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# LIPSCHITZ HARMONIC CAPACITY AND BILIPSCHITZ IMAGES OF CANTOR SETS

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Abstract. For bilipschitz images of Cantor sets in $\mathbb{R}^d$ we estimate the Lipschitz harmonic capacity and prove that this capacity is invariant under bilipschitz homeomorphisms. A crucial step of the proof is an estimate of the $L^2$ norms of the Riesz tranforms on $L^2(G,p)$ where $p$ is the natural probability measure on the Cantor set $E$ and $G \subset E$ has $p(G) > 0$.

## 1. Introduction

Let $Lip^1_{\text{loc}}$ be the set of locally Lipschitz real functions on Euclidean space $\mathbb{R}^d$, let $E$ be a compact subset of $\mathbb{R}^d$, and let

$$L(E,1) = \{f \in Lip^1_{\text{loc}} : \operatorname{supp}(\Delta f) \subset E,\ ||\nabla f||_\infty \le 1; \nabla f(\infty) = 0\}$$

be the set of locally Lipschitz functions harmonic on $\mathbb{R}^d \setminus E$ and normalized by the conditions $||\nabla f||_\infty \le 1$ and $\nabla f(\infty) = 0$. The *Lipschitz harmonic capacity* of $E$ is defined by

$$\kappa(E) = \sup\{|\langle \Delta f, 1\rangle| : f \in L(E,1)\}.$$

It was introduced by Paramonov [P] to study problems of $C^1$ approximation by harmonic functions in $\mathbb{R}^d$.

If $d = 2$ and if the Hausdorff measure $\Lambda_2(E) = 0$, then $f \in L(E,1)$ if and only if $F(z) = f_x - if_y$ is an analytic function on $\mathbb{C} \setminus E$ such that $\bar\partial F$ is real and $|F(z)| \le 1$. In that case it then follows from Green's theorem that $\kappa(E) = 2\pi\gamma_{\mathbb{R}}(E)$, where

$$\gamma_{\mathbb{R}}(E) = \sup\{|\lim_{z\to\infty} zF(z)| : F \text{ is analytic on } \mathbb{C}\setminus E,\ |F| \le 1,\ F(\infty) = 0,\ \bar\partial F \text{ real}\}$$

is the so called *real analytic capacity* of $E$. (See [P].) Moreover, by the main result of [T1], $\gamma_{\mathbb{R}}(E) \le \gamma(E) \le C\gamma_{\mathbb{R}}(E)$ where $\gamma$ is the analytic capacity of $E$ and $C$ is a constant.

Now let $T : \mathbb{R}^d \to \mathbb{R}^d$ be a bilipschitz homeomorphism:

$$A^{-1}|x-y| \le |Tx - Ty| \le A|x-y|. \tag{1}$$

This paper is concerned with the following conjecture.

**Conjecture 1.1.** *If $T$ is a bilipschitz homeomorphism, then*

$$\kappa(T(E)) \le C(A)\kappa(E),$$

*where $A$ is the constant in* (1).

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When $d = 2$ this conjecture was established in [T2] using the connection between analytic capacity and Menger curvature obtained in [T1]. The papers [T1] and [T2] were preceded by two papers [MTV] and [GV] that estimated the analytic capacity of planar Cantor sets and of their bilipschitz images. The recent paper [MT] estimated the Lipschitz harmonic capacity of certain Cantor sets in $\mathbb{R}^d$, and our purpose here is to establish Conjecture 1.1 for bilipschitz images of these Cantor sets. Thus in the language of fractions, this paper is to [MT] as paper [GV] was to [MTV] or paper [T2] was to [T1].

For fixed ratios $\lambda_n$ such that

$$2^{-\frac{d}{d-1}} \le \lambda_n \le \lambda_0 < \frac{1}{2}, \tag{2}$$

we write

$$\sigma_n = \prod_{k=0}^{n} \lambda_k,$$

and define the sets

$$E = \bigcap_{n=0}^{\infty} E_n,\ E_n = \bigcup_{|J|=n} Q_J^n, \tag{3}$$

where $J = (j_1, j_2, \dots, j_n)$ is a multi-index of length $n$ with $j_k \in \{1, 2, \dots 2^d\}$ and the $Q_J^n$ are compact sets such that

$$Q^{n+1}_{(J,j_{n+1})} \subset Q_J^n, \text{ for all } n \text{ and } J,$$

and such that for all $n$ and $J$,

$$c_1\sigma_n \le \operatorname{diam}(Q_J^n) \le c_2\sigma_n, \tag{4}$$

and

$$\operatorname{dist}(Q_J^n, Q_K^n) \ge c_3\sigma_n,\ J \ne K. \tag{5}$$

for positive constants $c_1, c_2$, and $c_3$.

When $Q_J^n$ is a cube with sides parallel to the coordinate axes and side-length $\sigma_n$ and

$$\{Q^{n+1}_{(J,j_{n+1})} \subset\ Q_J^n : j_{n+1} = 1, \dots, 2^d\}$$

consists of the $2^d$ corner subcubes of $Q_J^n$, the set defined by (3) is the Cantor set studied in [MT], and a set $E$ is the bilipschitz image of such a Cantor set if and only if $E$ satisfies (3), (4), and (5). Write

$$\theta_n = \frac{2^{-nd}}{\sigma_n^{d-1}}$$

and $\theta(Q) = \theta_n$ if $Q = Q_J^n$. Note that by (2),

$$\theta_{n+1} \le \theta_n.$$

For Cantor sets it was proved in [MT] that

$$C^{-1}\Big(\sum_{n=0}^{\infty} \theta_n^2\Big)^{-\frac{1}{2}} \le \kappa(E) \le C\Big(\sum_{n=0}^{\infty} \theta_n^2\Big)^{-\frac{1}{2}},$$

where $C$ depends only on the constant $\lambda_0$ in (2) and we extend their result to bilipschitz images of Cantor sets.

**Theorem 1.2.** *If $E$ is defined by* (3), (4), *and* (5), *then there is constant*

$$C = C(c_1, c_2, c_3, \lambda_0)$$

*such that*

$$C^{-1}\Big(\sum_{n=1}^{\infty}\theta_n^2\Big)^{-\frac{1}{2}} \le \kappa(E) \le C\Big(\sum_{n=1}^{\infty}\theta_n^2\Big)^{-\frac{1}{2}}.$$

The proof of Theorem 1.2 follows the reasoning in [MT], but with certain changes. In Section 2 we give some needed geometric properties of the sets $E$. In Section 3 we obtain $L^2$ estimates for the (truncated) Riesz transforms with respect to the probability measure $p$ on $E$ defined by $p(Q_J^n) = 2^{-nd}$ but restricted to a subset $G \subset E$ with $p(G) > 0$. In Section 4 we derive Theorem 1.2 from the $L^2$-estimates in section 3 by applying the dyadic $T(b)$ Theorem of M. Christ to a measure used in [MTV] and [MT].

## 2. The geometry of $E$

Fix $E$ such that (2) - (5) hold.

**Lemma 2.1.** *There is $c_4 = c_4(\lambda_0, c_1, c_2, c_3)$ such that for $j = 1, 2, \ldots, d$, and all $Q_J^n$*

$$\sup_{Q_J^n\cap E} x_j - \inf_{Q_J^n\cap E} x_j \ge c_4\sigma_n. \tag{6}$$

*Proof.* Write

$$w = \sup_{Q_J^n\cap E} x_j - \inf_{Q_J^n\cap E} x_j.$$

Let $\mathcal{P}$ be the hyperplane

$$x_j = \frac{1}{2}\big(\sup_{Q_J^n\cap E} x_j + \inf_{Q_J^n\cap E} x_j\big),$$

and let $\tilde{Q}_K^k$ be the orthogonal projection of $Q_K^k$ onto $\mathcal{P}$. If

$$w < \frac{c_3}{2}\sigma_{n+p}$$

then for $k = n+1, \cdots, n+p$, (5) and the Pythagorean Theorem give

$$\operatorname{dist}(\tilde{Q}_{J'}^k, \tilde{Q}_{J''}^k) \ge \frac{\sqrt{3}}{2}c_3\sigma_k$$

when $\tilde{Q}_{J'}^k \cup \tilde{Q}_{J''}^k \subset Q_J^n$. Consequently there are $(d-1)$-dimensional balls $B_{J'}^k$ with diameter comparable to the diameter of $\tilde{Q}_{J'}^k$ such that

$$\operatorname{dist}(\tilde{Q}_{J'}^k, B_{J'}^k) \le \frac{\sqrt{3}}{4}c_s\sigma_k$$

and

$$B_J^k \cap B_K^m = \varnothing, \text{ when } k \ge m.$$

Hence for constants $c_5 > c_6$ depending only on $d$ and $c_1, c_2$, and $c_3$,

$$\begin{aligned} c_5\sigma_n^{d-1} &\geq \Lambda_{d-1}\Big(\bigcup_{k=1}^{p}\bigcup_{|K|=k} B^{n+k}_{(J,K)}\Big) \\ &= \sum_{k=1}^{p}\sum_{|K|=k}\Lambda_{d-1}\Big(B^{n+k}_{(J,K)}\Big) \\ &\geq \sum_{k=1}^{p} c_6 2^{kd}\sigma_{n+k}^{d-1}, \end{aligned}$$

and by (2) this can only happen if $p \leq \frac{c_5}{c_6}$. Thus (6) holds with $c_4 = c_3 2^{\frac{-d}{d-1}\frac{c_5}{c_6}-1}$. □

Define the probability measure $p$ on $E$ by $p(Q_J^n) = 2^{-nd}$.

**Lemma 2.2.** *There exist $c_7, c_8$, and $0 < \gamma < 1$, depending only on $\lambda_0, c_1, c_2$, and $c_3$ such that for $j = 1, 2, \dots, d$, there exist at least $c_7 2^n$ disjoint slabs of the form*

$$S_k = \{a_k \leq x_j \leq b_k\}$$

*such that $b_k - a_k \leq c_7\sigma_n$, $p(S_k) < c_7\gamma^n$, and $p(\bigcup S_k) \geq c_8$.*

*Proof.* Condition (4) implies that there exist disjoint slabs $S_k$ satisfying all the conditions of the lemma except possibly $p(S_k) \leq c_7\gamma^n$. However, by Lemma 2.1 there exists $m_0$ such that if $m \leq n - m_0$, then for each $Q_J^m$ at most $2^d - 1$ cubes $Q_K^{m+1} \subset Q_J^m$ can meet $S_k$. Hence the number of $Q_L^n$ with $Q_L^n \cap S_k \neq \varnothing$ does not exceed $(2^d-1)^{(n-m_0)}2^{dm_0}$ and $p(S_k) \leq (1-2^{-d})^{n-m_0} \leq c_7\gamma^n$. □

## 3. The $L^2$ estimate

Let $E$ satisfy properties (2) - (5). For $x \in E$ we define $Q_x^n = Q_J^n$ to be the unique $Q_J^n$ such that $x \in Q_J^n$. If $f \in L^2(p)$ and $j = 1, 2, \dots, d$, we define the truncated Riesz transform as

$$R_N^j f(x) = \int_{y \notin Q_x^N} K_j(y-x) f(y)\,dp(y),$$

where $K_j(y-x) = \dfrac{(y-x)_j}{|y-x|^d}$. By (5) it is clear that $\|R_N^j\|_{L^2(p)} < \infty$.

**Theorem 3.1.** *Let $0 < \alpha < 1$ and let $G \subset E$ be a closed set such that $p(G) > \alpha$. There are constants $C_1(\alpha)$ and $C_2$, both depending on $\lambda_0$, $c_1$, $c_2$ and $c_3$, such that for all $N$ big enough,*

$$C_1\Big(\sum_{n=0}^{N}\theta_n^2\Big)^{\frac{1}{2}} \leq \|R_N^j\|_{L^2(G,p)} \leq C_2\Big(\sum_{n=0}^{N}\theta_n^2\Big)^{\frac{1}{2}}. \tag{7}$$

To begin we prove the upper bound in (7). Since the norm $\|R_N^j\|_{L^2(G,p)}$ increases with $G$ we may assume $G = E$, which also means $C_2$ does not depend on $\alpha$. The proof of the upper bound in (7) follows the paper [MT], but for convenience we repeat their argument. By the $T(1)$-Theorem for spaces of homogeneous type from [Ch1] we have

$$\|R_N^j\|_{L^2(p)} \leq C \sup_{n\leq N} \sup_{|J|=n} \frac{p(Q_J^n)}{\sigma_n^{d-1}} + C \sup_{n\leq N} \sup_{|J|=n} \frac{\|R_N^j(\chi_{Q_J^n})\|_{L^2(Q_J^n,p)}}{p(Q_J^n)^{\frac{1}{2}}}.$$

Therefore the upper bound in (7) will be an immediate consequence of the following two lemmas. For convenience we fix $j$, write $K(y-x) = K_j(y-x)$, and define

$$R_m f(x) = \int_{Q_x^m \setminus Q_x^{m+1}} K_j(y-x) f(y) dp(y).$$

**Lemma 3.2.** *If $n \leq m$, there is $c_9$ such that*

$$\|R_m \chi_{Q_J^n}\|_{L^2(Q_J^n,p)} \leq c_9 \theta_m p(Q_J^n)^{\frac{1}{2}}$$

*Proof.* For $y \in Q_x^m \setminus Q_x^{m+1}$, (5) gives

$$|K(y-x)| \leq \frac{1}{c_3^{d-1}\sigma_{m+1}^{d-1}}.$$

Hence by (2)

$$|R_m \chi_{Q_J^n}| \leq \frac{2^d}{c_3^{d-1}} \theta_m,$$

and

$$||R_m \chi_{Q_J^n}||_{L^2(Q_J^n,p)} \leq \frac{2^d}{{c_3}^{d-1}} \theta_m p(Q_J^n)^{\frac{1}{2}}.$$

□

**Lemma 3.3.** *There is a constant $C$ depending only on $\lambda_0$, $c_1$, $c_2$ and $c_3$ such that for all $N > n$ and all $J$,*

$$\|R_N^j \chi_{Q_J^n}\|^2_{L^2(Q_J^n,p)} \leq C \sum_{k=n}^{N} \theta_k^2 p(Q_J^n).$$

*Proof.* Fix $j = 1, \ldots, d$, then for $x \in Q_J^n$

$$R_N^j \chi_{Q_J^n}(x) = \sum_{m=n}^{N-1} R_m \chi_{Q_J^n}(x).$$

We claim that for $m \neq k$,

$$\left| \int R_m \chi_{Q_J^n} R_k \chi_{Q_J^n} dp \right| \leq C 2^{-|m-k|} \theta_m \theta_k p(Q_J^n). \tag{8}$$

Accepting (8) for the moment, we see from Lemma 3.2 that

$$\begin{aligned}\|R^j_N\chi_{Q^n_J}\|^2_{L^2(Q^n_J)} &= \|\sum_{m=n}^{N-1} R_m\chi_{Q^n_J}\|^2_{L^2(Q^n_J)}\\ &= \sum_{m=n}^{N-1}\|R_m\chi_{Q^n_J}\|^2_{L^2(Q^n_J)} + 2\sum_{n\le k<m\le N-1}\langle R_m\chi_{Q^n_J}, R_k\chi_{Q^n_J}\rangle\\ &\le C\sum_{m=n}^{N-1}\theta_m^2 p(Q^n_J),\end{aligned}$$

which gives the right-hand inequality in (7).

To prove (8) assume $n \le k < m \le N-1$. Then because the kernel $K$ is odd,

$$\int_{Q^m_K} R_m\chi_{Q^n_J}(x)dp(x) = \sum_{r\neq q}\int_{Q^{m+1}_{(K,r)}}\int_{Q^{m+1}_{(K,q)}} K(x-y)dp(y)dp(x) = 0,$$

so that for any $x^m_K \in Q^m_K$,

$$\int_{Q^m_K} R_m\chi_{Q^n_J}(x)R_k\chi_{Q^n_J}(x)dp(x) = \int_{Q^m_K} R_m\chi_{Q^n_J}(x)(R_k\chi_{Q^n_J}(x) - R_k\chi_{Q^n_J}(x^m_K))dp(x).$$

But when $x \in Q^m_K$, (4), (5) and (2) give

$$|R_k\chi_{Q^n_J}(x) - R_k\chi_{Q^n_J}(x^m_K)| \le C\frac{\sigma_m p(Q^k_x)}{\sigma^d_k} \le C\theta_k\frac{\sigma_m}{\sigma_k} \le C2^{-(m-k)}\theta_k.$$

Hence using Lemma 3.2

$$\begin{aligned}|\int R_m\chi_{Q^n_J}R_k\chi_{Q^n_J}dp| &\le C2^{-(m-k)}\theta_k\|R_m\chi_{Q^n_J}\|_{L^1(Q^n_J,p)}\\ &\le C2^{-(m-k)}\theta_k p(Q^n_J)^{\frac{1}{2}}\theta_m p(Q^n_J)^{\frac{1}{2}}\end{aligned}$$

and (8) holds. □

The proof of the lower bound in (7) also follows [MT] but with two alterations needed because $G \neq E$ and because the sets $Q^n_J$ may be incongruent. When $Q = Q^n_J$ we also write $n = n(Q)$, $Q \in \mathcal{D}_n$, and $\theta(Q) = \theta_n$.

Let $0 < \delta < 1$, fix $G$ and define $\mathcal{B}(\delta) = \{Q \in \bigcup_n \mathcal{D}_n : p(G\cap Q) < \delta p(Q)\}$.

**Lemma 3.4.** *Assume $\delta < \alpha$ and $p(G) \ge \alpha$.*
(a) *Then for all $n$,*

$$p(G \setminus \bigcup_{\mathcal{D}_n\cap\mathcal{B}(\delta)} Q^J_n) \ge p(G\setminus \bigcup_{\mathcal{B}(\delta)} Q) \ge \alpha - \delta.$$

(b) *For $N_0 \in \mathbb{N}$ there exists $M(N_0)$ such that whenever $Q \notin \mathcal{B}(\delta)$, there exist $Q' \subset Q$ with $n(Q') \le n(Q) + M$ such that for all $Q'' \subset Q'$ with $n(Q'') \le n(Q') + N_0$*

$$Q'' \notin \mathcal{B}(\frac{\delta}{2}).$$

*Proof.* To prove (a) let $\{Q_j\}$ be a family of maximal cubes in $\mathcal{B}(\delta)$, note that

$$p(G \cap \bigcup_{\mathcal{B}(\delta)} Q) \leq \sum p(G \cap Q_j) \leq \delta p(E) = \delta$$

and subtract this quantity from $p(G)$.

To prove (b) fix $N_0$ and suppose (b) is false for $N_0, \delta, Q$ and $M = 0$. Write $n = n(Q)$. Then there is $Q_1 \subset Q$ with $n(Q_1) \leq n + N_0$ and $Q_1 \in \mathcal{B}(\frac{\delta}{2})$. Set $\mathcal{F}_1 = \{Q_1\}$. Then $p(Q \setminus Q_1) \leq (1 - 2^{-N_0 d})p(Q) = \beta p(Q)$. Now assume (b) is also false for $N_0, \delta, Q$ and $M = N_0$ and write $Q \setminus Q_1 = \bigcup\{Q' : n(Q') = n(Q_1), Q' \neq Q_1\}$. Then for each $Q' \neq Q_1$ with $n(Q') = n(Q_1)$ there is $Q_2 \subset Q'$ with $n(Q_2) \leq n + 2N_0$ and $Q_2 \in \mathcal{B}(\frac{\delta}{2})$. Set $\mathcal{F}_2 = \{Q_2\}$. Then $p(Q \setminus \bigcup_{\mathcal{F}_1 \cup \mathcal{F}_2} Q_j) \leq \beta^2 p(Q)$. Further assume (b) is false for $N_0, \delta, Q$ and $M = 2N_0$ and repeat the above construction in each $Q' \setminus Q_2$. After $m$ steps we obtain families $\mathcal{F}_j$ of cubes $Q_j \in \mathcal{B}(\frac{\delta}{2})$ such that $\bigcup \mathcal{F}_j$ is disjoint and

$$p(Q \setminus \bigcup_{j=1}^{m} \bigcup_{\mathcal{F}_j} Q_j) \leq \beta^m p(Q)$$

and for $\beta^m < \frac{\delta}{2}$ we obtain $p(Q \cap G) \leq \frac{\delta}{2} \sum_{j=1}^m \sum_{\mathcal{F}_j} p(Q_j) + \beta^m p(Q) < \delta p(Q)$, which is a contradiction. We conclude that (b) holds for $M = mN_0$. □

For any $\delta < \alpha$ we say $Q' \in \mathcal{G}^*(\delta)$ if $Q'$ satisfies conclusion (b) of Lemma 3.4 for $N_0$ and $\delta$. Then by parts (b) and (a) of Lemma 3.4 we have:

**Lemma 3.5.** *Let $\delta = \frac{\alpha}{2}$ and assume $p(G) \geq \alpha$. Then*

$$\sum_{\mathcal{G}^*(\frac{\delta}{2})} \theta(Q')^2 p(Q' \cap G) \geq C(M) \sum_{Q \notin \mathcal{B}(\delta)} \theta(Q)^2 p(Q \cap G) \geq C(M, \alpha) \sum \theta_n^2.$$

Now let $A$ be a large constant. As in [MT], for $R \in \mathcal{D}$ we will define a family Stop$(R)$ of "stopping cubes" $Q \subset R$. We say $Q \in \text{Stop}_0(R)$ if $Q \subset R$ and $Q \notin \mathcal{B}(\frac{\delta}{2})$ and if

$$\inf_Q \Big| \int_{G \cap (R \setminus Q)} K(y - x) dp(y) \Big| \geq A\theta(R).$$

We also say $Q \in \text{Stop}_1(R)$ if $Q \subset R$ and $Q \notin \mathcal{B}(\frac{\delta}{2})$, if $\theta(Q) \leq \eta\theta(R)$ for constant $\eta$ to be chosen below, if $n(Q) \geq n(R) + N_1$ for constant $N_1$ to be chosen below, and if

$$P \in \text{Stop}_0(R) \Rightarrow n(P) \geq n(Q).$$

Then define

$$\text{Stop}(R) = \{Q \in \text{Stop}_0(R) \cup \text{Stop}_1(R) : Q \text{ is maximal}\}.$$

It follows from the last three conditions in the definition of $\text{Stop}_1(R)$ that either $\text{Stop}(R) \subset \text{Stop}_0(R)$ or $\text{Stop}(R) \subset \text{Stop}_1(R)$. Inductively we define $\text{Stop}^1(P) = \text{Stop}(P)$ and

$$\text{Stop}^k(P) = \bigcup\{\text{Stop}(Q) : Q \in \text{Stop}^{k-1}(P)\},$$

$$\text{Top} = \{P_0\} \cup \bigcup_{k \geq 1} \text{Stop}^k(P_0),$$

where $P_0$ is the unique cube in $\mathcal{D}_0$, and

$$P^{stp} = \bigcup_{\mathrm{Stop}(P)} Q.$$

**Remark.** The constants $N_0, N_1, A, \eta$ are chosen as follows. First we take $\delta = \alpha/2$. Then $N_1$ will be determined by Lemma 3.7, $\eta$ and $A$ will be determined by the proof of Lemma 3.8, and $N_0$, which depends on $A, \eta$, and $\delta$, will be determined by the proof of Lemma 3.6.

**Lemma 3.6.** *Let $\delta = \frac{\alpha}{2}$ and assume $p(G) \geq \alpha$. If $N_0 = N_0(A, \eta, \delta)$ is sufficiently large, then for all $Q \in \mathcal{G}^*(\frac{\delta}{2})$ there exists a cube $P \subset Q$ such that $P \in$ Top and $n(P) \leq n(Q) + N_0$.*

*Proof.* Let $Q \in \mathcal{G}^*(\frac{\delta}{2})$ and let $R$ be the smallest cube $R \in$ Top such that $Q \subset R$. We assume the conclusion of the lemma is false for $Q$. Thus $Q \notin$ Top, and $Q \notin \mathrm{Stop}(R)$. Hence by definition there is $x_0 \in Q$ such that

$$\Big| \int_{G \cap R \setminus Q} K(y - x_0) dp(y) \Big| \leq A\theta(R).$$

Then for $x \in Q$ (5) gives

$$\Big| \int_{G \cap R \setminus Q} (K(y - x) - K(y - x_0)) dp \Big| \leq C\sigma_{n(Q)} \sum_{k=n(R)}^{n(Q)-1} \frac{\theta_k}{\sigma_k} \leq C_1 \theta(R)$$

so that

$$\sup_Q \Big| \int_{G \cap R \setminus Q} K(y - x) dp(y) \Big| \leq (A + C_1)\theta(R). \tag{9}$$

Take $x^* \in Q \cap E$ with $x^*_j = \inf_Q x_j$ and let $Q^*$ be that $Q^* \subset Q$ such that $x^* \in Q^*$ and $n(Q^*) = n(Q) + N_0$. Then

$$K(y - x^*) \geq 0$$

for all $y \in Q$ and by Lemma 2.1 there is a constant $n_0$ such that if $n \leq n(Q^*) - n_0$, there exists $Q^n_J \subset (Q \setminus Q^*)$ such that

$$\inf_{y \in Q^n_J} K(y - x^*) \geq \frac{c}{\sigma_n^{d-1}}.$$

Because $\theta_{n+1} \leq \theta_n$ and because we assume the lemma is false for $Q$, we also have $\theta(Q^n_J) \geq \eta\theta(R)$ for every such $Q^n_J$. Hence by (5)

$$\int_{G \cap Q \setminus Q^*} K(y - x^*) dp(y) \geq (N_0 - n_0)\eta \frac{\delta}{2} \theta(R)$$

and by the proof of (9),

$$\inf_{Q^*} \int_{G \cap Q \setminus Q^*} K(y - x) dp(y) \geq ((N_0 - n_0)\eta \frac{\delta}{2} - C)\theta(R). \tag{10}$$

Taking $N_0 = N_0(A)$ sufficiently large and comparing (10) with (9) we conclude that $Q^* \in \mathrm{Stop}_0(R)$, which is a contradiction. □

Note that by Lemma 3.5 and Lemma 3.6 we have for all $P$,

$$\sum_{n=0}^{N} \theta_n^2 \leq C(\alpha) \sum_{n=0}^{N} \sum_{\mathcal{D}_n \setminus \mathcal{B}(\delta)} \theta(Q)^2 p(Q) \leq C'(\alpha) \sum_{\text{Top}} \theta(P)^2 p(G \cap P). \tag{11}$$

We define

$$\begin{aligned} K_P 1(x) &= \sum_{Q \in \text{Stop}(P)} \chi_{G\cap Q}(x) \int_{G \cap P \setminus Q} K(y-x) dp(y) \\ &+ \chi_{G \cap P \setminus P^{stp}}(x) \int_{G \cap P \setminus Q^N(x)} K(y-x) dp(y). \end{aligned}$$

By construction

$$\chi_G R_N 1 = \sum_{\text{Top}} K_P 1$$

and

$$\|R_N 1\|^2_{L^2(G)} = \sum_{\text{Top}} \|K_P 1\|^2_{L^2(G)} + \sum_{P,Q \in \text{Top}, P \neq Q} \langle K_P 1, K_Q 1 \rangle_{L^2(G)}.$$

**Lemma 3.7.** *If $N_1$ is chosen big enough, then for all $P \in \text{Top}$,*

$$\|K_P 1\|^2_{L^2(G)} \geq C^{-1} \theta(P)^2 p(G \cap P), \tag{12}$$

*where $C = C(\alpha)$, and*

$$\|K_P 1\|^2_{L^2(G)} \geq A^2 \theta(P)^2 p(G \cap P^{stp_0}), \tag{13}$$

*where*

$$P^{stp_0} = \bigcup \Big\{ Q : Q \in \text{Stop}(P) \cap \text{Stop}_0(P) \Big\}.$$

**Lemma 3.8.**

$$\sum_{P,Q \in \text{Top}, P \neq Q} |\langle K_P 1, K_Q 1 \rangle_{L^2(G)}| \leq C(A^{-1} + c(\eta)) \sum_{\text{Top}} \|K_P 1\|^2_{L^2(G)}, \tag{14}$$

*with $c(\eta) \to 0$ as $\eta \to 0$.*

Assuming Lemma 3.7 and Lemma 3.8 for the moment, we see that if $A$ is large and $\eta$ is small, then

$$\|R_N 1\|^2_{L^2(G)} \geq C^{-1} \sum_{\text{Top}} \theta(P)^2 p(G \cap P)$$

and then the lower bound in (7) follows from inequality (11).

To prove Lemma 3.7, first note that (13) follows from the definitions of $\text{Stop}_0(P)$ and $\text{Stop}(P)$. To prove (12), recall that $K = K_j$ for some $1 \leq j \leq d$. We apply Lemma 2.2 to $P$ with $\gamma^n \sim \alpha$ to obtain sets $S_1 \subset P$ and $S_2 \subset P$ such that

$$\sup_{S_1} x_j = a < \inf_{S_2} x_j$$

and

$$\text{Min}(p(G \cap S_1),\ p(G \cap S_2)) \geq c(\alpha) p(P).$$

We may assume that $S_1, S_2$ are much bigger that any stopping cube of $P$, because if there exists some $Q \in \mathrm{Stop}_0(P)$ with size similar to $S_1$ or $S_2$, then (12) follows from (13); and if we choose $N_1$ big enough, any cube $Q \in \mathrm{Stop}_1(P)$ will be much smaller that $S_1, S_2$. Then we get

$$\Big|\int_{S_2\cap G} K_P\chi_{S_1}(x)dp(x)\Big| \geq C^{-1}p(S_2\cap G)\frac{p(S_1\cap G)}{\mathrm{diam}(P)^{d-1}}.$$

Set

$$E_1 = P\cap\{x_j \leq a\} \text{ and } E_2 = P\cap\{x_j > a\}.$$

By its definition,

$$K_P 1 = \chi_G(x)\sum_k \chi_{Q_k}(x)\int_{G\cap P\setminus Q_k} K(y-x)dp(y)$$

where $\{Q_k\}$ is a cover of $P$ by disjoint cubes from $\mathcal{D}$. We also have

$$\begin{aligned} K_P 1(x) &= \chi_G(x)\sum_{i=1,2}\sum_k \chi_{Q_k}(x)\int_{G\cap E_i\setminus Q_k} K(y-x)dp(y) \\ &\equiv K_P\chi_{E_1}(x) + K_P\chi_{E_2}(x). \end{aligned}$$

Write $Q_k = Q(x)$ when $x \in Q_k$ and note that

$$y \notin Q(x) \Longleftrightarrow x \notin Q(y).$$

Hence by the antisymmetry $K(y-x) = -K(x-y)$ we have

$$\int_{G\cap E_2} K_P\chi_{E_2}(x)dp(x) = 0.$$

Therefore by the choices of $S_1, S_2, E_1$ and $E_2$,

$$\begin{aligned} (p(G\cap E_2))^{1/2}\|K_P 1\|_{L^2(G)} &\geq \Big|\int_{G\cap E_2} K_P 1(x)dp(x)\Big| \\ &= \Big|\int_{G\cap E_2} K_P\chi_{E_1}(x)dp(x)\Big| \\ &\geq \Big|\int_{S_2\cap G} K_P\chi_{S_1}(x)dp(x)\Big| \\ &\geq p(G\cap E_2)\frac{c(\alpha)p(G\cap P)}{\mathrm{diam}(P)^{d-1}}, \end{aligned}$$

which is (12).

To prove Lemma 3.8 we again follow [MT]. Suppose $P \neq Q \in \mathrm{Top}$ and $Q \subset P$. Let $P_Q \in \mathrm{Stop}(P)$ be such that $Q \subset P_Q \subset P$. By the antisymmetry of $K$ we have $\int_{Q\cap G} K_Q 1 dp = 0$ so that

$$\begin{aligned} \Big|\int_{Q\cap G} K_Q 1(x)K_P 1(x)dp\Big| &= \Big|\int_{Q\cap G} K_Q 1(x)(K_P 1(x) - K_P 1(x_Q))dp(x)\Big| \\ &\leq \|K_Q 1\|_{L^1(Q)}\sup_Q |K_P 1(x) - K_P 1(x_Q)|, \end{aligned}$$

where $x_Q$ is a fixed point from $Q$. But for any $x \in Q$, standard estimates yield

$$\begin{aligned}\left|K_P 1(x) - K_P 1(x_Q)\right| &\leq \int_{G\cap P\setminus P_Q} |K(y-x) - K(y-x_Q)|dp(y)\\ &\leq C\,\mathrm{diam}(Q)\int_{G\cap P\setminus P_Q}\frac{dp(y)}{|x-y|^d}\\ &\leq C\,\mathrm{diam}(Q)\sum_{P_Q\subset R\subset P}\frac{\theta(R)}{\mathrm{diam}(R)}.\end{aligned}$$

Assume first that $P_Q \in \mathrm{Stop}_0(P)$. Since $\theta(R) \leq \theta(P)$ in the last sum, we get

$$\left|K_P 1(x) - K_P 1(x_Q)\right| \leq C\,\frac{\mathrm{diam}(Q)}{\mathrm{diam}(P_Q)}\,\theta(P).$$

Hence by (13),

$$\left|\langle K_P 1, K_Q 1\rangle_{L^2(G,p)}\right| \leq \frac{C}{A}\frac{\mathrm{diam}(Q)}{\mathrm{diam}(P_Q)}\left(\frac{p(G\cap Q)}{p(G\cap P^{stp_0})}\right)^{1/2}\|K_Q 1\|_{L^2(G)}\|K_P 1\|_{L^2(G)},$$

when $P_Q \in \mathrm{Stop}_0(P)$.

Consider now the case $P_Q \in \mathrm{Stop}_1(P)$. This means that $\theta(P_Q) \leq \eta\theta(P)$. Then it follows from (2) that

$$\sum_{P_Q\subset R\subset P}\left\{\frac{\theta(R)}{\theta(P)} : \frac{\mathrm{diam}(P_Q)}{\mathrm{diam}(R)} \geq c_1(\eta)\right\} \leq c_2(\eta)\sum_{P_Q\subset R\subset P}\frac{\theta(R)}{\theta(P)}$$

so that

$$\mathrm{diam}(Q)\sum_{P_Q\subset R\subset P}\frac{\theta(R)}{\mathrm{diam}(R)} \leq c(\eta)\,\frac{\mathrm{diam}(Q)}{\mathrm{diam}(P_Q)}\,\theta(P) \qquad \text{with } c(\eta)\to 0 \text{ as } \eta \to 0.$$

Hence by (12),

$$\left|\langle K_P 1, K_Q 1\rangle_{L^2(G,p)}\right| \leq c(\eta)\,\frac{\mathrm{diam}(Q)}{\mathrm{diam}(P_Q)}\,\|K_Q 1\|_{L^2(G)}\|K_P 1\|_{L^2(G)},$$

when $P_Q \in \mathrm{Stop}_1(P)$. Thus (14) follows from Schur's lemma. □

## 4. Lipschitz harmonic capacity

In this section we will prove Theorem 1.2. We will assume that each cube $Q_J^n$ in the definition of the Cantor set $E$ (see (3)) contains a closed ball $B_J^n$ such that

$$c_1'\sigma_n \leq \mathrm{diam}(B_J^n).$$

This assumption comes for free from the definition of $E$ in Section 1. Indeed, one easily deduces that there exists a family of balls $B_J^n$ centered at $Q_J^n$ such that

$$c_1'\sigma_n \leq \mathrm{diam}(B_J^n) \leq c_2'\sigma_n,$$

and

$$\mathrm{dist}(B_J^n, B_K^n) \geq c_3'\sigma_n,\ \ J \neq K.$$

Then if one replaces the cubes $Q^n_J$ in the definition of $E$ by the sets

$$\tilde{Q}^n_J = \bigcup_{Q^m_K \subset Q^n_J} (Q^m_K \cup B^m_K),$$

$E$ does not change.

Given a real Radon measure $\mu$ and $f \in L^1(\mu)$, let

$$R_{\mu,\epsilon}(f d\mu)(x) = \int_{|y-x|>\epsilon} \frac{y-x}{|y-x|^d} f(y) d\mu(y)$$

be the (truncated) $(d-1)$-Riesz transform of $f \in L^1(\mu)$ with respect to the measure $\mu$ and set $\|R_\mu\|_{L^2(\mu)} = \sup_{\epsilon>0} \|R_{\mu,\epsilon}\|_{L^2(\mu)}$.

As in [MT], we need to introduce the following capacity of the sets $E_N$:

$$\kappa_p(E_N) = \sup\{\alpha : 0 \leq \alpha \leq 1, \|R_{\alpha\mu_N}\|_{L^2(\alpha\mu_N)} \leq 1\},$$

where $\mu_N$ is a probability measure on $E_N$ such that $\mu_N(Q^N_J) = 2^{-Nd}$.

The $L^2$ estimates from the previous section yield the following lemma.

**Lemma 4.1.**

$$\kappa_p(E_N) \approx \Big(\sum_{n=1}^N \theta_n^2\Big)^{-1/2}.$$

*Proof.* By Theorem 3.1 we have

$$\|R_{\alpha\mu_N}\|_{L^2(\alpha\mu_N)} = \alpha \|R_{\mu_N}\|_{L^2(\mu_N)} \approx \alpha \Big(\sum_{n=1}^N \theta_n^2\Big)^{1/2}.$$

The lemma follows because the sum above is $\geq 2^{-d}$. □

We will prove the following:

**Lemma 4.2.** *There exists an absolute constant $C_0$ such that for all $N \in \mathbb{N}$ we have*

$$\kappa(E_N) \leq C_0 \kappa_p(E_N). \tag{15}$$

Notice that Theorem 1.2 follows from Lemma 4.2 and

$$\kappa(E_N) \geq \kappa_+(E_N) \geq C^{-1} \kappa_p(E_N), \tag{16}$$

where

$$\kappa_+(E) = \sup\{|\langle \Delta f, 1\rangle| : \ f \in L(E,1), \Delta f = \mu \in M_+(E)\}$$

and $M_+(E)$ is the set of positive Borel measures supported on $E$. The first inequality in (16) is just a consequence of the definitions of $\kappa$ and $\kappa_+$ and the second inequality follows from a well known method that dualizes a weak (1,1) inequality (see Theorem 23 in [Ch2] and Theorem 2.2 in [MTV]. The original proof is from [DØ]).

In [Vo] it is shown that the capacities $\kappa$ and $\kappa_+$ are comparable for all subsets of $\mathbb{R}^d$, but we do not use that deep result.

For any $s > 0$, we write $\Lambda_s$ and $\Lambda_s^\infty$ for the $s$-dimensional Hausdorff measure and the $s$-dimensional Hausdorff content, respectively.

*Proof.* The arguments are similar to those in [MTV] and [MT], but a little more involved because our Cantor sets are not homogeneous. Also, instead of using the local $T(b)$-Theorem of M. Christ, we will run a stopping time argument in the spirit of [Ch1] and then use a dyadic $T(b)$-Theorem (see Theorem 20 in [Ch1]).

We set

$$S_n = \theta_1^2 + \theta_2^2 + \cdots + \theta_n^2.$$

Without loss of generality we can assume that for each $N > 1$ there exists $1 \leq M < N$ such that

$$S_M \leq \frac{S_N}{2} < S_{M+1}. \tag{17}$$

Otherwise $\frac{S_N}{2} < S_1$ and by Lemma 4.1 it follows that $\kappa_p(E_N) \geq C^{-1}\lambda_1^{d-1}$. By [P] we have

$$\kappa(E_N) \leq \kappa(E_1) \leq C\Lambda_{d-1}^{\infty}(E_1) \leq C\lambda_1^{d-1},$$

and if $C_0$ is chosen big enough the conclusion of the lemma will follow in this case.

Assuming (17), we will now prove (15) by induction on $N$. For $N = 1$ (15) holds clearly. The induction hypothesis is

$$\kappa(E_n) \leq C_0\kappa_p(E_n), \text{ for } 0 < n < N,$$

where the precise value of $C_0$ is to be determined later.

Notice that for $n \geq 0$, $(Q_K^N \cap E)_n$ is the $n-$th generation of the Cantor set $Q_K^N \cap E$, i.e. the union of $2^{nd}$ sets $Q_J^{n+N}$ satisfying properties (4) and (5) with $n$ replaced by $n+N$. Let $J^*$ be the multi-index of length $M$ such that

$$\kappa((Q_{J^*}^M \cap E)_{N-M}) = \max_{|J|=M} \kappa((Q_J^M \cap E)_{N-M}).$$

We distinguish two cases.

**Case 1:** For some absolute constant $A_0$ to be determined below,

$$\kappa((Q_{J^*}^M \cap E)_{N-M}) \geq A_0 2^{-Md}\kappa(E_N),$$

By the induction hypothesis (applied to $(Q_{J^*}^M \cap E)_{N-M}$) and by Lemma 4.1 we have that

$$\begin{aligned}\kappa(E_N) &\leq A_0^{-1}2^{Md}\kappa((Q_{J^*}^M \cap E)_{N-M}) \leq A_0^{-1}2^{Md}C_0\kappa_p((Q_{J^*}^M \cap E)_{N-M}) \\ &\leq A_0^{-1}C_0C2^{Md}\Big(\sum_{n=1}^{N-M}\Big(\frac{2^{-dn}}{\sigma_{M+n}^{d-1}}\Big)^2\Big)^{-1/2} = A_0^{-1}C_0C\Big(\sum_{n=M+1}^{N}\theta_n^2\Big)^{-1/2}.\end{aligned}$$

Now by using that $S_M \leq S_N/2$ is equivalent to $\sum_{n=1}^N \theta_n^2 \leq 2\sum_{n=M+1}^N \theta_n^2$ and Lemma 4.1 again, we obtain that

$$\kappa(E_N) \leq 2^{1/2}A_0^{-1}C_0C\Big(\sum_{n=1}^N \theta_n^2\Big)^{-1/2} \leq CA_0^{-1}C_0\kappa_p(E_N).$$

Hence if $A_0 = C$, we obtain (15).

**Case 2:** For the same constant $A_0$,

$$\kappa((Q_{J^*}^M \cap E)_{N-M}) \leq A_0 2^{-Md}\kappa(E_N). \tag{18}$$

Then if $\theta_{M+1}^2 > S_M$, $S_{M+1} = S_M + \theta_{M+1}^2 \approx \theta_{M+1}^2$. Therefore

$$\kappa_p(E_{M+1}) \approx S_{M+1}^{-1/2} \approx \theta_{M+1}^{-1} \geq C\Lambda_{d-1}^{\infty}(E_{M+1}).$$

Hence by (17),

$$\kappa(E_N) \leq \kappa(E_{M+1}) \leq C\Lambda_{d-1}^{\infty}(E_{M+1}) \leq C\kappa_p(E_{M+1}) \approx \kappa_p(E_N),$$

which is (15) if $C_0$ is chosen big enough.

On the other hand, if $\theta_{M+1}^2 \leq S_M$, then $S_{M+1} \approx S_M \approx S_N$. Recall that we are assuming that each cube $Q_J^M$ contains some ball $B_J^M$ with comparable diameter. Moreover, we may suppose that all the balls $B_J^M$, $J = 1, \ldots, 2^{Md}$, have the same diameter $d_M$. We set

$$\tilde{E}_M = \bigcup_{|J|=M} B_J^M.$$

We consider now the measure

$$\sigma = \kappa(E_N)\mu'_M,$$

where $\mu'_M$ is defined by

$$\mu'_M(K) = \sum_J \frac{\Lambda_{d-1}(K \cap \partial B_J^M)}{\Lambda_{d-1}(\partial \tilde{E}_M)},$$

for compact sets $K$. Clearly $\sigma(\tilde{E}_M) = \kappa(E_N)$.

Note that the measure $\sigma$ is doubling and has $(d-1)-$growth. To verify this, one uses that

$$\kappa(E_N) \leq \kappa(E_M) \leq C\Lambda_{d-1}^{\infty}(E_M) \leq C\Lambda_{d-1}(\partial\tilde{E}_M)$$

and $\mu'_M(Q_K^n) = 2^{-nd}$ for all $0 \leq n \leq M$ (see (4.8) and (4.9) of [MT]).

We will show that there exists a good set $G \subset \tilde{E}_M$ with $\sigma(G) \approx \sigma(\tilde{E}_M)$ such that $R_{\sigma_{|G}}$ is bounded on $L^2(\sigma_{|G})$ with absolute constants. From this fact, by Theorem 3.1 we have

$$\|R_{\sigma_{|G}}\|_{L^2(\sigma_{|G})} \approx \kappa(E_N)S_M^{1/2} \leq C.$$

So by Lemma 4.1 we infer

$$\kappa(E_N) \leq CS_M^{-1/2} \leq CS_N^{-1/2} \approx C\kappa_p(E_N),$$

which proves the lemma.

To establish the existence of the set $G$, we run a stopping time argument. First we construct a set $E'$ and a doubling measure $\sigma'$ on $E'$. The pair $(E', \sigma')$ is endowed with a system of dyadic cubes $\mathcal{Q}(E')$, where

$$\mathcal{Q}(E') = \{Q_\beta^k \subset E' : \beta \in \mathbb{N},\ k \in \mathbb{N}\}$$

(see Theorem 11 in [Ch1]). We also define a function $b'$ on $E'$, dyadic para-accretive with respect to this system of dyadic cubes, i.e. for every $Q_\beta^k \in \mathcal{Q}(E')$, there exists $Q_\gamma^l \in \mathcal{Q}(E')$, $Q_\gamma^l \subset Q_\beta^k$, with $l \leq k + N$ and

$$\left|\int_{Q_\gamma^l} b' d\sigma'\right| \geq c\sigma'(Q_\gamma^l)$$

for some fixed constants $c > 0$ and $N \in \mathbb{N}$, and such that the function $R(b'd\sigma')$ belongs to dyadic BMO$(\sigma')$. Therefore, the $(d-1)$-Riesz transform $R$ associated to

$\sigma'$ will be bounded on $L^2(E', \sigma')$ by the $T(b)-$theorem on a space of homogeneous type (see Theorem 20 in [Ch1]). Our set $G$ will be contained in $E' \cap \tilde{E}_M$.

Now we turn to the construction of the set $E'$ and the measure $\sigma'$. By definition there exists a distribution $T$ supported on $E_N$ such that

$$\kappa(E_N) \leq C|\langle T, 1\rangle|$$

and

$$\|RT\|_{L^\infty(\mathbb{R}^d)} \leq 1.$$

We replace the distribution $T$ with a real measure $\nu$ supported on $E_N$ such that $\kappa(E_N) \leq C|\nu(E_N)|$ and $\|R\nu\|_{L^\infty}(\mathbb{R}^d) \leq 1$. (The measure $\nu$ exists because of Volberg's theorem ([Vo]), but in the special case of $E_N$ considered here $\nu$ can be constructed directly by setting $\nu = \sum_{|J|=N} \nu_J$ with $\nu_J = h_J \chi_{\partial B_J^N} \Lambda_{d-1}$ and $h_J$ smooth on $\partial B_J^N$ such that for all polynomials $P$ of degree at most $d$, $\int P(x) d\nu_J = T(P(x)\varphi_J(x))$, where $\varphi_J$ is smooth and $\varphi_J = \chi_{Q_J^N}$ on $E_N$. See [P].)

The definition of $\sigma$ implies that

$$|\nu(E_N)| \geq C^{-1}\sigma(\tilde{E}_M) > \epsilon_0 \sigma(\tilde{E}_M), \tag{19}$$

where $\epsilon_0$ is a sufficiently small constant to be fixed later. Notice that for a fixed generation $n$, $0 \leq n \leq M$, there exists at least one cube $Q_K^n$, such that $|\nu(Q_K^n)| > \epsilon_0 \sigma(Q_K^n)$, since otherwise for $0 \leq n \leq M$

$$|\nu(E_N)| \leq \sum_{|K|=n} \epsilon_0 \sigma(Q_K^n) = \epsilon_0 \sum_{|J|=M} \sigma(B_J^M) = \epsilon_0 \sigma(\tilde{E}_M),$$

which contradicts (19).

We now run a stopping-time procedure. Let $\epsilon > 0$ be another constant to be chosen later, much smaller than $\epsilon_0$. We check whether or not the condition

$$|\nu(Q_J^1)| \leq \epsilon \sigma(Q_J^1) \tag{20}$$

holds for the cubes $Q_J^1$. If (20) holds for the cube $Q_J^1$, we call it stopping-time cube. If (20) does not hold for $Q_J^1$, we consider the children $Q_K^2$ of $Q_J^1$ and call each such $Q_K^2$ with (20) a stopping-time cube. We continue this procedure through generation $M$, but we do not consider the cubes of later generations. We obtain in this way a collection of pairwise disjoint stopping-time cubes $\{P_\gamma\}_\gamma$, where $P_\gamma = Q_J^n$, for some $0 \leq n \leq M$ and by definition each $P_\gamma$ satisfies condition (20) with $Q_J^1$ replaced by $P_\gamma$.

Consider now the function

$$b = \sum_{|J|=M} \frac{\nu(Q_J^M)}{\sigma(B_J^M)} \chi_{B_J^M}.$$

The function $b$ has the following three important properties:

(1) for $0 \leq n \leq M$, $\displaystyle\int_{Q_K^n} b d\sigma = \nu(Q_K^n)$.
(2) $\|b\|_\infty \leq C$.
(3) For any $0 \leq n \leq M$,

$$\|R(b\chi_{Q_K^n} d\sigma)\|_{L^\infty(\mathbb{R}^d)} \leq C. \tag{21}$$

To show that $b$ is bounded it is enough to verify that

$$|\nu(Q_J^M)| \leq C\sigma(B_J^M), \text{ for } |J| = M. \tag{22}$$

Inequality (22) can be shown by localizing the potential $\nu * x/|x|^d$ (see [P] and [MPrV]) and using (18), namely

$$|\nu(Q_J^M)| \leq C\kappa((Q_J^M \cap E)_{N-M}) \leq CA_0 2^{-Md}\kappa(E_N) = CA_0\sigma(B_J^M).$$

To see (21), notice that

$$\|R(\chi_{B_J^M}\, d\sigma)\|_{L^\infty(\mathbb{R}^d)} \leq C\frac{\kappa(E_M)}{\Lambda_{d-1}(\partial E_M)}\|R(\chi_{B_J^M}\, d\Lambda_{d-1})\|_{L^\infty(\mathbb{R}^d)} \leq C. \tag{23}$$

Since $\|R(\chi_{Q_K^n}\, d\nu)\|_{L^\infty(\mathbb{R}^d)} \leq C$, again by localization ([P]), in order to show (21) we only need to estimate the following differences for $0 \leq n < M$,

$$R(b\chi_{Q_K^n}\, d\sigma)(x) - R(\chi_{Q_K^n}\, d\nu)(x) = \sum_{Q_J^M \subset Q_K^n} R\alpha_J^M(x),$$

where $\alpha_J^M = \dfrac{\nu(Q_J^M)}{\sigma(B_J^M)}\chi_{B_J^M}\, d\sigma - \chi_{Q_J^M}\, d\nu$. Since $\int d\alpha_J^M = 0$, $\|R\alpha_J^M\|_{L^\infty(\mathbb{R}^d)} \leq C$ and for $|x - c(B_J^M)| > c\sigma_M$,

$$|R(\alpha_J^M)(x)| \leq C\frac{\sigma_M^d}{\operatorname{dist}(x, Q_J^M)^d},$$

(21) follows.

At this point one can finish the proof by applying Theorem 7.1 of Volberg [Vo] with the function $b$, but we will give a direct argument based on [Ch1]. We thank the referee for the route through Theorem 7.1 of [Vo].

Given a cube $Q_J^n$, $0 \leq n \leq M$, set

$$\tilde{Q}_J^n = \bigcup_{B_J^M \cap Q_J^n \neq \emptyset} B_J^M.$$

Notice that $\operatorname{diam}(\tilde{Q}_J^n) = c\sigma_n \approx \operatorname{diam}(Q_J^n)$ and $\sigma_{|Q_J^n} = \sigma_{|\tilde{Q}_J^n}$. By (19) and (20) we have

$$\begin{aligned}
\sigma(\tilde{E}_M \setminus \bigcup_\gamma \tilde{P}_\gamma) &\geq \frac{1}{C}\int_{\tilde{E}_M \setminus \bigcup_\gamma \tilde{P}_\gamma} |b| d\sigma \\
&\geq \frac{1}{C}\left|\int_{\tilde{E}_M} b d\sigma\right| - \frac{1}{C}\sum_\gamma \left|\int_{P_\gamma} b d\sigma\right| \\
&> \frac{1}{C}(\epsilon_0\sigma(\tilde{E}_M) - \epsilon\sum_\gamma \sigma(P_\gamma)).
\end{aligned}$$

Therefore, for $\eta = \dfrac{\epsilon_0 - \epsilon}{C - \epsilon}$,

$$\sum_\gamma \sigma(P_\gamma) \leq (1-\eta)\sigma(\tilde{E}_M). \tag{24}$$

We can now define our good set $G \subset \tilde{E}_M$. Set

$$G = \tilde{E}_M \setminus \bigcup_\gamma \tilde{P}_\gamma.$$

By (24), $\eta\sigma(\tilde{E}_M) \leq \sigma(G) \leq \sigma(\tilde{E}_M)$. We want to construct the set $E'$, by excising from $\tilde{E}_M$ the union of the stopping time cubes $\tilde{P}_\gamma$, and replacing each $\tilde{P}_\gamma$ by a union of two spheres. For each stopping time cube $\tilde{P}_\gamma$, set

$$S_\gamma = \partial B^1_\gamma \cup \partial B^2_\gamma,$$

where $B^j_\gamma$, $j = 1, 2$ are two balls with center $c(S_\gamma) := c(B^1_\gamma) = c(B^2_\gamma) \in P_\gamma$ and such that

$$2\text{diam}(B^1_\gamma) = \text{diam}(B^2_\gamma) = \begin{cases} \frac{c}{2}\sigma_n & \text{if } P_\gamma = Q^n_J, \text{ for some } 0 \leq n < M, \\ d_M & \text{if } P_\gamma = Q^M_J. \end{cases}$$

Set

$$E' = G \cup \bigcup_\gamma S_\gamma = \Big(\tilde{E}_M \setminus \bigcup_\gamma \tilde{P}_\gamma\Big) \cup \bigcup_\gamma S_\gamma,$$

and define a measure $\sigma'$ on $E'$ as follows:

$$\sigma' = \begin{cases} \sigma & \text{on } G \\ \dfrac{\sigma(P_\gamma)}{2}\Big(\dfrac{\Lambda_{d-1|\partial B^1_\gamma}}{\Lambda_{d-1}(\partial B^1_\gamma)} + \dfrac{\Lambda_{d-1|\partial B^2_\gamma}}{\Lambda_{d-1}(\partial B^2_\gamma)}\Big) & \text{on } S_\gamma. \end{cases}$$

Using that $\sigma$ is doubling and has $(d-1)-$growth it is easy to see that $\sigma'$ also satisfies these two properties.

For a system of dyadic cubes in $E'$ satisfying the required properties (see Theorem 11 in [Ch1]), we take all cubes $\tilde{Q}^n_J$, $0 \leq n \leq M$, which are not contained in any stopping time cube $\tilde{P}_\gamma$, together with each $S_\gamma$, together with each $\partial B^j_\gamma$, $j = 1, 2$ comprising $S_\gamma$, together with subsets of the two spheres,... and repeat.

We will now modify the function $b$ on the union $\cup_\gamma S_\gamma$ in order to obtain a new function $b'$ defined on $E'$, bounded and dyadic para-accretive with respect to the system of dyadic cubes defined above. Let

$$b'(x) = \begin{cases} b(x) & \text{if } x \in G \\ g_\gamma(x) = c^1_\gamma \chi_{\partial B^1_\gamma}(x) - c^2_\gamma \chi_{\partial B^2_\gamma}(x) & \text{on } S_\gamma, \end{cases}$$

where

$$c^1_\gamma = 2\omega_\gamma, \;\; c^2_\gamma = 2\omega_\gamma\Big(1 - \frac{|\nu(P_\gamma)|}{\sigma(P_\gamma)}\Big) \;\text{ and }\; \omega_\gamma = \begin{cases} \frac{\nu(P_\gamma)}{|\nu(P_\gamma)|} & \text{if } |\nu(P_\gamma)| \neq 0 \\ 1 & \text{otherwise.} \end{cases}$$

Notice that the coefficients $c^j_\gamma$, $j = 1, 2$, are defined so that

$$\int_{S_\gamma} g_\gamma d\sigma' = \int_{P_\gamma} b d\sigma = \nu(P_\gamma), \tag{25}$$

and $|c^1_\gamma| = 2$ and $2(1-\epsilon) \leq |c^2_\gamma| \leq 2$, because $P_\gamma$ is a stopping time cube. The function $b'$ is bounded because of the upper bound on the coefficients $c^j_\gamma$, $j = 1, 2$ and the fact that $\|b\|_\infty \leq C$.

For future reference, notice that, for every dyadic cube $Q$ in $E'$, such that $Q \nsubseteq S_\gamma$ for all $\gamma$, there is a non-stopping time cube $Q^*$ ($Q^* = \tilde{Q}^n_K$ for some $1 \leq n \leq M$) uniquely associated to $Q$ by the identity

$$Q = (Q^* \setminus \bigcup_{\tilde{P}_\gamma \subset Q^*} \tilde{P}_\gamma) \cup (\bigcup_{\tilde{P}_\gamma \subset Q^*} S_\gamma). \tag{26}$$

Moreover one has $\mathrm{diam}(Q) \approx \mathrm{diam}(Q^*)$ and

$$\sigma'(Q) = \sigma(Q^*) - \sum_{\tilde{P}_\gamma \subset Q^*} \sigma(\tilde{P}_\gamma) + \sum_{\tilde{P}_\gamma \subset Q^*} \sigma'(S_\gamma) = \sigma(Q^*). \tag{27}$$

We will check now that, by construction, the function $b'$ is dyadic para-accretive with respect to the system of dyadic cubes in $E'$:

If for some $\gamma$, $Q \subseteq S_\gamma$, the para-accretivity of $b'$ follows from the definition of $g_\gamma$ and the lower bound on $|c^j_\gamma|$, $j = 1, 2$. Recall that, when examining the para-accretivity condition on $S_\gamma$, although identity (25) holds, we have a satisfactory lower bound on the integral over each child $\partial B^j_\gamma$ of $S_\gamma$, which turns to be enough for $b'$ to be dyadic para-accretive.

Otherwise, let $Q^*$ be non-stopping time cube defined in (26). Then due to (25) and (27) we can write

$$\left|\int_Q b' d\sigma'\right| = \left|\int_{Q^*} b d\sigma\right| \geq \epsilon\sigma(Q^*) = \epsilon\sigma'(Q).$$

We must still show that $R(b'\sigma')$ belongs to dyadic $BMO(\sigma')$. It is enough to show the following $L^1-$ inequality

$$\|R(b'\chi_Q)\|_{L^1(\sigma'_Q)} \leq C\sigma'(Q), \tag{28}$$

for every dyadic cube in $E'$.

Let $Q$ be some dyadic cube in $E'$. We distinguish between two cases:

**Case 1:** For some $\gamma$, $Q \subseteq S_\gamma$. Then (28) follows from the boundedness of the coefficients $|c^j_\gamma|$, $j = 1, 2$, $\sigma(P_\gamma) \leq C\mathrm{diam}(P_\gamma)^{d-1}$ and $\Lambda_{d-1}(S_\gamma) \approx \mathrm{diam}(P_\gamma)^{d-1}$.

**Case 2:** Otherwise, $Q = (Q^* \setminus \bigcup_{\tilde{P}_\gamma \subset Q^*} \tilde{P}_\gamma) \cup (\bigcup_{\tilde{P}_\gamma \subset Q^*} S_\gamma)$ for some non-stopping $Q^* = \tilde{Q}^n_K$, $1 \le n \le M$. Due to (25) we can write

$$\begin{aligned} R(b'\chi_Q)(y) &= R(b\chi_{Q^*})(y) \\ &+ \sum_{\gamma:\tilde{P}_\gamma \subset Q^*} \int_{S_\gamma} g_\gamma(x)\Big(K(x-y) - K(c(S_\gamma)-y)\Big)d\sigma'(x) \\ &+ \sum_{\gamma:\tilde{P}_\gamma \subset Q^*} \int_{P_\gamma} b(x)\Big(K(c(S_\gamma)-y) - K(x-y)\Big)d\sigma(x) \\ &= A + B + C. \end{aligned}$$

By (21) (or (23) if $Q^* = B^M_J$), $\|R(b\chi_{Q^*})\|_{L^\infty(\mathbb{R}^d)} \le C$. Hence

$$\int_Q |A| d\sigma' \le C\sigma'(Q).$$

We deal now with term $B$. Set

$$B1 = \int_{Q\setminus S_\gamma} \Big| \int_{S_\gamma} g_\gamma(x)\Big(K(x-y) - K(c(S_\gamma)-y)\Big)d\sigma'(x)\Big| d\sigma'(y)$$

and

$$B2 = \int_{S_\gamma} \Big| \int_{S_\gamma} g_\gamma(x)\Big(K(x-y) - K(c(S_\gamma)-y)\Big)d\sigma'(x)\Big| d\sigma'(y).$$

For $B1$, let $g(Q) \in \mathbb{N}$ be such that $\text{diam}(Q) \approx \sigma_{g(Q)}$ and $P_\gamma = Q^n_J$ for some $0 \le n \le M$. Observe that $\text{diam}(S_\gamma) \approx \text{diam}(P_\gamma) \approx \sigma_n$. Denote by $Q^i$, $g(Q) \le i \le n$, the cubes in $E'$ contained in $Q$ and containing $S_\gamma$ such that $\text{diam}(Q^i) \approx \sigma_i$ (note that the $Q^i$ are either $\tilde{Q}^i_J$s or unions of spheres replacing the stopping time cubes of generation $i$). Then by the boundedness of $g_\gamma$, the $(d-1)$-growth of $\sigma'$ and the upper bound in (2),

$$\begin{aligned} B1 &\le C\sigma'(S_\gamma) \sum_{i=g(Q)}^{n-1} \int_{Q^i \setminus Q^{i+1}} \frac{\sigma_n}{\sigma_i^d}\, d\sigma' \\ &\le C\sigma'(S_\gamma) \sum_{i=g(Q)}^{n-1} \frac{\sigma_n}{\sigma_i} \le C\sigma'(S_\gamma) \sum_i 2^{-i} \le C\sigma'(S_\gamma). \end{aligned}$$

For $B2$ argue like in the previous case, i.e. (28) for $Q = S_\gamma$, to get that $B2 \le C\sigma'(S_\gamma)$. Therefore by $\sigma'(S_\gamma) = \sigma(P_\gamma)$, the packing condition (24) (with $\tilde{E}_M$ replaced by $Q^*$) and (27) we get that $\int_Q |B| d\sigma' \le C\sigma'(Q)$.

Similar arguments work to show $\int_Q |C| d\sigma' \le C\sigma'(Q)$. Therefore we are done. □

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## References


[Ch1] M. Christ, *A $T(b)$ Theorem with remarks on analytic capacity and the Cauchy integral,* Colloq. Math. **60/61** (1990), 601–628.

[Ch2] ———, *Lectures on singular integral operators,* CBMS Regional Conference Series in Mathematics 77, Providence, RI, 1990.

[DØ] A. M. Davie and B. Øksendal, *Analytic capacity and differentiability properties of finely harmonic functions,* Acta Math. **149** (1982) 127–152.

[GV] J. Garnett and J. Verdera, *Analytic capacity, bilipschitz maps and Cantor sets,* Math. Res. Lett. 10 (2003), no. 4, 515–522.

[MPrV] J. Mateu, L. Prat, and J. Verdera, *The capacity associated to signed Riesz kernels, and Wolff potentials,* J. Reine Angew. Math. **578** (2005) 201–223.

[MT] J. Mateu and X. Tolsa, *Riesz transforms and harmonic $Lip_1$-capacity in Cantor sets,* Proc. London Math. Soc. **89** (2004), no. 3, 676–696.

[MTV] J. Mateu, X. Tolsa and J. Verdera, *The planar Cantor sets of zero analytic capacity and the local $T(b)$ theorem*, J. Amer. Math. Soc. **16** (2003), no. 1, 19–28.

[P] P. V. Paramonov, *Harmonic approximations in the $\mathcal{C}^1$-norm,* Math. USSR-Sb. **71** (1992) 183–207.

[T1] X. Tolsa, *Painlevé's problem and the semiadditivity of analytic capacity,* Acta Math. **190** (2003) 105–149.

[T2] ———, *Bilipschitz maps, analytic capacity and the Cauchy integral,* Ann. of Math. **162** (2005) 1241–1302.

[Vo] A. Volberg, *Calderón-Zygmund capacities and operators on nonhomogeneous spaces,* CBMS Regional Conference Series in Mathematics, 100, Providence, RI, 2003.



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