$ . We will use the notation introduced in Lemmas 3.1 and 3.2 so the central fibre  $\mathcal{Y}_0$  has two components  $E$  and  $F$ , and the normalisation  $\tilde{F}$  of  $F$  is the blowup  $\pi: \tilde{F} \rightarrow D \times D$  at the four points  $(q, q), (q, r), (r, q), (r, r)$ .

Fix a line bundle  $L$  on  $\mathcal{C}/\mathcal{C}_0$  that has degree 1 on each of the fibres. As  $\mathcal{C}$  is assumed to be smooth and  $\mathcal{C}_0$  is irreducible,  $L$  extends uniquely to a line bundle  $L'$  on all of  $\mathcal{C}$ . Take a meromorphic section of  $L'$  whose support does not contain the node  $p$  of  $\mathcal{C}_0$ . By shrinking  $T$  if necessary we may assume furthermore that the support of  $s$  does not contain any fibre  $\mathcal{C}_t$ , and we write this support as  $\sum_i a_i D_i$  for some divisors  $D_i \subset \mathcal{C}$  that do not contain  $p$ . For each  $i$  define a divisor on  $\mathcal{Y}$  by

$$G_i = \{(u, v) \in \mathcal{Y}_t \text{ for some } t \text{ and either } u \in D_i \text{ or } v \in D_i\},$$

and set

$$G = \sum_i a_i G_i.$$

Clearly  $G$  is invariant under the natural involution on  $\mathcal{Y}$ . In fact for all  $t \neq 0$  the numerical class of  $G|_{\mathcal{Y}_t}$  is the pullback of  $x$  under  $\sigma_{C_t}: C_t \times C_t \rightarrow C_t^{(2)}$ . Moreover  $G$  is trivial along  $E$  and the numerical class of the pullback of  $G|_F$  to  $\tilde{F}$  has class  $\pi^* \sigma_D^* x$ .

Let  $\mathcal{D} \subset \mathcal{Y}$  be the proper transform of the diagonal in  $\mathcal{C} \times_T \mathcal{C}$  as in Lemma 3.2 and define an  $\mathbb{R}$ -divisor on  $\mathcal{Y}$  by

$$H = (a + b)G - b(\mathcal{D} + E). \quad (3.4)$$

We claim that  $H|_{\mathcal{Y}_0}$  is nef. To see this note first from (3.1, 3.2) that  $E|_E$  has class  $-(2\alpha + 2\beta)$  and  $\mathcal{D}|_E$  has class  $\alpha$ . Thus  $H|_E$  has class  $b(\alpha + 2\beta)$  which is clearly nef as both  $\alpha$  and  $\beta$  are nef and  $b > 0$ . To show that  $H|_F$  is nef consider its pullback to  $\tilde{F}$  which using (3.1, 3.2) has numerical class

$$\begin{aligned} (a + b)\pi^* \sigma_D^* x - b(\pi^* \Delta_D - \tilde{e}_{qq} - \tilde{e}_{rr}) - b(\tilde{e}_{qq} + \tilde{e}_{rr} + \tilde{e}_{rq} + \tilde{e}_{rq}) \\ = \pi^* \sigma_D^* ((a + b)x - b(\delta/2)) - b(\tilde{e}_{rq} + \tilde{e}_{rq}) \end{aligned} \quad (3.5)$$

since  $\sigma_D^*(\delta/2) = \Delta_D$ . Now  $\sigma_D^{-1}(q + r) = \{(q, r), (r, q)\}$  so (3.3) and Lemma 2.6 imply

$$\epsilon((q, r), (r, q); D \times D, \sigma_D^*((a + b)x - b(\delta/2))) \geq b.$$

But by (3.5) this means exactly that the pullback of  $H|_F$  to  $\tilde{F}$  is nef and thus  $H|_F$  is nef as well.

Hence  $H|_{\mathcal{Y}_0}$  is nef and by semicontinuity so is  $H|_{\mathcal{Y}_t}$  for very general  $t$ . But  $H|_{\mathcal{Y}_t}$  has class  $\sigma_{C_t}^*((a + b)x - b(\delta/2))$  so  $(a + b)x - b(\delta/2) \in N^1(C_t^{(2)})$  is nef for very general  $t$  which proves that  $\tau(C_t) \leq \frac{a}{b}$ . By (2.1) the same inequality holds for any very general curve of genus  $g$ .  $\square$

*Proof of Theorem 1.6.* The fact that  $(a + b)x - b(\delta/2) \in N^1(D^{(2)})$  is nef comes from Theorem 1.2 since  $\epsilon_r(\cdot) \leq \epsilon_1(\cdot)$ . Essentially the result we want comes from the degeneration  $\mathcal{Y}$  described in the proof of Theorem 1.2 and semicontinuity of Seshadri constants in families.

To describe this more explicitly we continue the notation from the above proof. Pick  $m$  sections  $s_1, \dots, s_m$  of  $\mathcal{C} \times_T \mathcal{C} \rightarrow T$  that meet  $\mathcal{C}_0 \times \mathcal{C}_0$  at  $m$  very general points (so in particular these points are not equal to  $(p, p)$ ). Let  $V_i$  be the image of  $s_i$  and

$V'_i$  be the image of  $V_i$  under the involution. We denote the proper transform of  $V_i$  and  $V'_i$  in  $\mathcal{Y}$  by  $W_i$  and  $W'_i$  and let  $W = \bigcup_i W_i \cup W'_i$ . By shrinking  $T$  if necessary we may assume that  $W$  meets each fibre  $\mathcal{Y}_t = C_t \times C_t$  at a collection of  $2m$  distinct points. Note that in the central fibre  $\mathcal{Y}_0$  these points are all in the component  $F$ .

Now let  $\pi: \mathcal{Y}' \rightarrow \mathcal{Y}$  be the blowup along  $W$  with exceptional divisor  $E'$ . The central fibre of  $\mathcal{Y}'$  has components  $E$  and  $F'$  where the normalisation  $\tilde{F}'$  of  $F'$  is the blowup of  $\tilde{F}$  at  $m$  very general points and their image under the involution. Thus  $\tilde{F}'$  is the blowup of  $D \times D$  at the points  $(q, q), (q, r), (r, q), (r, r)$  and at a further  $2m$  points.

Set  $c = \epsilon_m(D^{(2)}, (a+b)x - (\delta/2))$  and consider the  $\mathbb{R}$ -divisor

$$H' = \pi^*H - cE'$$

where  $H$  is the divisor defined in (3.4). Then exactly as in the proof of Theorem 1.2,  $H'|_E$  is nef and the hypothesis on the Seshadri constant and Lemma 2.6 imply that the pullback of  $H'|_{F'}$  to  $\tilde{F}'$  is also nef. Thus  $H'|_{\mathcal{Y}'_t}$  is nef for very general  $t$  and using (2.6) once again proves the theorem.  $\square$

*Proof of Corollary 1.7.* Fix  $g \geq 1$ . Let  $h$  be the class of the hyperplane in  $\mathbb{P}^2$  and set  $b = \epsilon_g(\mathbb{P}^2, h)$ . The second symmetric product of a genus 0 curve is  $\mathbb{P}^2$  and under this identification  $x = (\delta/2) = h$ . We have

$$\epsilon_g((\mathbb{P}^1)^{(2)}, (1+b)x - b(\delta/2)) = \epsilon_g(\mathbb{P}^2, h) = b.$$

Thus repeated use of Theorem 1.6 yields for a very general curve  $D$  of genus  $g - 1$

$$\epsilon_1(C^{(2)}, (1+b)x - b(\delta/2)) \geq b,$$

and so the result follows from Theorem 1.2.  $\square$

#### 4. Application to the case $g = 5$

We now prove that if  $C$  is a very general curve of genus 5 then  $\tau(C) \leq 16/7$ . This is done by estimating the Seshadri constant at a very general point  $p$  of  $D^{(2)}$  where  $D$  is a very general curve of genus 4. By (2.2) we know that  $\tau(D) = 2$ . Set  $a = 16, b = 7$  and  $L = (a+b)x - b(\delta/2) \in N^1(D^{(2)})$  which is ample. By (1.2) it is sufficient to show the following

*Claim: If  $p$  is a very general point in  $D^{(2)}$  then*

$$\epsilon(p; D^{(2)}, L) \geq b = 7. \tag{4.1}$$

The proof of the claim will use the ideas of Ein-Lazarsfeld [6]. Rather than using the main result of that paper we get an improvement by using their techniques and special properties of the particular surface  $D^{(2)}$ . In particular we will need the following lemma.

**Lemma 4.2** (Ein-Lazarsfeld [6]). *Let  $X$  be a smooth surface and  $L$  be an integral ample line bundle (or class) on  $X$ . Suppose  $\{p_t \in E_t\}_{t \in T}$  is a one-parameter family consisting of a point  $p_t$  in a curve  $E_t \subset X$  such that  $\text{mult}_{p_t} E_t \geq m$  for all  $t$ . Suppose in addition that  $E = E_0$  is reduced and irreducible and moreover that the Kodaira-Spencer class of this family is non-zero. Then  $E^2 \geq m(m - 1)$ .*

Now suppose  $E \subset D^{(2)}$  is a reduced irreducible curve passing through a very general point  $p$  with numerical class  $(n + \gamma)x - \gamma(\delta/2)$ .

*Claim:* We have  $L.E \geq 7$  and if  $(n, \gamma) \notin \{(1, 0), (3, 1), (5, 2)\}$  then  $L.E \geq 4b = 28$ .

To see this note that  $L.E = an - 4b\gamma = 16n - 28\gamma$  so certainly if  $\gamma \leq 0$  then  $L.E \geq 28$  unless  $(n, \gamma) = (1, 0)$ . So suppose that  $\gamma > 0$ . As  $\tau(D) = 2$  we must have  $n \geq 2\gamma$  with equality if and only if  $E$  has zero self intersection.

Now for each fixed  $\gamma \geq 0$  there are at most finitely many irreducible curves  $E$  with numerical class  $(n + \gamma)x - \gamma(\delta/2)$  and self-intersection zero. Since  $p$  is assumed to be very general we may assume there is no irreducible curve of zero self-intersection through  $p$ , and so we in fact have  $n \geq 2\gamma + 1$ . Then it is easily checked that  $L.E \geq 7$  and  $L.E \geq 28$  except when  $(n, \gamma) \in \{(3, 1), (5, 2)\}$ .

We now finish the proof of (4.1). Suppose for contradiction that  $\epsilon(p; D^{(2)}, L) < 7$  for a very general  $p \in D^{(2)}$ . Then through a very general point  $p$  there exists a reduced irreducible curve  $E$  with  $m = \text{mult}_p E$  and

$$\frac{L.E}{m} < b = 7. \quad (4.3)$$

As in [6] the collection of pairs  $(p, E)$  consisting of a point  $p$  in an irreducible curve  $E$  such that  $\text{mult}_p(E) > L.E/7$  consists of a countable collection of algebraic families and the proof will be completed by showing that any such family with  $p$  a very general point is discrete.

To this end suppose for contradiction that there is a family  $\{p_t \in E_t\}_{t \in T}$  with  $E_t$  reduced and irreducible and  $\text{mult}_{p_t} E_t > L.E_t/7$  for all  $t$ . Set  $(p, E) = (p_0, E_0)$ . Since  $p$  is assumed to be very general we have from the above that  $L.E_0 \geq 7$  so  $\text{mult}_{p_t} > 1$ . Since  $\text{mult}_y E_t = 1$  for a general point  $y$  of  $E_t$  we deduce that the curves  $E_t$  are moving in a non-trivial family and thus from (4.2) we have that  $E^2 \geq m(m-1)$ .

*Case 1:*  $(n, \gamma) = (1, 0)$ . If  $E$  is not the irreducible curve  $x_p = \{p + q | q \in C\}$  then  $m \leq E.x = 1$  so  $m = 1$  which is impossible by (4.3). On the other hand if  $E = x_p$  then  $m = 1$  and  $L.E = a$  which again is absurd.

*Case 2:*  $(n, \gamma) = (3, 1)$  (resp.  $(n, \gamma) = (5, 2)$ ). As  $m(m-1) \leq E^2 = 5$  (resp.  $m(m-1) \leq E^2 = 9$ ) we have  $m \leq 2$  (resp.  $m \leq 3$ ). But this implies  $\frac{L.E}{m} \geq \frac{3a-4b}{2} = 10 \geq b$  (resp.  $\frac{L.E}{m} \geq \frac{5a-8b}{3} = 8 \geq b$ ) which in both cases is impossible by (4.3).

*Case 3:*  $(n, \gamma) \neq (1, 0), (3, 1), (5, 2)$ . Here we follow [6] but first note that by the previous claim in this case  $L.E \geq 4b$  which by (4.3) implies that  $m \geq 5$ . Again from (4.3) we have  $L.E < 7m$  so  $L.E \leq 7m - 1$ . Thus by the Hodge index theorem

$$m(m-1) \leq E^2 \leq \frac{(L.E)^2}{L^2} \leq \frac{(7m-1)^2}{60}$$

which is impossible for  $m \geq 5$ .

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### References

- [1] E. Arbarello, M. Cornalba, P. Griffiths, and J. Harris, *Geometry of algebraic curves. Vol. I*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], **267** Springer-Verlag, New York, (1985)
- [2] T. Bauer, *Seshadri constants on algebraic surfaces*, Math. Ann. **313** (1999), no. 3, 547–583.
- [3] F. Campana and T. Peternell, *Algebraicity of the ample cone of projective varieties*, J. Reine Angew. Math. **407** (1990) 160–166.
- [4] C. Ciliberto and A. Kouvidakis, *On the symmetric product of a curve with general moduli*, Geom. Dedicata **78** (1999), no. 3, 327–343.
- [5] L. Ein, O. Küchle, and R. Lazarsfeld, *Local positivity of ample line bundles*, J. Differential Geom. **42** (1995), no. 2, 193–219.
- [6] L. Ein and R. Lazarsfeld, *Seshadri constants on smooth surfaces*, Astérisque **218** (1993) 177–186.
- [7] B. Harbourne and J. Roé, *Multipoint Seshadri constants on  $\mathbb{P}^2$* , Rend. Sem. Mat. Univ. Politec. Torino **63** (2005), no. 1, 99–102.
- [8] A. Kouvidakis, *Divisors on symmetric products of curves*, Trans. Amer. Math. Soc. **337** (1993), no. 1, 117–128.
- [9] ———, *On some results of Morita and their application to questions of ampleness*, Math. Z. **241** (2002), no. 1, 17–33.
- [10] R. Lazarsfeld, *Positivity in algebraic geometry. I.*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge, Springer-Verlag, Berlin (2004)
- [11] K. Oguiso, *Seshadri constants in a family of surfaces*, Math. Ann. **323** (2002), no. 4, 625–631.
- [12] G. Pacienza and F. Polizzi, *On a degeneration of the symmetric product of a curve with general moduli*, Matematiche (Catania) **56** (2001), no. 2, 297–307 (2003). PRAGMATIC, 2001 (Catania).
- [13] B. Strycharz-Szemberg and T. Szemberg, *Remarks on the Nagata conjecture*, Serdica Math. J. **30** (2004), no. 2-3, 405–430.
- [14] T. Szemberg and H. Tutaj-Gasińska, *General blow-ups of the projective plane*, Proc. Amer. Math. Soc. **130** (2002), no. 9, 2515–2524 (electronic).
- [15] G. Xu, *Curves in  $\mathbb{P}^2$  and symplectic packings*, Math. Ann. **299** (1994), no. 4, 609–613.

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