Math. Res. Lett. **14** (2006), *no.* 1, 49–62

SHARP GAGLIARDO-NIRENBERG INEQUALITIES VIA SYMMETRIZATION

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ABSTRACT. We prove Gagliardo-Nirenberg inequalities using certain new symmetrization inequalities. Our methods are elementary and can be applied in a very general context.

1. Introduction

Recently, the Sobolev and Gagliardo-Nirenberg inequalities have been sharpened and extended in different directions. In particular, we mention the works of Cohen-Meyer-Oru [13], Cohen-DeVore-Petrushev-Xu [11], Cohen-Dahmen-Daubechies-DeVore [12], where it is shown that the Gagliardo-Nirenberg-Sobolev inequality[1]

$$\|f\|_{L^{n'}} \leq C \|\nabla f\|_{L^1}$$

can be sharpened to

$$\|f\|_{L^{n'}} \leq C \|\nabla f\|_{L^1}^{\frac{n-1}{n}} \|f\|_{B_{\infty,\infty}^{-(n-1)}}^{\frac{1}{n}} \tag{1.1}$$

or even to

$$\|f\|_{L^{n'}} \leq C \|\nabla f\|_{BV}^{\frac{n-1}{n}} \|f\|_{B_{\infty,\infty}^{-(n-1)}}^{\frac{1}{n}}, \tag{1.2}$$

where $B_{\infty,\infty}^{-(n-1)}$ is the homogeneous Besov space of indices $(-(n-1), \infty, \infty)$ (see section 2 below).

Ledoux [22] developed a new method to treat (1.1) and obtained the following extension:

$$\|f\|_{L^q} \leq C \|\nabla f\|_{L^p}^{\theta} \|f\|_{B_{\infty}^{-\theta/(1-\theta),\infty}}^{1-\theta}, \ 1 \leq p < q < \infty, \theta = \frac{p}{q}. \tag{1.3}$$

A special case of (1.3) ($p = 2$, $q = 4$) was obtained by Meyer-Rivière [30], who proved

$$\|f\|_{L^4} \leq C \|\nabla f\|_2^{1/2} \|f\|_{B_{\infty}^{-1,\infty}}^{1/2}, \tag{1.4}$$

en route to obtaining

$$\|\nabla f\|_{L^4} \leq C \left\|\nabla^2 f\right\|_2^{1/2} \|f\|_{BMO}^{1/2}. \tag{1.5}$$

Received by the editors June 17, 2006.

2000 *Mathematics Subject Classification.* Mathematics Subject Classification Primary: 46E30, 26D10.

Key words and phrases. Gagliardo, Nirenberg, symmetrization, embedding.

The first author is supported in part by MTM2004-02299 and by CURE 2005SGR00556.

[1]Here and in what follows all the results are stated for function spaces on $\mathbb{R}^n$ unless explicit mention to the contrary.

Indeed, recalling[2] that $\nabla : BMO \rightarrow BMO^{-1} \subset B_{\infty}^{-1,\infty}$ (cf. [23, Chapter 16]) we see that (1.5) follows from (1.4) applied to ∇f. The inequality (1.5) is stronger than the classical Gagliardo-Nirenberg estimate[3]

$$\|\nabla f\|_{L^4} \leq C \left\|\nabla^2 f\right\|_2^{1/2} \|f\|_{L^\infty}^{1/2}.$$

There are versions of (1.5) for higher order derivatives and other L^p spaces. For example, Strzelecki [37] (cf. also [33]) established that[4]

$$\|\nabla f\|_{L^{2p}} \leq C \|f\|_{BMO}^{1/2} \left\|\nabla^2 f\right\|_{L^p}^{1/2}, 1 < p < \infty,\ f \in W_p^2 \cap BMO, \tag{1.6}$$

or, more generally, $1 < p < \infty, 1 \leq m < k, \theta = \frac{m}{k}, q = \frac{k}{m}p$,

$$\|\nabla^m f\|_{L^q} \leq C \left\|\nabla^k f\right\|_{L^p}^{1-\theta} \|f\|_{BMO}^{\theta},\ f \in W_p^k \cap BMO.$$

In a closely related development, Rivière-Strzelecki (cf. [34] and [37]) have obtained non linear versions of Gagliardo-Nirenberg inequalities: for smooth compactly supported f they prove

$$\int_{\mathbb{R}^n} |\nabla f|^{p+2} \leq C \|f\|_{BMO}^2 \int_{\mathbb{R}^n} \left|\nabla^2 f\right|^2 |\nabla f|^{p-2}. \tag{1.7}$$

As an application these authors obtained regularity results for solutions of nonlinear degenerate elliptic systems of the form

$$-div(|\nabla u|^{p-2} \nabla u) = G(x, u, \nabla u),$$

where G grows like $|\nabla f|^p$.

Once again, the inequality underlying (1.7) involves the Besov space $B_{\infty}^{-1,\infty}$ (cf. [34]), [37])

$$\int_{\mathbb{R}^n} |f|^{p+2} \leq C \|f\|_{B_{\infty}^{-1,\infty}}^2 \int_{\mathbb{R}^n} |\nabla f|^2 |f|^{p-2}.$$

Except for Ledoux, all these authors use sophisticated tools that come from harmonic analysis: Littlewood-Paley theory, wavelets (cf. [13], [11], [12], [30]), connection with interpolation theory (cf. [11], [12]), duality between H^1 and BMO (cf. [37]). Ledoux's methods [22] stand out since they are elementary and based only on the use of some new, but straightforward, Poincaré inequalities. Nevertheless, Ledoux has been able to extend (1.3) to general settings, where, in particular, the original methods are not available.

In this paper, we develop a symmetrization approach to the sharp Gagliardo-Nirenberg inequalities described above. Our approach hinges on interpolation of the Ledoux-Poincaré inequalities. For example, in Theorem 1 (i) below, we show that for $f \in (W_1^1 + W_\infty^1) \cap B_{\infty,\infty}^{\alpha}, \alpha < 0$, we have

$$f^{**}(s) \leq c(|\alpha|, n) |\nabla f|^{**}(s)^{\frac{|\alpha|}{1+|\alpha|}} \|f\|_{B_{\infty,\infty}^{\alpha}}^{\frac{1}{1+|\alpha|}}, \tag{1.8}$$

[2] BMO^{-1} is the space of derivatives of functions in BMO.

[3] For stronger pointwise Gagliardo-Nirenberg inequalities see also Maz'ya-Shaposhnikova [29].

[4] Observe that (1.6) follows from (1.3) applied with $q = 2p, \theta = 1/2$.

(here $f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)ds$, and f^* is the non-increasing rearrangement of f). In fact, a similar inequality holds if we replace $B^{\alpha}_{\infty,\infty}$ by suitable Morrey spaces (cf. Theorem 1 (ii) below).

Let X be a rearrangement invariant space (see section 2 below). From (1.8) it follows that

$$\|f\|_X \leq c \left\| |\nabla f|^{**} \right\|_{X_{\frac{|\alpha|}{1+|\alpha|}}}^{\frac{|\alpha|}{1+|\alpha|}} \|f\|_{B^{\alpha}_{\infty,\infty}}^{\frac{1}{1+|\alpha|}},$$

where if $a > 0$

$$X_a = \{f : |f|^a \in X, \text{ with } \|f\|_{X_a} = \||f|^a\|_X^{1/a}\}.$$

In particular, for $X = L^q$ and $\alpha = 1 - n/p$, we recover (1.3) for $p > 1$. The case $p = 1$ also follows by first observing that (1.8) readily implies the weak-type inequality $(1, n/(n-1))$, and, consequently, the strong type $(1, n/(n-1))$ follows by Maz'ya's truncation principle (cf. [18] and also [22]).

In this fashion we can also handle the nonlinear inequalities of Rivière-Strzelecki type. To give a flavor of the results, we show here[5] an easy approach to an inequality closely related to (1.7). Indeed, suppose that $p \geq 2$. Then

$$\begin{aligned}\int_{\mathbb{R}^n} |f|^{p+2} &= \int_0^\infty (f^*)^p (f^*)^2 \\ &\leq c \|f\|_{B^{-1}_{\infty\infty}} \int_0^\infty (f^*)^p |\nabla f|^{**} \quad \text{(by (1.8))} \\ &\leq c \|f\|_{B^{-1}_{\infty\infty}} \int_0^\infty (f^*(t))^{\frac{p+2}{2}} \left(\frac{1}{t} \int_0^t (f^*(s))^{\frac{p-2}{2}} |\nabla f|^* (s) ds \right) dt.\end{aligned}$$

Therefore, by Cauchy-Schwartz and Hardy's inequalities,

$$\int_{\mathbb{R}^n} |f|^{p+2} \leq c \|f\|_{B^{-1}_{\infty\infty}} (\int_{\mathbb{R}^n} |f|^{p+2})^{1/2} (\int_0^\infty (f^*(s))^{p-2} |\nabla f|^* (s)^2 dt)^{1/2},$$

and, rearranging terms, we have obtained

$$\int_{\mathbb{R}^n} |f|^{p+2} \leq C \|f\|^2_{B^{-1}_{\infty\infty}} \int_0^\infty (f^*(s))^{p-2} |\nabla f|^* (s)^2 dt.$$

Applying the last inequality to ∇f gives

$$\int_{\mathbb{R}^n} |\nabla f|^{p+2} \leq c \|f\|^2_{BMO} \int_0^\infty (|\nabla f|^*)^{p-2} (|\nabla^2 f|^*)^2.$$

The motivation for the next result comes from our recent work on sharp Sobolev inequalities. In this connection, note that the new Gagliardo-Nirenberg inequalities provide a sharpening of the classical Sobolev inequalities. For example, if $1 \leq p < n$, $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$, and $\theta = \frac{p}{p^*}$, then (1.3) reads

$$\|f\|_{p^*} \leq c \|\nabla f\|_p^{p/p^*} \|f\|^{1-p/p^*}_{B^{\frac{p}{p-p^*}}_{\infty,\infty}}, \tag{1.9}$$

[5] A somewhat more sophisticated argument involving integration by parts as in [14] or [37] combined with the symmetrization inequality (1.8) gives (1.7) and more (see Theorem 4 below).

and, therefore, the classical Sobolev inequality

$$\|f\|_{p^*} \leq c\,\|\nabla f\|_p$$

follows from the fact that $L^{p^*} \subset B_{\infty,\infty}^{\frac{p}{p-p^*}}$. However, as is well known, the sharpest Sobolev inequality involves Lorentz spaces, specifically, for $1 \leq p \leq n$, $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$,

$$\|f\|_{L(p^*,p)} \leq c\,\|\nabla f\|_{L^p}\,,$$

where

$$\|f\|_{L(q,r)} = \left(\int_0^\infty \left(f^{**}(t)t^{\frac{1}{q}}\right)^r \frac{dt}{t}\right)^{1/r},\ 1 < q < \infty,\ 0 \leq r < \infty. \tag{1.10}$$

The case $p < n$ is, of course, classical, but the limiting case $p = n$, improving on [27], [8], [19] and the references therein, was formulated in this fashion only recently in Bastero-Milman-Ruiz [2]. It requires defining the limiting Lorentz spaces $L(\infty, q)$, by replacing in (1.10) f^{**} by the oscillation $f_o^*(t) := f^{**}(t) - f^*(t)$. The idea of using the oscillation $f_o^*(t)$ in connection with embeddings seems to have originated in the work of Garsia and Rodemich [17]. The limiting space $L(\infty, \infty)$ was introduced[6] in [4], where it is shown[7] that

$$BMO \subset L(\infty, \infty),$$

and, in fact, on a cube

$$L(\infty,\infty) \text{ is the rearrangement invariant hull of } BMO.$$

To obtain sharp Gagliardo-Nirenberg-Sobolev inequalities with Lorentz spaces we need to sharpen (1.3). We refine the symmetrization inequality (1.8) as follows (cf. Theorem 1 (iii) below):

$$f^{**}(s) \leq c(|\alpha|\,, n)\,|\nabla f|^{**}(s)^{\frac{|\alpha|}{1+|\alpha|}} \left[\left(\sup_{t>0} t^{-\frac{\alpha}{2}}\,|P_t f(\cdot)|\right)^{**}(s)\right]^{\frac{1}{1+|\alpha|}}. \tag{1.11}$$

Assume $|\alpha| < 1 - \frac{n}{p}$. Then from (1.11) and Hölder inequality it follows that for all $0 < \phi \leq 1$ we have

$$\|f\|_{L(p^*,\phi p)} \leq c\,\|\nabla f\|_{L^p}^{\frac{|\alpha|}{1+|\alpha|}} \left\|\left(\sup_{t>0} t^{-\frac{\alpha}{2}}\,|P_t f(\cdot)|\right)\right\|_{L\left(\frac{pn}{n-(1+|\alpha|)p},\frac{\phi p}{1+|\alpha|(1-\phi)}\right)}^{\frac{|\alpha|}{1+|\alpha|}}.$$

We can interpret the condition $\left\|\left(\sup_{t>0} t^{-\frac{\alpha}{2}}\,|P_t f(\cdot)|\right)\right\|_{L\left(\frac{pn}{n-(1+|\alpha|)p},\frac{\phi p}{1+|\alpha|(1-\phi)}\right)} < \infty$ as a Triebel-Lizorkin condition corresponding to the space $F^{\alpha}_{L\left(\frac{pn}{n-(1+|\alpha|)p},\frac{\phi p}{1+|\alpha|(1-\phi)}\right),\infty}$ (cf. Jawerth [20] for an account of the basic theory of Triebel-Lizorkin spaces). The Gagliardo-Nirenberg effect is achieved since with weaker integrability conditions on the gradient, namely $\|\nabla f\|_{L^p} < \infty$, we improve the optimal integrability to $\|f\|_{L(p^*,p)} \leq \|f\|_{L(p^*,\phi p)} < \infty$ if we assume the extra Triebel-Lizorkin condition.

[6]For more on spaces defined in terms of oscillation see [10] and [32].

[7]The containment $BMO \subset L(\infty,\infty)$ is proved in [4] when the domain is a cube; the general result for $f \in L^1 + L^\infty$ is in [35].

The study of the limiting case $p = n$ led us to consider the following symmetrization inequality (cf. [2] and more recently [25] for a more general formulation):

$$f_o^*(t) \leq ct^{1/n}(\nabla f)^{**}(t),\ f \in C_0^\infty,$$

which we can rewrite as

$$f_o^*(t) \leq c[t^{1/n}(\nabla f)^{**}(t)]^\theta f_o^*(t)^{1-\theta},\ 0 \leq \theta \leq 1. \tag{1.12}$$

Suppose that $\frac{1}{r} = \frac{(1-\theta)}{p^*} + \frac{\theta}{s}, \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, 0 \leq \theta \leq 1$. Then, from (1.12) and Hölder's inequality we get the classical Gagliardo-Nirenberg inequality

$$\|f\|_{L(r,q)} \leq c \|\nabla f\|_{L(p,q_0)}^{1-\theta} \|f\|_{L(s,q_1)}^\theta, \tag{1.13}$$

in its formulation for Lorentz spaces. Note that when $\theta = 0$ and $q_0 = q = p, r = p^*$, (1.13) gives back the sharp Sobolev inequality (1.9) up the end point $p = n$. The same method works for domains using the symmetrization inequalities of [25], and the fractional case follows likewise from the symmetrization inequalities in [24].

In conclusion, we should mention that there is a well known connection between Gagliardo-Nirenberg inequalities and interpolation/extrapolation which is expressed in part by the equivalence between inequalities of the form

$$\|x\|_Z \leq \|x\|_{X_0}^{1-\theta} \|x\|_{X_1}^\theta$$

and the embedding

$$(X_0, X_1)_{\theta,1} \subset Z. \tag{1.14}$$

This is well known (cf. [5]) and is exploited for example in [12]. The connection with extrapolation (cf. [21]) takes into account the constants of embeddings of the type (1.14). (In this connection see also [6], [7], [28], [31], the references therein, as well as our forthcoming paper [26], where we also deal with the connection with the Calderón product method (see also the discussion preceding (1.13)).

2. Preliminaries

In this section we consider a brief description of the spaces we shall deal with in this paper.

Let $P_t = e^{t\triangle}$, $t \geq 0$, be the heat semigroup on $\mathbb{R}^n$. For $\alpha \in \mathbb{R}$, $\alpha < 0$, the homogeneous Besov space $B^\alpha_{\infty,\infty}$ (see for example [22] and the references quoted therein) is the space of tempered distributions f on $\mathbb{R}^n$ for which the Besov norm

$$\|f\|_{B^\alpha_{\infty,\infty}} = \sup_{t>0} t^{-\alpha/2} \|P_t f\|_\infty$$

is finite.

For $\alpha < 0$, the Morrey spaces M^α_∞ are defined by the condition

$$\|f\|_{M^\alpha_\infty} := \sup_{r>0,\ x\in\mathbb{R}^n} r^{-\alpha} |f_r(x)| < \infty,$$

where we let

$$f_r(x) = \frac{1}{|B(x,r)|} \int_{B(x,r)} f(z)dz, \quad r > 0,\ x \in \mathbb{R}^n.$$

A rearrangement invariant space (r.i. space), X is a Banach function space of Lebesgue measurable functions $\mathbb{R}^n$ endowed with a norm $\|\cdot\|_X$ that satisfies the Fatou property and is such that, if $f \in X$, and $g^* = f^*$, then $g \in X$ and $\|g\|_X = \|f\|_X$.

Some important examples of r.i. spaces are the L^p spaces, Lorentz spaces $L(p,q)$ and Orlicz spaces (see [3]).

Every r.i. space X has a representation as a function space on $X\hat{}(0,\infty)$ such that[8]

$$\|f\|_X = \|f^*\|_{X\hat{}(0,|\Omega|)}.$$

Since it will simplify our considerations, we assume throughout what follows that we are working with r.i. spaces such that[9]

$$\|f^*\|_X \simeq \|f^{**}\|_X,$$

where as usual, the symbol $f \simeq g$ will indicate the existence of a universal constant $c > 0$ (independent of all parameters involved) so that $(1/c)f \leq g \leq c\,f$, while the symbol $f \preceq g$ means that $f \leq c\,g$.

3. Gagliardo-Nirenberg inequalities with Besov spaces of negative order

The purpose of this section is to establish the following:

Theorem 1. *Let $\alpha < 0$. Then*

(i) for every $f \in (W^{1,1} + W^{1,\infty}) \cap B^{\alpha}_{\infty,\infty}$, we have

$$f^{**}(s) \leq c(|\alpha|, n) \left|\nabla f\right|^{**}(s)^{\frac{|\alpha|}{1+|\alpha|}} \|f\|_{B^{\alpha}_{\infty,\infty}}^{\frac{1}{1+|\alpha|}};$$

(ii) for every $f \in (W^1_1 + W^1_\infty) \cap M^{\alpha}_{\infty}$ we have

$$f^{**}(s) \leq c(|\alpha|, n) \left|\nabla f\right|^{**}(s)^{\frac{|\alpha|}{1+|\alpha|}} \|f\|_{M^{\alpha}_{\infty}}^{\frac{1}{1+|\alpha|}}.$$

(iii) Let $f \in (W^1_1 + W^1_\infty)$ be such that $\left(\sup_{t>0} t^{-\frac{\alpha}{2}} \left|P_t f(\cdot)\right|\right) \in L^1 + L^\infty$. Then,

$$f^{**}(s) \leq c(|\alpha|, n) \left|\nabla f\right|^{**}(s)^{\frac{|\alpha|}{1+|\alpha|}} \left[\left(\sup_{t>0} t^{-\frac{\alpha}{2}} \left|P_t f(\cdot)\right|\right)^{**}(s)\right]^{\frac{1}{1+|\alpha|}}, s > 0.$$

We give the proof of Theorem 1 below, but, before that, we need to develop a suitable interpolation tool. Indeed, a crucial step in the proof of these results is the computation of certain $K-$functionals associated with homogeneous Sobolev spaces, which we shall treat in the next subsection.

3.1. Interpolation of Sobolev spaces. The $K-$functional for the pair (W^1_1, W^1_∞) has been computed by DeVore-Scherer [16],

$$K(t, f; W^1_1, W^1_\infty) \approx \sum_{|\alpha| \leq 1} \int_0^t (D^\alpha f)^*(s)ds. \tag{3.1}$$

However, in our development we need a precise estimate of the $K-$functional for the corresponding homogeneous pair $(\mathring{W}^1_1, \mathring{W}^1_\infty)$, which is not explicitly stated in the literature. In connection with this problem, we refer to Coulhon-Auscher [1], Section

[8]Since the measure space will be always clear from the context, it is convenient to "drop the hat" and use the same letter X to indicate the different versions of the space X that we use.

[9]This condition means that X is not too close to L^1 (see [3] for a description of these classes of r.i. spaces.)

1.3, where the problem[10] is explicitly mentioned. We now show that the proof given in [9] can be suitably modified to prove (3.1).

Theorem 2. *Let* $f \in W_1^1 + W_\infty^1$, *and let*

$$K(t,f) = K(t,f;\mathring{W}_1^1,\mathring{W}_\infty^1) = \inf_{\substack{f=f_0+f_1 \\ f_0\in W_1^1,\ f_1\in W_\infty^1}} \left(\||\nabla f_0|\|_1 + t\,\||\nabla f_1|\|_\infty \right).$$

Then, with constants independent of f, *we have*

$$K(t,f) \simeq \int_0^t |\nabla f|^*(s)ds. \tag{3.2}$$

Proof. Obviously

$$\int_0^t |\nabla f|^*(s)ds \le K(t,f).$$

To prove the reverse inequality we follow the proof in [9] for the non homogeneous case. Assume first that $f \in C_0^\infty$, and let $t > 0$ be given. For each multi-index α, with $|\alpha| = 1$, let

$$E_\alpha = \left\{ x : M(D^\alpha f)(x) > \left(M(D^\alpha f)\right)^*(t) \right\}, \quad E = \bigcup_{|\alpha|=1} E_\alpha,$$

where M is the maximal operator of Hardy-Littlewood. It is clear that E is open and $|E| \le nt$.

Consider the function $\bar{f} = f\chi_{E^c}$. We will show below that

$$\sup_{x,y\in E^c} \frac{\left|\bar{f}(x) - \bar{f}(y)\right|}{|x-y|} \le c \sum_{|\alpha|=1} M(D^\alpha f)^*(t). \tag{3.3}$$

Moreover, since we obviously have $\left\|\bar{f}\right\|_\infty \le \|f\|_\infty$, we see that $\bar{f} \in Lip(1,E^c)$. This given, we can use Whitney's extension theorem (see [36, Theorem 3, pag 174]) to produce $f_0 \in Lip(1,\mathbb{R}^n) = W^{1,\infty}(\mathbb{R}^n)$ with $f_0(x) = \bar{f}(x)$, $D^\alpha f_0(x) = D^\alpha \bar{f}(x)$, $|\alpha| = 1$, $x \in E^c$, and $\|f_0\|_{W^{1,\infty}} \le \left\|\bar{f}\right\|_{Lip(1,E^c)}$. Set $f_1 = f - f_0$.

A perusal of the proof ([36, Theorem 3, pag 174]) shows that

$$\sup_{x,y\in\mathbb{R}^n} \frac{|f_0(x) - f_0(y)|}{|x-y|} \le c' \sum_{|\alpha|=1} \left(M(D^\alpha f)\right)^*(t),$$

with $c' = c'(n)$. Thus,

$$t\,\|\nabla f_0\|_\infty \le c' t \sum_{|\alpha|=1} \left(M(D^\alpha f)\right)^*(t) \preceq \sum_{|\alpha|=1} \int_0^t (D^\alpha f)^*(s)ds. \tag{3.4}$$

We also have

$$\||\nabla f_1|\|_1 \le \sum_{|\alpha|=1} \|D^\alpha f \chi_E\|_1 + \sum_{|\alpha|=1} \|D^\alpha f_0 \chi_E\|_1 = I_1 + I_2.$$

[10] Indeed, using our formula for the $K-$functional of $(\mathring{W}_1^1, \mathring{W}_\infty^1)$ simplifies the development of some portions of [1].

Since $|E| \leq nt$,

$$I_1 \leq n \sum_{|\alpha|=1} \int_0^t (D^\alpha f)^*(s)ds.$$

On the other hand by (3.4) we get

$$\begin{aligned} I_2 &= \sum_{|\alpha|=1} \int_E |D^\alpha f_0| (x)dx \\ &\leq \left(\sup_{x,y \in \mathbb{R}^n} \frac{|f_0(x) - f_0(y)|}{|x-y|} \right) |E| \\ &\preceq \sum_{|\alpha|=1} (M(D^\alpha f))^* (t)t \\ &\preceq \sum_{|\alpha|=1} \int_0^t (D^\alpha f)^*(s)ds. \end{aligned}$$

Therefore

$$\begin{aligned} K(t,f) &\preceq \|\nabla f_1\|_1 + t \|\nabla f_0\|_\infty \\ &\preceq \sum_{|\alpha|=1} \int_0^t (D^\alpha f)^*(s)ds. \end{aligned}$$

Having established (3.2) for $f \in C_0^\infty$, we can easily show that (3.2) does in fact hold for all $f \in W_1^1 + W_\infty^1$, using the arguments given in [16], and we shall omit the details.

Let us now prove (3.3). Let $x_1, x_2 \in E^c$, $r = |x_1 - x_2|$. Let Q_i $i = 1, 2$, be cubes centered at x_i, $i = 1, 2$, with sides parallel to the coordinate axes with length equal to $4r/3$. Then there exists a constant c_n, independent of x_i, such that

$$|Q_1 \cap Q_2| \geq c_n |Q_i|, \qquad i = 1, 2. \tag{3.5}$$

Let

$$H_i = \left\{ y \in Q_i : \frac{|f(x_i) - f(y)|}{r} > \lambda(t)N \right\}, \qquad i = 1, 2,$$

where $\lambda(t) = \sum_{|\alpha|=1} M(D^\alpha f)^*(t)$, and N is a positive fixed number to be determined precisely later.

Using Chebyshev's inequality and Lemma 2.5 of [9], we get

$$\begin{aligned} |H_i| &\leq \frac{1}{\lambda(t)N} \int_{Q_i} \frac{|f(x_i) - f(y)|}{r} dy \leq \frac{c}{\lambda(t)N} \sum_{|\alpha|=1} M(D^\alpha f)(x_i) |Q_i| \\ &\leq \frac{c}{N} |Q_i| \qquad (\text{since } x_i \in E^c). \end{aligned}$$

Now choose N so big that $C/N < c_n/2$, where c_n is the constant appearing in (3.5). Therefore, $|H_1 \cap H_2| < |Q_1 \cap Q_2|$. Let $z \in (Q_1 \cap Q_2) \setminus (H_1 \cup H_2)$. Then

$$|f(x_i) - f(z)| \leq r\lambda(t)N, \qquad i = 1, 2,$$

and, consequently,

$$\left| \bar{f}(x_1) - \bar{f}(x_2) \right| = |f(x_1) - f(x_2)| \leq c\lambda(t) |x_1 - x_2|, \qquad \forall x_1, x_2 \in E^c,$$

as we wished to show. □

3.1.1. *Proof of Theorem 1.*

Proof. **(i)**. Fix $t>0$. We claim that for all $s>0$

$$(f-P_tf)^{**}(s)\le ct^{1/2}(\nabla f)^{**}(s), \tag{3.6}$$

where $c>0$ depends only on n.

Assuming the validity of (3.6), we obtain

$$\begin{aligned} f^{**}(s) &\le (f-P_tf)^{**}(s)+(P_tf)^{**}(s)\\ &\preceq t^{1/2}(\nabla f)^{**}(s)+t^{\frac{\alpha}{2}}t^{-\frac{\alpha}{2}}(P_tf)^{**}(s)\\ &\le t^{1/2}(\nabla f)^{**}(s)+t^{\frac{\alpha}{2}}\sup_{t>0}\left(t^{-\frac{\alpha}{2}}\sup_{s>0}(P_tf)^{**}(s)\right)\\ &= t^{1/2}(\nabla f)^{**}(s)+t^{\frac{\alpha}{2}}\sup_{t>0}t^{-\alpha/2}\|P_tf\|_\infty\\ &= t^{1/2}(\nabla f)^{**}(s)+t^{\frac{\alpha}{2}}\|f\|_{B^{\alpha}_{\infty,\infty}}. \end{aligned}$$

It follows that

$$\begin{aligned} f^{**}(s) &\le \inf_{t>0}\left(t^{1/2}(\nabla f)^{**}(s)+t^{\frac{\alpha}{2}}\|f\|_{B^{\alpha}_{\infty,\infty}}\right)\\ &\le c(|\alpha|,n)\,|\nabla f|^{**}(s)^{\frac{|\alpha|}{1+|\alpha|}}\|f\|_{B^{\alpha}_{\infty,\infty}}^{\frac{1}{1+|\alpha|}}. \end{aligned}$$

We now prove (3.6). Our main tool here is the pseudo-Poincaré inequalities of [22], namely for f in W_1^1 (resp. $f\in W_\infty^1$), we have for all $t>0$,

$$\|f-P_tf\|_1\le c(n)t^{1/2}\|\nabla f\|_1 \quad (\text{resp.}\quad \|f-P_tf\|_\infty\le c(n)t^{1/2}\|\nabla f\|_\infty). \tag{3.7}$$

We indicate the proofs for the sake of completeness. Write

$$f-P_tf=\int (f(\cdot)-f(\cdot-y))\,p_t(y)dy,$$

where $p_t(y)$ is the Gauss-Weirstrass kernel. Then

$$\|f-P_tf\|_1\le\int\|f(\cdot)-f(\cdot-y)\|_1\,p_t(y)dy.$$

Moreover, since for $f\in W_1^1$ we have $\|f(\cdot)-f(\cdot-y)\|_1\le|y|\,\|\nabla f\|_1$, the conclusion follows with $c(n)=\int|y|\,e^{-\frac{|y|^2}{2}}(2\pi)^{-n/2}dy$. (The case involving L^∞−norms is similar).

Let $f\in W_1^1+W_\infty^1$. Then for any decomposition $f=f_0+f_1$, with $f_0\in W_1^1$, $f_1\in W_\infty^1$, we have

$$f-P_tf=(f_0-P_tf_0)+(f_1-P_tf_1).$$

Let $s>0$. Then by (3.7)

$$\|f_0-P_tf_0\|_1+s\,\|f_1-P_tf_1\|_\infty\le c(n)^{1/2}t^{1/2}(\|\nabla f_0\|_1+s\,\|\nabla f_1\|_\infty).$$

If we combine the last inequality with the formula for the $K-$functional for the pair (L^1, L^∞) (cf. [3, Chapter 5, Theorem 1.6]), we get

$$\begin{aligned} s\left(f-P_t f\right)^{**}(s) &= \inf_{\substack{f-P_t f=g_0+g_1 \\ g_0\in L^1,\ g_1\in L^\infty}} \left(\|g_0\|_1 + s\|g_1\|_\infty\right) \\ &\leq \inf_{\substack{f=f_0+f_1 \\ f_0\in W^{1,1},\ f_1\in W^{1,\infty}}} \left(\|f_0-P_t f_0\|_1 + s\|f_1-P_t f_1\|_\infty\right) \\ &\leq c(n)^{1/2}t^{1/2} \inf_{\substack{f=f_0+f_1 \\ f_0\in W_1^1, f_1\in W_\infty^1}} \left(\|\nabla f_0\|_1 + s\|\nabla f_1\|_\infty\right) \\ &= c(n)^{1/2}t^{1/2}K(s,f). \end{aligned}$$

At this point, we can invoke Theorem 2 to obtain

$$s\left(f-P_t f\right)^{**}(s) \preceq t^{1/2}s\left(\nabla f\right)^{**}(s),$$

and (3.6) follows, thus concluding the proof of Theorem 1.

(ii). We only sketch the proof since it follows *mutatis mutandis*. Once again the key ingredients are known pseudo-Poincaré inequalities for averages (cf. [15]): If $f \in W_1^1$ (or $f \in W_\infty^1$), then for all $t > 0$,

$$\|f-f_r\|_1 \leq c(n)r\|\nabla f\|_1 \quad (\text{resp.} \quad \|f-f_r\|_\infty \leq c(n)r\|\nabla f\|_\infty).$$

From this point onward, the proof is the same as (i).

(iii). By the triangle inequality

$$|f(x)| \leq |f(x)-P_t f(x)| + t^{\alpha/2}\sup_{t>0} t^{-\frac{\alpha}{2}}\left|P_t f(x)\right|.$$

Taking rearrangements

$$\begin{aligned} f^{**}(s) &\leq \left(f-P_t f\right)^{**}(s) + t^{\alpha/2}\left(\sup_{t>0} t^{-\frac{\alpha}{2}}\left|P_t f(\cdot)\right|\right)^{**}(s) \\ &\preceq t^{1/2}\left(\nabla f\right)^{**}(s) + t^{\alpha/2}\left(\sup_{t>0} t^{-\frac{\alpha}{2}}\left|P_t f(\cdot)\right|\right)^{**}(s). \end{aligned}$$

Consequently,

$$\begin{aligned} f^{**}(s) &\leq \inf_{t>0}\left(t^{1/2}\left(\nabla f\right)^{**}(s) + t^{\alpha/2}\left(\sup_{t>0} t^{-\frac{\alpha}{2}}\left|P_t f(\cdot)\right|\right)^{**}(s)\right) \\ &\leq c(|\alpha|,n)\left|\nabla f\right|^{**}(s)^{\frac{|\alpha|}{1+|\alpha|}}\left[\left(\sup_{t>0} t^{-\frac{\alpha}{2}}\left|P_t f(\cdot)\right|\right)^{**}(s)\right]^{\frac{1}{1+|\alpha|}}, \end{aligned}$$

as we wished to show. □

3.2. Gagliardo-Nirenberg inequalities with BMO terms. In this brief section we indicate how we can iterate our inequalities to recover, without effort, the results in Strzelecki [37]. As a matter of fact, we give a slight extension and formulate our result in the setting of r.i. spaces.

Given $k \in \mathbb{N}$, we denote by $d^k g$ the vector $(\partial^\beta g)_{|\beta|=k}$ of all derivatives of order $|\beta| = k$.

Given a r.i. space X , $W^{k,X}$ is the Sobolev space

$$W^{k,X} = \left\{f : \partial^\beta f \in X, \text{ for all } \beta, \ |\beta| \leq k\right\},$$

provided with the norm

$$\|f\|_{W^{k,X}} = \sum_{0\leq|\beta|\leq k} \left\|\partial^\beta f\right\|_X .$$

Theorem 3. *Let X be an r.i space. Let $1 \leq m < k$, with $k, m \in \mathbb{N}$. Then*

$$\|d^m f\|_{X(\frac{k}{m})} \preceq \left\|d^k f\right\|_X^{\frac{m}{k}} \|f\|_{BMO}^{1-\frac{m}{k}}, f \in W^{k,X}(\mathbb{R}^n) \cap BMO.$$

Proof. Let $f \in W^{k,X}(\mathbb{R}^n) \cap BMO$, $|\beta| = m$ and consider $\partial^\beta f$. Then

$$\partial^\beta f \in B^{-m}_{\infty,\infty} \text{ with } \left\|\partial^\beta f\right\|_{B^{-m}_{\infty,\infty}} \preceq \|f\|_{BMO} .$$

By Theorem 1,

$$\left|\partial^\beta f\right|^{**}(s) \preceq \left|\nabla\partial^\beta f\right|^{**}(s)^{\frac{m}{1+m}} \left\|\partial^\beta f\right\|_{B^{-m}_{\infty,\infty}}^{\frac{1}{1+m}} \preceq \left|\nabla\partial^\beta f\right|^{**}(s)^{\frac{m}{1+m}} \|f\|_{BMO}^{\frac{1}{1+m}} .$$

Therefore,

$$|d^m f|^{**}(s) \leq \sum_{|\beta|=m} \left|\partial^\beta f\right|^{**}(s) \preceq \sum_{|\beta|=m} \left|\nabla\partial^\beta f\right|^{**}(s)^{\frac{m}{1+m}} \|f\|_{BMO}^{\frac{1}{1+m}} \tag{3.8}$$

$$\preceq \left|d^{m+1} f\right|^{**}(s)^{\frac{m}{1+m}} \|f\|_{BMO}^{\frac{1}{1+m}} . \tag{3.9}$$

Let now $|\beta| = m + 1$ and consider

$$\partial^\beta f \in B^{-(m+1)}_{\infty,\infty} \text{ with } \left\|\partial^\beta f\right\|_{B^{-(m+1)}_{\infty,\infty}} \leq \|f\|_{BMO} .$$

Then, using the same argument as before,

$$\left|d^{m+1} f\right|^{**}(s) \preceq \left|d^{m+2} f\right|^{**}(s)^{\frac{m+1}{m+2}} \|f\|_{BMO}^{\frac{1}{m+2}} . \tag{3.10}$$

Combining (3.8) and (3.10) we have that

$$\begin{aligned} |d^m f|^{**}(s) &\preceq \left(\left|d^{m+2} f\right|^{**}(s)^{\frac{m+1}{m+2}} \|f\|_{BMO}^{\frac{1}{m+2}}\right)^{\frac{m}{m+1}} \|f\|_{BMO}^{\frac{1}{m+1}} \\ &= \left|d^{m+2} f\right|^{**}(s)^{\frac{m}{m+2}} \|f\|_{BMO}^{\frac{2}{m+2}} . \end{aligned}$$

Let $k = m + j$. Iterating the previous process $j - 2$ times, we get

$$|d^m f|^{**}(s) \preceq \left|d^k f\right|^{**}(s)^{\frac{m}{k}} \|f\|_{BMO}^{\frac{k-m}{k}} ,$$

which implies

$$\|d^m f\|_{X(\frac{k}{m})} \preceq \left\|d^k f\right\|_X^{\frac{m}{k}} \|f\|_{BMO}^{1-\frac{m}{k}} ,$$

as we wished to show. □

3.3. Nonlinear Gagliardo-Nirenberg inequalities. We provide a new approach and extensions to the work of Rivière-Strzelecki (cf. [34] and [37]) mentioned in the Introduction. Our approach is amenable to further generalizations.

Theorem 4. *Let $p \geq 2$. Then, for every $f \in C_0^\infty(\mathbb{R}^n)$,*

$$\int_{\mathbb{R}^n} |f|^{p+1} \preceq \|f\|_{B^{-1}_{\infty,\infty}} \int_{\mathbb{R}^n} \left|\nabla^2 f\right|^2 |\nabla f|^{p-2}.$$

Consequently,

$$\int_{\mathbb{R}^n} |\nabla f|^{p+1} \preceq \|f\|_{BMO} \int_{\mathbb{R}^n} \left|\nabla^2 f\right|^2 |\nabla f|^{p-2}. \tag{3.11}$$

Proof. Let $f \in C_0^\infty(\mathbb{R}^n)$. Since $p+1 \geq 2$, integrating by parts, we obtain

$$\||\nabla f|\|^{p+1} = -\int_{\mathbb{R}^n} div\left(|\nabla f|^{p-1} \nabla f\right) f.$$

From

$$\left|div\left(|\nabla f|^{p-1} \nabla f\right)\right| \leq (p-1+\sqrt{n}) |\nabla f|^{p-1} \left|\nabla^2 f\right|$$

we get

$$\||\nabla f|\|^{p+1} \leq (p-1+\sqrt{n}) \int_{\mathbb{R}^n} |\nabla f|^{p-1} \left|\nabla^2 f\right| |f|.$$

Let $I = \int_{\mathbb{R}^n} |\nabla f|^{p-1} \left|\nabla^2 f\right| |f|$. Then

$$\begin{aligned} I &= \int_0^\infty \left(|\nabla f|^{p-1} \left|\nabla^2 f\right| |f|\right)^*(s)ds = \int_0^\infty \left(|\nabla f|^{\frac{p-2}{2}+\frac{p}{2}} \left|\nabla^2 f\right| |f|\right)^*(s)ds \\ &\leq \int_0^\infty \left(|\nabla f|^{\frac{p}{2}}\right)^*(s) f^*(s) \left(|\nabla f|^{\frac{p-2}{2}} \left|\nabla^2 f\right|\right)^*(s)ds \\ &= \int_0^\infty |\nabla f|^*(s)^{\frac{p}{2}} f^*(s) \left(|\nabla f|^{\frac{p-2}{2}} \left|\nabla^2 f\right|\right)^*(s)ds \\ &\leq \int_0^\infty |\nabla f|^{**}(s)^{\frac{p}{2}} f^{**}(s) \left(|\nabla f|^{\frac{p-2}{2}} \left|\nabla^2 f\right|\right)^*(s)ds. \end{aligned}$$

Therefore, by Theorem 1 and Hölder's inequality,

$$\begin{aligned} I &\preceq \|f\|^{1/2}_{B^{-1}_{\infty,\infty}} \int_0^\infty |\nabla f|^{**}(s)^{\frac{p+1}{2}} \left(|\nabla f|^{\frac{p-2}{2}} \left|\nabla^2 f\right|\right)^*(s)ds \\ &\leq \|f\|^{1/2}_{B^{-1}_{\infty,\infty}} \left(\int_0^\infty |\nabla f|^{**}(s)^{p+1}\right)^{1/2} \left(\int_0^\infty \left(\left(|\nabla f|^{\frac{p-2}{2}} \left|\nabla^2 f\right|\right)^*(s)\right)^2 ds\right)^{1/2} \\ &\preceq \|f\|^{1/2}_{B^{-1}_{\infty,\infty}} \left(\int_{\mathbb{R}^n} |\nabla f|^{p+1}\right)^{1/2} \left(\int_{\mathbb{R}^n} \left|\nabla^2 f\right|^2 |\nabla f|^{p-2}\right)^{1/2} \end{aligned}$$

and the result follows. □

Remark 1. *An analogous result holds for Morrey spaces and can be proved using the second part of Theorem 1. We leave the details to the interested reader.*

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