Math. Res. Lett. **13** (2006), *no.* 5, 813–823

ALL SIEGEL HECKE EIGENSYSTEMS (MOD p) ARE CUSPIDAL

Alexandru Ghitza

Abstract. Fix integers $g \geq 1$ and $N \geq 3$, and a prime p not dividing N. We show that the systems of Hecke eigenvalues occurring in the spaces of Siegel modular forms (mod p) of dimension g, level N, and varying weight, are the same as the systems occurring in the spaces of Siegel *cusp forms* with the same parameters and varying weight. In particular, in the case $g = 1$, this says that the Hecke eigensystems (mod p) coming from classical modular forms are the same as those coming from cusp forms. The proof uses both the main theorem of [Ghi04] and a modification of the techniques used there, namely restriction to the superspecial locus.

1. Introduction

This paper is concerned with the systems of Hecke eigenvalues coming from modular forms in positive characteristic. The main result is that imposing the condition of cuspidality has no effect on the set of eigensystems that can be obtained, at least if we allow ourselves to change the weight of the form. We prove this in the context of Siegel modular forms, but the method should apply to forms coming from other Shimura varieties Y of PEL type, i.e. those arising as moduli spaces of abelian varieties with specified polarizations, endomorphisms, and level structures. For the reader's convenience, here are the key properties of Y that we use: Y should have an arithmetic Satake compactification X, normal and of finite type over $\operatorname{Spec} \mathbb{Z}[\frac{1}{N}]$ for an appropriate N, containing Y as a dense open subscheme, and such that the superspecial locus of $Y \otimes \overline{\mathbb{F}}_p$ is non-empty. Moreover, the Hodge line bundle $\underline{\omega}$ on Y should extend to an ample line bundle on X. According to §V.0 of [FC90], most of these properties (except perhaps for the one regarding the non-emptiness of the superspecial locus) hold for Shimura varieties of PEL type.

In §2 we prove the main result in the case of elliptic modular forms ($g = 1$). We have two reasons for doing this: first, it will give an idea of the proof of the general case unclouded by technical complications; second, the elliptic case is simple enough as to allow us to give an effective version of our result, something that we cannot accomplish in general.

The rest of the paper deals with the case $g > 1$. In §3 we review properties of the arithmetic Satake compactification and use them to give our definition of Siegel cusp forms. In the process we give a proof of the Köcher principle for the arithmetic Satake compactification, which basically says that any Siegel modular form extends to the Satake boundary. This is undoubtably known to the experts, but it does not seem

Received by the editors November 30, 2005.
2000 *Mathematics Subject Classification.* Primary: 11F46.
Key words and phrases. Siegel modular forms, Hecke eigenvalues, Köcher principle, Siegel cusp forms.

to have ever been written down in this setting. Our proof of the Köcher principle should apply to any Shimura variety of PEL type whose boundary inside the Satake compactification has everywhere codimension at least 2.

In §5 we review the definition and properties of superspecial forms. Section 6 contains the proof of our main result. Finally, in §7 we show that our notion of Siegel cusp form (defined using the Satake compactification) agrees with that introduced by Chai-Faltings in [FC90] (based on the toroidal compactifications).

2. The elliptic case: $g = 1$

Let p be a prime, and let $N \geq 3$ be an integer not divisible by p. We denote by $M_k(N)$ the space of modular forms (mod p) of weight k and level $\Gamma(N)$, and by $S_k(N)$ the subspace of cusp forms. We are interested in the action of Hecke operators T_ℓ ($\ell \nmid pN$) on the spaces $M_k(N)$ and $S_k(N)$. There is a Hecke-equivariant injection

$$M_k(N) \hookrightarrow M_{k+p-1}(N)$$

given by multiplication by the Hasse invariant A. The *filtration* of f is the smallest integer w such that there exists $\tilde{f} \in M_w(N)$ with

$$f = \tilde{f} \cdot (\text{some nonnegative power of } A).$$

In this section we apply Serre's ideas from [Ser96] to obtain a proof of the following

Theorem 1. *Fix a prime p and a level $N \geq 3$, $p \nmid N$. Then all Hecke eigensystems (mod p) are cuspidal. More precisely, let Φ be the eigensystem associated to some $f \in M_k(N)$, and let w be the filtration of f.*

- (a) *Suppose $p > 2$, or $p = 2$ and $w > 0$. Then there exists $f' \in S_w(N)$ or $S_{w+p^2-1}(N)$ such that Φ is associated to f'. Moreover, the first situation (existence of such $f' \in S_w(N)$) occurs if and only if $f \in S_k(N)$.*
- (b) *If $p = 2$ and $w = 0$, then there exists $f' \in S_6(N)$ such that Φ is associated to f'.*

Remark. Let $N = 3$, $p = 2$, $w = 0$. Then $S_0(3) = S_3(3) = 0$, so in the situation of (b) we cannot do as well as in (a).

Proof. We fix the level N and often drop it from our notation.

Let $\underline{\omega}$ be the Hodge (line) bundle on $X = X(N) \otimes \overline{\mathbb{F}}_p$, so that $M_k = \mathrm{H}^0(X, \underline{\omega}^{\otimes k})$, and let $A \in M_{p-1}$ be the Hasse invariant. Let $\mathscr{S}\mathscr{S}_k$ denote the cokernel of multiplication by A, i.e.

$$0 \longrightarrow \underline{\omega}^{\otimes(k-p+1)} \xrightarrow{\times A} \underline{\omega}^{\otimes k} \longrightarrow \mathscr{S}\mathscr{S}_k \longrightarrow 0.$$

Define

$$SS_k := \mathrm{H}^0(X, \mathscr{S}\mathscr{S}_k) = \mathrm{H}^0(\Sigma, \mathscr{S}\mathscr{S}_k|_\Sigma),$$

where Σ denotes the supersingular locus of X. Using the fact that every supersingular elliptic curve over $\overline{\mathbb{F}}_p$ has a canonical $\mathbb{F}_{p^2}$-structure, one easily sees that $SS_k = SS_{k+p^2-1}$ for all k. Also SS_k has a natural (away from Np) Hecke action, coming from the fact that ℓ-isogenies preserve supersingularity (if $\ell \nmid Np$).

Serre shows (in particular) that if the eigensystem Φ is associated to $f \in M_k$ and w is the filtration of f, then Φ also occurs in SS_w. So it is enough for us to show that any eigensystem Φ occurring in SS_w also occurs in S_w or S_{w+p^2-1}. To see this, let Δ denote the divisor of cusps on X and consider the exact sequence

$$0 \longrightarrow \underline{\omega}^{\otimes(w-p+1)}(-\Delta) \xrightarrow{\times A} \underline{\omega}^{\otimes w}(-\Delta) \longrightarrow \mathscr{S}\mathscr{S}_w(-\Delta) \longrightarrow 0.$$

Taking global sections, we get

$$0 \longrightarrow S_{w-p+1} \xrightarrow{\times A} S_w \longrightarrow SS_w \longrightarrow \mathrm{H}^1(X, \underline{\omega}^{\otimes(w-p+1)}(-\Delta)), \tag{1}$$

where we used the identifications

$$\mathrm{H}^0(X, \mathscr{S}\mathscr{S}_w(-\Delta)) = \mathrm{H}^0(\Sigma, \mathscr{S}\mathscr{S}_w(-\Delta)|_\Sigma) = \mathrm{H}^0(\Sigma, \mathscr{S}\mathscr{S}_w|_\Sigma) = SS_w$$

since $\Delta \cap \Sigma = \emptyset$.

The canonical sheaf of X is Ω^1_X, which is isomorphic (by Kodaira-Spencer) to $\underline{\omega}^{\otimes 2}(-\Delta)$. So by Serre duality we have

$$\mathrm{H}^1(X, \underline{\omega}^{\otimes m}(-\Delta)) \cong \mathrm{H}^0\left(X, \left(\underline{\omega}^{\otimes m}(-\Delta)\right)^\vee \otimes \underline{\omega}^{\otimes 2}(-\Delta)\right)^\vee \cong M^\vee_{2-m},$$

where $\cdot^\vee$ denotes the dual. So if $m > 2$ we conclude that $\mathrm{H}^1(X, \underline{\omega}^{\otimes m}(-\Delta)) = 0$. In particular, if $w > p+1$ we know that the map $S_w \to SS_w$ from (1) is surjective, and therefore Φ occurs in S_w.

If $p = 2$ and $w = 0$, we have $w + 2(p^2-1) = 6 > 3 = p+1$, so the map $S_6 \to SS_6 = SS_0$ is surjective, and therefore Φ occurs in S_6. This settles (b).

In the situation of (a), $w + p^2 - 1 > p + 1$ so the map

$$S_{w+p^2-1} \longrightarrow SS_{w+p^2-1} = SS_w$$

is surjective, and therefore Φ occurs in S_{w+p^2-1}.

It remains to prove the last statement of (a). Saying that w is the filtration of f implies that if we put $n = (k-w)/(p-1)$ then there exists $\tilde{f} \in M_w$ such that $A^n\tilde{f} = f$. Since the divisor Σ of A is disjoint from the cusps Δ, we have that $f \in S_k$ if and only if $\tilde{f} \in S_w$. So on one hand if $f \in S_k$ then $\tilde{f} \in S_w$ and f and $\tilde{f}$ have the same eigensystem, so we may put $f' = \tilde{f}$ and we are done. Conversely, suppose $f' \in S_w$, then f' and $\tilde{f}$ have the same eigensystem, hence also the same Fourier coefficients. Moreover, they have the same weight w, so by the q-expansion principle $\tilde{f} = f'$. So $\tilde{f} \in S_w$, therefore $f \in S_k$. □

Corollary 2. *If $f \in M_k(N)$ is an eigenform and has filtration $w > p+1$, then f is a cusp form.*

Proof. This follows from the proof of Theorem 1: if $w > p+1$ then the restriction-to-Σ map $S_w \to SS_w$ is surjective, so there exists $f' \in S_w$ with the same eigensystem as f. But then the last statement of the Theorem tells us that f is a cusp form. □

We conclude this section with some explicit numerical examples, in which the cusp eigenforms we exhibit are taken from W. Stein's database [MFD].

Example 1. Let $p = 5$, $N = 1$, and let $f = E_4$ be the Eisenstein series of weight 4. We know that f is a Hecke eigenform with eigensystem

$$(1 + \ell^3)_{\ell \neq 5} = (4, 3, 4, 2, 3, 4, 0, 3, 0, 2, 4, \dots)$$

Of course, f is nothing but the Hasse invariant (mod 5), so its filtration is $w = 0$. Since f is not a cusp form, Theorem 1 predicts the existence of a cuspidal eigenform

f' of weight $w + p^2 - 1 = 24$ with the same eigensystem. Indeed, there is a cusp eigenform (mod 5) of weight 24 with q-expansion

$$\begin{aligned} f'(q) &= q + 4q^2 + 3q^3 + 3q^4 + 2q^6 + 4q^7 + 2q^9 + 2q^{11} + 4q^{12} + 3q^{13} + q^{14} + q^{16} + 4q^{17} + 3q^{18} \\ &\quad + 2q^{21} + 3q^{22} + 3q^{23} + 2q^{26} + 2q^{28} + 2q^{31} + 4q^{32} + q^{33} + q^{34} + q^{36} + 4q^{37} + O(q^{38}). \end{aligned}$$

Similarly, E_6 has filtration 6 and eigensystem

$$(1 + \ell^5)_{\ell \neq 5} = (3, 4, 3, 2, 4, 3, 0, 4, 0, 2, 3, \dots).$$

There exists a cusp eigenform (mod 5) of weight 30 with q-expansion

$$\begin{aligned} &q + 3q^2 + 4q^3 + 2q^4 + 2q^6 + 3q^7 + 3q^9 + 2q^{11} + 3q^{12} + 4q^{13} + 4q^{14} + q^{16} + 3q^{17} + 4q^{18} \\ &+ 2q^{21} + q^{22} + 4q^{23} + 2q^{26} + q^{28} + 2q^{31} + 3q^{32} + 3q^{33} + 4q^{34} + q^{36} + 3q^{37} + O(q^{38}). \end{aligned}$$

Example 2. Let $p = 7$, $N = 1$. We consider the Eisenstein series of weights 4, 6, and 8:

(a) $f_1 = E_4$ has filtration 4 and eigensystem

$$(1 + \ell^3)_{\ell \neq 7} = (2, 0, 0, 2, 0, 0, 0, 2, 2, 0, 2, \dots)$$

There is a cusp eigenform (mod 7) of weight 52 with q-expansion

$$\begin{aligned} f_1'(q) &= q + 2q^2 + 3q^4 + 4q^8 + q^9 + 2q^{11} + 5q^{16} + 2q^{18} + 4q^{22} \\ &\quad + 2q^{23} + q^{25} + 2q^{29} + 6q^{32} + 3q^{36} + 2q^{37} + O(q^{38}). \end{aligned}$$

(b) $f_2 = E_6$ has filtration 0 and eigensystem

$$(1 + \ell^5)_{\ell \neq 7} = (5, 6, 4, 3, 0, 6, 4, 5, 2, 6, 5, \dots)$$

There is a cusp eigenform (mod 7) of weight 48 with q-expansion

$$\begin{aligned} f_2'(q) &= q + 5q^2 + 6q^3 + 4q^5 + 2q^6 + q^8 + 3q^9 + 6q^{10} + 3q^{11} + 3q^{15} + 5q^{16} + 6q^{17} + q^{18} + 4q^{19} \\ &\quad + q^{22} + 5q^{23} + 6q^{24} + 6q^{25} + 2q^{27} + 2q^{29} + q^{30} + 6q^{31} + 4q^{33} + 2q^{34} + 5q^{37} + O(q^{38}). \end{aligned}$$

(c) $f_3 = E_8$ has filtration 8 and eigensystem

$$(1 + \ell^7)_{\ell \neq 7} = (3, 4, 6, 5, 0, 4, 6, 3, 2, 4, 3, \dots)$$

There is a cusp eigenform (mod 7) of weight 56 with q-expansion

$$\begin{aligned} f_3'(q) &= q + 3q^2 + 4q^3 + 6q^5 + 5q^6 + q^8 + 6q^9 + 4q^{10} + 5q^{11} + 3q^{15} + 3q^{16} + 4q^{17} + 4q^{18} + 6q^{19} \\ &\quad + q^{22} + 3q^{23} + 4q^{24} + 3q^{25} + 5q^{27} + 2q^{29} + 2q^{30} + 4q^{31} + 6q^{33} + 5q^{34} + 3q^{37} + O(q^{38}). \end{aligned}$$

3. The arithmetic Satake compactification

For the remainder of the paper we assume that $g > 1$.

Fix an integer $N \geq 3$, and let $\mathscr{A}_{g,N}$ denote the moduli space of g-dimensional principally polarized abelian varieties with symplectic level N structure.

There are several ways to compactify $\mathscr{A}_{g,N}$; we will work with the arithmetic Satake (also known as minimal) compactification $\mathscr{A}_{g,N}^*$, whose existence and properties are described in Theorem V.2.5 of [FC90]. For now, we just need to know that $\mathscr{A}_{g,N}^*$ is a normal scheme, proper and of finite type over $\operatorname{Spec} \mathbb{Z}[\frac{1}{N}]$, containing $\mathscr{A}_{g,N}$ as a dense open subscheme.

Note that in the classical case $g = 1$, these are the usual modular curves $\mathscr{A}_{1,N} = Y(N)$ and $\mathscr{A}_{1,N}^* = X(N)$.

There is a universal abelian scheme

$$\begin{array}{ccc} A^{\mathrm{univ}} & & \Omega^1_{A^{\mathrm{univ}}/\mathscr{A}_{g,N}} \\ \Big\downarrow \pi & & \wr\!\downarrow \\ \mathscr{A}_{g,N} & & \mathbb{E} := \pi_* \Omega^1_{A^{\mathrm{univ}}/\mathscr{A}_{g,N}}, \end{array}$$

and we can therefore define the *Hodge bundle* $\mathbb{E}$ on $\mathscr{A}_{g,N}$. It has rank g, so given a representation ρ of the algebraic group GL_g, we can define the *twist* of $\mathbb{E}$ by ρ, by applying ρ to the transition functions of $\mathbb{E}$. The result is denoted $\mathbb{E}_\rho$ and it has rank equal to the dimension of ρ.

We want to see how much of this can be extended to the minimal compactification $\mathscr{A}^*_{g,N}$. For this we need the following technical result:

Theorem 3. *Let X be a locally noetherian scheme which is locally of finite type over a quotient R_0/I of a regular ring R_0. Let U be an open subset of X such that the complement Z of U has everywhere codimension at least 2 in X. Let $i : U \hookrightarrow X$ denote the canonical inclusion. Let $\mathscr{F}$ be a torsion-free coherent $\mathscr{O}_U$-module. Then $i_*\mathscr{F}$ is a coherent $\mathscr{O}_X$-module. If, moreover, X is normal and $\mathscr{F}$ is reflexive, so is $i_*\mathscr{F}$, and it is the unique reflexive coherent sheaf on X extending $\mathscr{F}$.*

Proof. We start by noticing that X is *locally embeddable in a regular scheme*, i.e. that any point $x \in X$ has an open neighborhood isomorphic to a subscheme of a regular scheme. This follows directly from the fact that X is locally of finite type over R_0/I, i.e. locally embeddable in affine space over R_0/I. In turn, this is embeddable in a regular scheme, namely affine space over the regular ring R_0.

Since $\mathscr{F}$ is torsion-free, the support of $\mathscr{F}$ is all of U. Also U is dense in X, so $\bar{U} = X$. Since Z is everywhere of codimension at least 2 in X, we have that for any irreducible component U' of $\bar{U} = X$,

$$\mathrm{codim}(Z \cap U', U') \geq 2.$$

So we may now apply the following result with $n = 1$ to conclude that $i_*\mathscr{F}$ is a coherent $\mathscr{O}_X$-module:

> **Proposition VIII.3.2 in** [SGA2]**:** Let X be a locally noetherian scheme which is locally embeddable in a regular scheme. Let U be an open subscheme of X and let $i : U \to X$ be the canonical embedding. Let $n \in \mathbb{Z}$, and let $\mathscr{F}$ be a coherent Cohen-Macaulay $\mathscr{O}_U$-module. Then the following are equivalent:
> (a) The sheaf $\mathrm{R}^p\, i_*\mathscr{F}$ is coherent for all $p < n$.
> (b) Let S denote the support of $\mathscr{F}$ and let $\bar{S}$ be the closure of S in X. For any irreducible component S' of $\bar{S}$, we have
> $$\mathrm{codim}(S' \cap (X - U), S') > n.$$

Finally, to prove the statement about reflexivity, we employ an argument used by Serre in the complex-analytic category (see Proposition 7 of [Ser66]). First notice that $i_*\mathscr{O}_U = \mathscr{O}_X$: let V be an open subset of X and let $f \in \mathscr{O}_U(U \cap V)$. We know that $U \cap V$ is dense in V and that its complement $Z \cap V$ has everywhere codimension at least 2 in V. So f defines a rational function on V; assume that f has at least one pole. Consider the locus D of its poles; by assumption $D \subset Z \cap V$, but on the other hand D is a Weil divisor on V, so it has codimension one, which is absurd since $Z \cap V$ has codimension at least 2.

Next we claim that if $\mathscr{R}$ is a reflexive sheaf on X then

$$i_* i^* \mathscr{R} = \mathscr{R}.$$

To see this, let $\mathscr{D} = \mathscr{R}^\vee$. Since $\mathscr{R}$ is reflexive we have $\mathscr{R} = \mathscr{D}^\vee = \mathscr{H}om(\mathscr{D}, \mathscr{O}_X)$, so $i^*\mathscr{R} = \mathscr{H}om(i^*\mathscr{D}, \mathscr{O}_U)$. Therefore

$$i_* i^* \mathscr{R} = i_* \mathscr{H}om(i^*\mathscr{D}, \mathscr{O}_U) = \mathscr{H}om(\mathscr{D}, i_*\mathscr{O}_U) = \mathscr{H}om(\mathscr{D}, \mathscr{O}_X) = \mathscr{D}^\vee = \mathscr{R},$$

as claimed.

Now assume $\mathscr{F}$ is reflexive, and let $\mathscr{G}$ be a coherent sheaf on X extending $\mathscr{F}$. Let $\mathscr{G}^{\vee\vee}$ be the bidual of $\mathscr{G}$, then $\mathscr{G}^{\vee\vee}$ extends $\mathscr{F}$ and is reflexive. Hence $\mathscr{G}^{\vee\vee} = i_* i^* \mathscr{G}^{\vee\vee} = i_*\mathscr{F}$, from which we conclude that $i_*\mathscr{F}$ is reflexive and that if $\mathscr{G}$ is reflexive then $\mathscr{G} = i_*\mathscr{F}$. □

In particular, we can apply Theorem 3 with $X = \mathscr{A}^*_{g,N}$, $U = \mathscr{A}_{g,N}$, $\mathscr{F} = \mathbb{E}_\rho$ (we know that the codimension of $X - U$ in X is g, so by our assumptions at least 2).

We will denote the pushforward $i_*\mathbb{E}_\rho$ by $\mathbb{E}^*_\rho$. We stress the fact that $\mathbb{E}^*_\rho$ is in general only a coherent sheaf on $\mathscr{A}^*_{g,N}$, not necessarily locally free. This causes some complications in working with the minimal compactification, but as we shall see they are not essential.

A notable exception to this caveat, of which we will make crucial use in our main argument, is the following result (part of Theorem V.2.5 in [FC90]):

Fact 4 (Chai-Faltings)**.** *The invertible sheaf* $\underline{\omega} := \mathbb{E}_{\det}$ *on* $\mathscr{A}_{g,N}$ *extends to an invertible sheaf* $\underline{\omega}^*$ *on* $\mathscr{A}^*_{g,N}$ *relatively ample over* $\operatorname{Spec}\mathbb{Z}$.

We have the following result:

Proposition 5 (Köcher principle)**.** *If* $g > 1$*, then for any* ρ *and for any* $\mathbb{Z}[\frac{1}{N}]$*-module* M *there is a natural identification*

$$\mathrm{H}^0(\mathscr{A}^*_{g,N}, \mathbb{E}^*_\rho \otimes M) = \mathrm{H}^0(\mathscr{A}_{g,N}, \mathbb{E}_\rho \otimes M).$$

Proof. This is presumably well-known but we prove it here for lack of a reference. See Theorem 10.14 of [BB66] for the complex-analytic version of a more general result.

Any $\mathbb{Z}[\frac{1}{N}]$-module M is a direct limit of finitely generated $\mathbb{Z}[\frac{1}{N}]$-modules; since cohomology and tensor products commute with direct limits, we may safely assume that M is finitely generated. Using the classification of finitely generated $\mathbb{Z}[\frac{1}{N}]$-modules and the additivity of cohomology, we may further reduce to the case where $M = \mathbb{Z}[\frac{1}{N}]$ or $M = \mathbb{Z}/n\mathbb{Z}$ for some integer n coprime to N. In both cases M is actually a ring, which we denote by R.

Let $X = \mathscr{A}^*_{g,N} \otimes R$ and $Y = \mathscr{A}_{g,N} \otimes R$. Since $\mathbb{E}^*_\rho = i_*\mathbb{E}_\rho$, we have

$$\mathrm{H}^0(X, i_*\mathbb{E}_\rho) = (i_*\mathbb{E}_\rho)(X) = \mathbb{E}_\rho(Y) = \mathrm{H}^0(Y, \mathbb{E}_\rho),$$

as desired. □

Given a $\mathbb{Z}[\frac{1}{N}]$-module M, the space of *Siegel modular forms* of weight ρ and level N with coefficients in M is

$$M_\rho(M) := \mathrm{H}^0(\mathscr{A}_{g,N}, \mathbb{E}_\rho \otimes M).$$

(We do not include N in the notation since we will always work with a fixed N.)

We define the *cusps* to be the boundary

$$\Delta := \mathscr{A}^*_{g,N} - \mathscr{A}_{g,N}.$$

We want to define a notion of cusp form in this setting. We start with the short exact sequence of $\mathscr{O}_{\mathscr{A}^*_{g,N}}$-modules that defines the ideal sheaf $\mathscr{I}_\Delta$ of $\iota : \Delta \hookrightarrow \mathscr{A}^*_{g,N}$:

$$0 \longrightarrow \mathscr{I}_\Delta \longrightarrow \mathscr{O}_{\mathscr{A}^*_{g,N}} \longrightarrow \iota_* \mathscr{O}_\Delta \longrightarrow 0,$$

and we tensor it with $\mathbb{E}^*_\rho$; since $\mathbb{E}^*_\rho$ is not necessarily locally free, we only get

$$\mathscr{I}_\Delta \otimes \mathbb{E}^*_\rho \longrightarrow \mathbb{E}^*_\rho \longrightarrow \mathbb{E}^*_\rho|_\Delta \longrightarrow 0.$$

Define

$$\mathscr{S}_\rho := \ker\left(\mathbb{E}^*_\rho \longrightarrow \mathbb{E}^*_\rho|_\Delta\right).$$

In other words, for any open $U \subset \mathscr{A}^*_{g,N}$, $\mathscr{S}_\rho(U)$ consists of the sections of $\mathbb{E}_\rho$ over U that vanish at the cusps Δ.

It is then natural to define the space of *Siegel cusp forms* of weight ρ and level N with coefficients in a $\mathbb{Z}[\frac{1}{N}]$-module M to be

$$S_\rho(M) := \mathrm{H}^0(\mathscr{A}^*_{g,N}, \mathscr{S}_\rho \otimes M).$$

Note that as a result of this definition and of the Köcher principle, $S_\rho(M)$ is a subset of $M_\rho(M)$.

4. Hecke eigensystems (mod p)

We now fix a prime p not dividing N, and set

$$U := \mathscr{A}_{g,N} \otimes \overline{\mathbb{F}}_p, \quad X := \mathscr{A}^*_{g,N} \otimes \overline{\mathbb{F}}_p, \quad M_\rho := M_\rho(\overline{\mathbb{F}}_p), \quad S_\rho := S_\rho(\overline{\mathbb{F}}_p).$$

There is a Hecke action on M_ρ given by the Hecke operators corresponding to the primes not dividing Np. They are essentially induced by isogenies of degree coprime to Np (for their exact definition, see §2.2.2 and §3.2.1 of [Ghi04]). We denote by $\mathscr{H}$ the $\mathbb{Z}$-algebra generated by these operators. It is known to be commutative (see Satz IV.1.13 of [Fre83]).

If V is any $\overline{\mathbb{F}}_p$-vector space with an action of $\mathscr{H}$, an element $v \in V$ which is a common eigenvector for $\mathscr{H}$ defines an algebra homomorphism $\Phi : \mathscr{H} \to \overline{\mathbb{F}}_p$ given by

$$Tv = \Phi(T)v, \quad \text{for all } T \in \mathscr{H}.$$

This Φ is called the *eigensystem* associated to v.

With this terminology, a *Hecke eigensystem* (mod p) is one associated to an element of M_ρ. The action of $\mathscr{H}$ restricts to S_ρ; this follows from the fact that S_ρ is the space of global sections of the coherent sheaf $\mathscr{S}_\rho$ on X (see §2.2.2 of [Ghi04]). We say that Φ is *cuspidal* if it is associated to an element of S_ρ.

We can now state our main result:

Theorem 6. *Fix the characteristic $p > 0$, the dimension $g \geq 2$, and the level $N \geq 3$, $p \nmid N$. Then all Hecke eigensystems (mod p) are cuspidal. That is, for any Φ associated to some $f \in M_\rho$ there exists some $f' \in S_{\rho'}$ such that Φ is associated to f'.*

The idea of the proof is this: we use a result of [Ghi04] saying that any eigensystem (mod p) is superspecial (see the next section for the definition), and then we show that any superspecial eigensystem is cuspidal.

5. Superspecial forms

A g-dimensional abelian variety A over $\overline{\mathbb{F}}_p$ is said to be *superspecial* if

$$\dim_{\overline{\mathbb{F}}_p} \mathrm{Hom}(\alpha_p, A) = g,$$

where α_p is the kernel of multiplication by p on the additive group $\mathbb{G}_a$ over $\overline{\mathbb{F}}_p$. Equivalently, A is $\overline{\mathbb{F}}_p$-isomorphic to E^g, where E is any supersingular elliptic curve.

Let $\Sigma \subset X$ denote the set of superspecial points. It has a number of remarkable properties, including

- It is finite.
- It is closed under isogenies of degree coprime to Np.
- Any superspecial A has a canonical and functorial $\mathbb{F}_{p^2}$-structure (see Proposition 6 in [Ghi04]); in particular it makes sense to talk about the space of $\mathbb{F}_{p^2}$-rational differentials on A, and it turns out that a principal polarization on A induces a natural hermitian form on this space. Thus if we are interested in hermitian bases, the change-of-basis group is

  $$\mathrm{GU}_g(\mathbb{F}_{p^2}) := \{M \in \mathrm{GL}_g(\mathbb{F}_{p^2}) : {}^t\overline{M}M = \gamma(M)I \text{ for some } \gamma(M) \in \mathbb{F}_{p^2}^\times\},$$

  where the "conjugation" $\bar{\cdot} : \mathbb{F}_{p^2} \to \mathbb{F}_{p^2}$ is $a \mapsto \overline{a} = a^p$.

This suggests the following definition. Given a finite-dimensional $\overline{\mathbb{F}}_p$-representation

$$\tau : \mathrm{GU}_g(\mathbb{F}_{p^2}) \longrightarrow W,$$

we set

$$SS_\tau := \{f : [A, \lambda, \alpha, \eta] \longrightarrow W \text{ such that } f([A, \lambda, \alpha, M\eta]) = \tau(M)^{-1} f([A, \lambda, \alpha, \eta]) \text{ for all } M \in \mathrm{GU}_g(\mathbb{F}_{p^2})\},$$

where $[A, \lambda, \alpha, \eta]$ denotes the $\overline{\mathbb{F}}_p$-isomorphism class of the quadruple, and

- (A, λ) is a superspecial principally polarized abelian variety over $\overline{\mathbb{F}}_p$;
- α is a symplectic level N structure on (A, λ);
- η is a hermitian basis of invariant $\mathbb{F}_{p^2}$-rational differentials on (A, λ).

We refer to SS_τ as the space of *superspecial forms* of weight τ. The aforementioned properties of Σ imply that SS_τ admits an action of the Hecke algebra $\mathscr{H}$ and that it has the following periodicity property:

$$SS_{\tau \otimes \det^{\otimes p^2-1}} = SS_\tau \quad \text{for all } \tau.$$

An eigensystem Φ associated to some $f \in SS_\tau$ is said to be *superspecial*.

6. Proof of the main result

We now prove Theorem 6.

It is part of the proof of Theorem 28 in [Ghi04] (more precisely, the first paragraph on page 380 loc. cit.) that any eigensystem (mod p) is superspecial. Therefore it suffices to show that any superspecial eigensystem is cuspidal.

Let $\mathscr{I}_\Sigma$ be the ideal sheaf of $j : \Sigma \hookrightarrow X$; it is defined by the short exact sequence of $\mathscr{O}_X$-modules

$$0 \longrightarrow \mathscr{I}_\Sigma \longrightarrow \mathscr{O}_X \longrightarrow j_*\mathscr{O}_\Sigma \longrightarrow 0.$$

Upon tensoring with the coherent sheaf $\mathscr{S}_\rho$ introduced towards the end of §3, we get

$$\mathscr{I}_\Sigma \otimes \mathscr{S}_\rho \longrightarrow \mathscr{S}_\rho \longrightarrow \mathscr{S}_\rho|_\Sigma \longrightarrow 0. \tag{2}$$

We can easily pass from $\mathscr{S}_\rho|_\Sigma$ to $\mathbb{E}_\rho|_\Sigma$; since restriction to U is exact and $\Delta \cap U = \emptyset$, the short exact sequence defining $\mathscr{S}_\rho$

$$0 \longrightarrow \mathscr{S}_\rho \longrightarrow \mathbb{E}^*_\rho \longrightarrow \mathbb{E}^*_\rho|_\Delta \longrightarrow 0$$

gives an isomorphism $\mathscr{S}_\rho|_U \cong \mathbb{E}^*_\rho|_U$. In particular, $\mathscr{S}_\rho|_\Sigma \cong \mathbb{E}^*_\rho|_\Sigma \cong \mathbb{E}_\rho|_\Sigma$, the latter isomorphism coming from $\Sigma \cap \Delta = \emptyset$.

Therefore the surjective map from the sequence (2) gives

$$\mathscr{S}_\rho \longrightarrow \mathbb{E}_\rho|_\Sigma \longrightarrow 0.$$

Let $\mathscr{K}_\rho$ denote its kernel:

$$0 \longrightarrow \mathscr{K}_\rho \longrightarrow \mathscr{S}_\rho \longrightarrow \mathbb{E}_\rho|_\Sigma \longrightarrow 0. \tag{3}$$

This yields a long exact sequence

$$0 \longrightarrow \mathrm{H}^0(X, \mathscr{K}_\rho) \longrightarrow S_\rho \longrightarrow \mathrm{H}^0(\Sigma, \mathbb{E}_\rho|_\Sigma) \longrightarrow \mathrm{H}^1(X, \mathscr{K}_\rho).$$

But it is easily seen that

$$\mathrm{H}^0(\Sigma, \mathbb{E}_\rho|_\Sigma) = SS_{\operatorname{Res}\rho},$$

where $\operatorname{Res}\rho$ denotes the restriction of ρ to the finite group $\mathrm{GU}_g(\mathbb{F}_{p^2})$.

Therefore we have a map (which we think of as restriction of cusp forms to the superspecial locus)

$$r_\rho : S_\rho \to SS_{\operatorname{Res}\rho},$$

which is Hecke-equivariant and whose cokernel is contained in $\mathrm{H}^1(X, \mathscr{K}_\rho)$. Recall that Fact 4 says that $\underline{\omega}^*$ is a line bundle; by tensoring the short exact sequence (3) with $\underline{\omega}^*$, it is easy to see that

$$\mathscr{K}_{\rho\otimes\det} = \mathscr{K}_\rho \otimes \underline{\omega}^*.$$

Since $\underline{\omega}^*$ is an ample line bundle on the projective scheme X over $\overline{\mathbb{F}}_p$, we know from a theorem of Serre (Theorem III.5.2 in [Har77]) that for k large enough we have

$$\mathrm{H}^1(X, \mathscr{K}_{\rho\otimes\det^k}) = \mathrm{H}^1(X, \mathscr{K}_\rho \otimes (\underline{\omega}^*)^{\otimes k}) = 0.$$

Thus for k large enough we know that $r_{\rho\otimes\det^k}$ is surjective.

Now suppose we start with a superspecial eigensystem Φ, say associated to some $f \in SS_\tau$. By Corollary 27 of [Ghi04], we can extend τ to $\mathrm{GL}_g(\overline{\mathbb{F}}_p)$, i.e. there exists a rational representation

$$\rho : \mathrm{GL}_g(\overline{\mathbb{F}}_p) \longrightarrow \mathrm{GL}(V) \text{ such that } \tau \subset \operatorname{Res}\rho.$$

This means that $SS_\tau \subset SS_{\mathrm{Res}\,\rho}$. Now by the periodicity property of SS_τ we have

$$SS_{\mathrm{Res}\,\rho} = SS_{\mathrm{Res}\,\rho\otimes\det^{k(p^2-1)}} \text{ for all } k.$$

So we can pick k large enough such that

$$r_{\rho\otimes\det^{k(p^2-1)}} : S_{\rho\otimes\det^{k(p^2-1)}} \longrightarrow SS_{\mathrm{Res}\,\rho\otimes\det^{k(p^2-1)}} = SS_{\mathrm{Res}\,\rho} \supset SS_\tau$$

is surjective. Therefore by a simple linear algebra argument we conclude that Φ is associated to some $f' \in S_{\rho\otimes\det^{k(p^2-1)}}$.

7. Comparison with cusp forms à la Chai-Faltings

Siegel cusp forms were already defined in an algebraic-geometric way by Chai and Faltings (see pp. 144–147 of [FC90]), at least in the scalar case, i.e. for $\rho = \det^{\otimes k}$. Since their definition is based on the toroidal compactifications of $\mathscr{A}_{g,N}$, it is not immediately obvious that it agrees with ours, and this last section is devoted to showing this.

Chai and Faltings define so-called arithmetic toroidal compactifications $\bar{\mathscr{A}}_{g,N}$ of $\mathscr{A}_{g,N}$. These depend on certain combinatorial data, and have various nice properties summarized in Theorem IV.6.7 of [FC90]. Most importantly, they are moduli spaces and thus one has a Hodge bundle $\bar{\mathbb{E}}$ and its twisted versions $\bar{\mathbb{E}}_\rho$, which we define in the same way as we did $\mathbb{E}$ and $\mathbb{E}_\rho$ in §3. In this setting, Chai and Faltings define Siegel cusp forms with coefficients in a $\mathbb{Z}[\frac{1}{N}]$-module M to be

$$\mathrm{H}^0(\bar{\mathscr{A}}_{g,N}, \mathcal{I}_{\bar\Delta} \otimes \bar{\mathbb{E}}_\rho \otimes M), \quad \text{where } \mathcal{I}_{\bar\Delta} \text{ is the ideal sheaf of } \bar\Delta = \bar{\mathscr{A}}_{g,N} - \mathscr{A}_{g,N}.$$

In other words, cusp forms are global sections of $\bar{\mathbb{E}}_\rho \otimes M$ that vanish along the boundary $\bar\Delta$ of $\bar{\mathscr{A}}_{g,N}$. Moreover, this turns out to be independent of the choice of toroidal compactification.

The key to comparing the two notions of cusp forms is the following fact (part of Theorem V.2.5 of [FC90]): a toroidal compactification is related to the minimal compactification by a canonical morphism $\bar\pi : \bar{\mathscr{A}}_{g,N} \to \mathscr{A}^*_{g,N}$ restricting to the identity on the open dense subscheme $\mathscr{A}_{g,N}$. Two facts about $\bar\pi$ are important for our purposes:

- The boundary $\bar\Delta$ is the scheme-theoretic preimage of $\Delta \subset \mathscr{A}^*_{g,N}$ under $\bar\pi$: this follows from the detailed description of the interaction between $\bar\pi$ and the stratifications of $\bar{\mathscr{A}}_{g,N}$ and $\mathscr{A}^*_{g,N}$ (see Theorem V.2.5(6) of [FC90]).
- The pullback $\bar\pi^*(\mathbb{E}^*_\rho)$ is $\bar{\mathbb{E}}_\rho$: this can be seen easily from the fact that both are reflexive coherent sheaves on $\bar{\mathscr{A}}_{g,N}$ extending $\mathbb{E}_\rho$ on $\mathscr{A}_{g,N}$, together with the uniqueness argument from the end of Theorem 3.

Using these it is easy to obtain the following result, whose proof we leave to the reader:

Proposition 7. *The canonical morphism $\bar\pi : \bar{\mathscr{A}}_{g,N} \to \mathscr{A}^*_{g,N}$ induces via pullback an isomorphism*

$$\bar\pi^* : S_\rho(M) \xrightarrow{\cong} \mathrm{H}^0(\bar{\mathscr{A}}_{g,N}, \mathscr{I}_{\bar\Delta} \otimes \bar{\mathbb{E}}_\rho \otimes M)$$

for any $\mathbb{Z}[\frac{1}{N}]$-module M.

Acknowledgements

This paper originated in a conversation between the author and A. J. de Jong, whom we thank heartily. Many thanks go to A. Archibald for pointing out a serious bug in an earlier version of the paper, to B. Conrad and P. Kassaei for helping with the proof of Theorem 3, and to E. Goren and F. Oort for useful clarifications and suggestions. It is also a pleasure to thank the referee for the detailed comments and corrections. This research was funded by the Centre Interuniversitaire en Calcul Mathématique Algébrique (CICMA) and the Fonds Québecois de la Recherche sur la Nature et les Technologies (FQRNT).

References

[BB66] W. L. Baily, Jr. and A. Borel, *Compactification of arithmetic quotients of bounded symmetric domains*, Ann. of Math. (2) **84** (1966) 442–528.

[FC90] G. Faltings and C.-L. Chai, Degeneration of abelian varieties. Springer-Verlag, Berlin, 1990.

[Fre83] E. Freitag, *Siegelsche Modulfunktionen*, Grundlehren der Mathematischen Wissenschaften Vol 254. Springer-Verlag, Berlin, 1983.

[Ghi04] A. Ghitza, *Hecke eigenvalues of Siegel modular forms (mod p) and of algebraic modular forms*, J. Number Theory **106** (2004), no. 2, 345–384.

[SGA2] A. Grothendieck, *Cohomologie locale des faisceaux cohérents et théorèmes de Lefschetz locaux et globaux (SGA* 2*)*. North-Holland Publishing Co., Amsterdam, 1968. Augmenté d'un exposé par Michèle Raynaud, Séminaire de Géométrie Algébrique du Bois-Marie, 1962, Advanced Studies in Pure Mathematics, Vol. 2.

[Har77] R. Hartshorne, *Algebraic geometry*, Graduate texts in mathematics Vol. 52. Springer-Verlag, New York, 1977.

[Ser66] J-P. Serre. *Prolongement de faisceaux analytiques cohérents*, Ann. Inst. Fourier (Grenoble) **16** (1966) 363–374.

[Ser96] ———, *Two letters on quaternions and modular forms (mod p)*, Israel J. Math. **95** (1996) 281–299.

[MFD] W. Stein. *The Modular Forms Database*, `http://modular.math.washington.edu/Tables`, 2004.

Department of Mathematics and Statistics, McGill University, 805 Sherbrooke St. West, Montréal, Québec, H3A 2K6, CANADA

E-mail address: `aghitza@alum.mit.edu`