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A MODULARITY LIFTING THEOREM FOR WEIGHT TWO HILBERT MODULAR FORMS

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Abstract. We prove a modularity lifting theorem for potentially Barostti-Tate representations over totally real fields, generalising recent results of Kisin.

1. Introduction

In [Kis04] Mark Kisin introduced a number of new techniques for proving modularity lifting theorems, and was able to prove a very general lifting theorem for potentially Barsotti-Tate representations over $\mathbb{Q}$. In [Kis05] this was generalised to the case of p-adic representations of the absolute Galois group of a totally real field in which p splits completely. In this note, we further generalise this result to:

Theorem. *Let $p > 2$, let F be a totally real field in which p is unramified, and let E be a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}$. Let $\rho : G_F \to \mathrm{GL}_2(\mathcal{O})$ be a continuous representation unramified outside of a finite set of primes, with determinant a finite order character times the p-adic cyclotomic character. Suppose that*

(1) *ρ is potentially Barsotti-Tate at each $v|p$.*
(2) *$\overline{\rho}$ is modular.*
(3) *$\overline{\rho}|_{F(\zeta_p)}$ is absolutely irreducible.*

Then ρ is modular.

We emphasise that the techniques we use are entirely those of Kisin. Our only new contributions are some minor technical improvements; specifically, we are able to prove a more general connectedness result than Kisin for certain local deformation rings, and we replace an appeal to a result of Raynaud by a computation with Breuil modules with descent data.

The motivation for studying this problem was the work reported on in [Gee06], where we apply the main theorem of this paper to the conjectures of [BDJ05]. In these applications it is crucial to have a lifting theorem valid for F in which p is unramified, rather than just totally split.

2. Connected components

Firstly, we recall some definitions and theorems from [Kis04]. We make no attempt to put these results in context, and the interested reader should consult section 1 of [Kis04] for a more balanced perspective on this material.

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Let $p > 2$ be prime. Let k be a finite extension of $\mathbb{F}_p$ of cardinality $q = p^r$, and let $W = W(k)$, $K_0 = W(k)[1/p]$. Let K be a totally ramified extension of K_0 of degree e. We let $\mathfrak{S} = W[[u]]$, equipped with a Frobenius map ϕ given by $u \mapsto u^p$, and the natural Frobenius on W. Fix an algebraic closure $\overline{K}$ of K, and fix a uniformiser π of K. Let $E(u)$ denote the minimal polynomial of π over K_0.

Let $'(\mathrm{Mod}/\mathfrak{S})$ denote the category of $\mathfrak{S}$-modules $\mathfrak{M}$ equipped with a ϕ-semilinear map $\phi : \mathfrak{M} \to \mathfrak{M}$ such that the cokernel of $\phi^*(\mathfrak{M}) \to \mathfrak{M}$ is killed by $E(u)$. For any $\mathbb{Z}_p$-algebra A, set $\mathfrak{S}_A = \mathfrak{S} \otimes_{\mathbb{Z}_p} A$. Denote by $'(\mathrm{Mod}/\mathfrak{S})_A$ the category of pairs $(\mathfrak{M}, \iota)$ where $\mathfrak{M}$ is in $'(\mathrm{Mod}/\mathfrak{S})$ and $\iota : A \to \mathrm{End}(\mathfrak{M})$ is a map of $\mathbb{Z}_p$-algebras.

We let $(\mathrm{Mod\,FI}/\mathfrak{S})_A$ denote the full subcategory of $'(\mathrm{Mod}/\mathfrak{S})_A$ consisting of objects $\mathfrak{M}$ such that $\mathfrak{M}$ is a projective $\mathfrak{S}_A$-module of finite rank.

Choose elements $\pi_n \in \overline{K}$ $(n \geq 0)$ so that $\pi_0 = \pi$ and $\pi_{n+1}^p = \pi_n$. Let $K_\infty = \bigcup_{n \geq 1} K(\pi_n)$. Let $\mathcal{O}_{\mathcal{E}}$ be the p-adic completion of $\mathfrak{S}[1/u]$. Let $\mathrm{Rep}_{\mathbb{Z}_p}(G_{K_\infty})$ denote the category of continuous representations of G_{K_∞} on finite $\mathbb{Z}_p$-algebras. Let $\Phi\,\mathrm{M}_{\mathcal{O}_{\mathcal{E}}}$ denote the category of finite $\mathcal{O}$-modules M equipped with a ϕ-semilinear map $M \to M$ such that the induced map $\phi^* M \to M$ is an isomorphism. Then there is a functor

$$T : \Phi\,\mathrm{M}_{\mathcal{O}_{\mathcal{E}}} \to \mathrm{Rep}_{Z_p}(G_{K_\infty})$$

which is in fact an equivalence of abelian categories (see section 1.1.12 of [Kis04]). Let A be a finite $\mathbb{Z}_p$-algebra, and let $\mathrm{Rep}'_A(G_{K_\infty})$ denote the category of continuous representations of G_{K_∞} on finite A-modules, and let $\mathrm{Rep}_A(G_{K_\infty})$ denote the full subcategory of objects which are free as A-modules. Let $\Phi\,\mathrm{M}_{\mathcal{O}_{\mathcal{E}},A}$ denote the category whose objects are objects of $\Phi\,\mathrm{M}_{\mathcal{O}_{\mathcal{E}}}$ equipped with an action of A.

Lemma 2.1. *The functor T above induces an equivalence of abelian categories*

$$T_A : \Phi\,\mathrm{M}_{\mathcal{O}_{\mathcal{E}},A} \to \mathrm{Rep}'_A(G_{K_\infty}).$$

The functor T_A induces a functor

$$T_{\mathfrak{S},A} : (\mathrm{Mod\,FI}/\mathfrak{S})_A \to \mathrm{Rep}_A(G_{K_\infty});\ \mathfrak{M} \mapsto T_A(\mathcal{O}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{M}).$$

Proof. Lemmas 1.2.7 and 1.2.9 of [Kis04]. □

Fix $\mathbb{F}$ a finite extension of $\mathbb{F}_p$, and a continuous representation of G_K on a 2-dimensional $\mathbb{F}$-vector space $V_{\mathbb{F}}$. We suppose that $V_{\mathbb{F}}$ is the generic fibre of a finite flat group scheme, and let $M_{\mathbb{F}}$ denote the preimage of $V_{\mathbb{F}}(-1)$ under the equivalence $T_{\mathbb{F}}$ of Lemma 2.1.

In fact, from now on we assume that the action of G_K on $V_{\mathbb{F}}$ is trivial, that $k \subset \mathbb{F}$, and that $k \neq \mathbb{F}_p$. In applications we will reduce to this case by base change.

Recall from Corollary 2.1.13 of [Kis04] that we have a projective scheme $\mathcal{GR}_{V_{\mathbb{F}},0}$, such that for any finite extension $\mathbb{F}'$ of $\mathbb{F}$, the set of isomorphism classes of finite flat models of $V_{\mathbb{F}'} = V_{\mathbb{F}} \otimes_{\mathbb{F}} \mathbb{F}'$ is in natural bijection with $\mathcal{GR}_{V_{\mathbb{F}},0}(\mathbb{F}')$. We work below with the closed subscheme $\mathcal{GR}^{\mathrm{v}}_{V_{\mathbb{F}},0}$ of $\mathcal{GR}_{V_{\mathbb{F}},0}$, defined in Lemma 2.4.3 of [Kis04], which parameterises isomorphism classes of finite flat models of $V_{\mathbb{F}'}$ with cyclotomic determinant.

As in section 2.4.4 of [Kis04], if $\mathbb{F}^{\mathrm{sep}}$ is the residue field of K_0^{sep}, and $\sigma \in \mathrm{Gal}(K_0/\mathbb{Q}_p)$, we denote by $\epsilon_\sigma \in k \otimes_{\mathbb{F}_p} \mathbb{F}'$ the idempotent which is 1 on the kernel of the map $1 \otimes \sigma : k \otimes_{\mathbb{F}_p} \mathbb{F}' \to \mathbb{F}^{\mathrm{sep}}$ corresponding to σ, and 0 on the other maximal ideals of $k \otimes_{\mathbb{F}_p} \mathbb{F}'$.

Lemma 2.2. *If $\mathbb{F}'$ is a finite extension of $\mathbb{F}$, then the elements of $\mathcal{GR}^{\mathbf{v}}_{V_{\mathbb{F}},0}(\mathbb{F}')$ naturally correspond to free $k \otimes_{\mathbb{F}_p} \mathbb{F}'[[u]]$-submodules $\mathcal{M}_{\mathbb{F}'} \subset M_{\mathbb{F}'} := M_{\mathbb{F}} \otimes_{\mathbb{F}} \mathbb{F}'$ of rank 2 such that:*

(1) *$\mathcal{M}_{\mathbb{F}'}$ is ϕ-stable.*
(2) *For some (so any) choice of $k \otimes_{\mathbb{F}_p} \mathbb{F}'[[u]]$-basis for $\mathcal{M}_{\mathbb{F}'}$, for each σ the map*

$$\phi : \epsilon_\sigma \mathcal{M}_{\mathbb{F}'} \to \epsilon_{\sigma\circ\phi^{-1}} \mathcal{M}_{\mathbb{F}'}$$

has determinant αu^e, $\alpha \in \mathbb{F}'[[u]]^\times$.

Proof. This follows just as in the proofs of Lemma 2.5.1 and Proposition 2.2.5 of [Kis04]. More precisely, the method of the proof of Proposition 2.2.5 of [Kis04] allows one to "decompose" the determinant condition into the condition that for each σ we have

$$\dim_{\mathbb{F}'}(\epsilon_{\sigma\circ\phi^{-1}} \mathcal{M}_{\mathbb{F}'}/\phi(\epsilon_\sigma \mathcal{M}_{\mathbb{F}'})) = e,$$

and then an identical argument to that in the proof of Lemma 2.5.1 [Kis04] shows that this condition is equivalent to the stated one. □

We now number the elements of $\mathrm{Gal}(K_0/\mathbb{Q}_p)$ as $\sigma_1, \ldots, \sigma_r$, in such a way that $\sigma_{i+1} = \sigma_i \circ \phi^{-1}$ (where we identify σ_{r+1} with σ_1). For any sublattice $\mathfrak{M}_{\mathbb{F}}$ in $(Mod/\mathfrak{S})_{\mathbb{F}}$ and any $(A_1, \ldots, A_r) \in \mathcal{M}_2(\mathbb{F}((u)))^r$, we write $\mathfrak{M}_{\mathbb{F}} \sim A$ if there exist bases $\{\mathbf{e}_1^i, \mathbf{e}_2^i\}$ for $\epsilon_{\sigma_i} \mathcal{M}_{\mathbb{F}}$ such that

$$\phi \begin{pmatrix} \mathbf{e}_1^i \\ \mathbf{e}_2^i \end{pmatrix} = A_i \begin{pmatrix} \mathbf{e}_1^{i+1} \\ \mathbf{e}_2^{i+1} \end{pmatrix}.$$

If we have fixed such a choice of basis, then for any $(B_1, \ldots, B_r) \in \mathrm{GL}_2(k_r((u)))^r$ we denote by $B\mathfrak{M}$ the module generated by $\left\langle B_i \begin{pmatrix} \mathbf{e}_1^i \\ \mathbf{e}_2^i \end{pmatrix} \right\rangle$, and consider $B\mathfrak{M}$ with respect to the basis given by these entries.

Proposition 2.3. *Let $\mathbb{F}'/\mathbb{F}$ be a finite extension. Suppose that $x_1, x_2 \in \mathcal{GR}^{\mathbf{v}}_{V_{\mathbb{F}},0}(\mathbb{F}')$ and that the corresponding objects of $(\mathrm{Mod}/\S)_{\mathbb{F}'}$, $\mathfrak{M}_{\mathbb{F}',1}$ and $\mathfrak{M}_{\mathbb{F}',2}$ are both non-ordinary. Then (the images of) x_1 and x_2 both lie on the same connected component of $\mathcal{GR}^{\mathbf{v}}_{V_{\mathbb{F}},0}(\mathbb{F}')$.*

Proof. Replacing $V_{\mathbb{F}}$ by $\mathbb{F}' \otimes_{\mathbb{F}} v_{\mathbb{F}}$, we may assume that $\mathbb{F}' = \mathbb{F}$. Suppose that $\mathfrak{M}_{\mathbb{F},1} \sim A$. Then $\mathfrak{M}_{\mathbb{F},2} = B \cdot \mathfrak{M}_{\mathbb{F},1}$ for some $B \in \mathrm{GL}_2(k_r((u)))^r$, and $\mathfrak{M}_{\mathbb{F},2} \sim (\phi(B_i) \cdot A_i \cdot B_{i+1}^{-1})$. Each B_i is uniquely determined up to left multiplication by elements of $\mathrm{GL}_2(\mathbb{F}[[u]])$, so by the Iwasawa decomposition we may assume that each B_i is upper triangular. By Lemma 2.2, $\det \phi(B_i) \det B_{i+1}^{-1} \in \mathbb{F}[[u]]^\times$ for all i, which implies that $\det(B_i) \in \mathbb{F}[[u]]^\times$ for all i, so that the diagonal elements of B_i are $\mu_1^i u^{-a_i}$, $\mu_2^i u^{a_i}$ for μ_1^i, $\mu_2^i \in \mathbb{F}[[u]]^\times$, $a_i \in \mathbb{Z}$. Replacing B_i with $\mathrm{diag}(\mu_1^i, \mu_2^i)^{-1} B_i$, we may assume that B_i has diagonal entries u^{-a_i} and u^{a_i}.

We now show that x_1 and x_2 are connected by a chain of rational curves, using the following lemma:

Lemma 2.4. *Suppose that (N_i) are nilpotent elements of $M_2(\mathbb{F}((u)))$ such that $\mathfrak{M}_{\mathbb{F},2} = (1 + N) \cdot \mathfrak{M}_{\mathbb{F},1}$. If $\phi(N_i) A N_{i+1}^{\mathrm{ad}} \in M_2(F[[u]])$ for all i, then there is a map $\mathbb{P}^1 \to \mathcal{GR}^{\mathbf{v}}_{V_{\mathbb{F}},0}$ sending 0 to x_1 and 1 to x_2.*

Proof. Exactly as in the proof of Lemma 2.5.7 of [Kis04]. □

In fact, we will only apply this lemma in situations where all but one of the N_i are zero, so that the condition of the lemma is automatically satisfied.

Lemma 2.5. *With respect to some basis, $\phi : M_{\mathbb{F}} \to M_{\mathbb{F}}$ is given by $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.*

Proof. This is immediate from the definition of $M_{\mathbb{F}}$ (recall that we have assumed that the action of G_K on $V_{\mathbb{F}}$ is trivial). □

Let v_1, v_2 be a basis as in the lemma, and let $\mathfrak{M}_{\mathbb{F}}$ be the sub-$k \otimes_{\mathbb{F}_p} \mathbb{F}[[u]]$-module generated by $u^{e/(p-1)}v_1$ and v_2 (note that the assumption that the action of G_K on $V_{\mathbb{F}}$ is trivial guarantees that $e|(p-1)$). Then $\mathfrak{M}_{\mathbb{F}}$ corresponds to an object of $\mathcal{GR}^{\mathbf{v}}_{V_{\mathbb{F}},0}(\mathbb{F}')$, and $\mathfrak{M}_{\mathbb{F}} \sim (A_i)$ where each $A_i = \left(\begin{smallmatrix} u^e & 0 \\ 0 & 1 \end{smallmatrix}\right)$, so that $\mathfrak{M}_{\mathbb{F}}$ is ordinary.

Furthermore, every object of $\mathcal{GR}^{\mathbf{v}}_{V_{\mathbb{F}},0}(\mathbb{F}')$ is given by $B \cdot \mathfrak{M}_{\mathbb{F}}$ for some $B = (B_i)$, where $B_i = \left(\begin{smallmatrix} u^{-a_i} & v_i \\ 0 & u^{a_i} \end{smallmatrix}\right)$, and $\phi(B_i)A_iB_{i+1}^{-1} \in M_2(\mathbb{F}[[u]])$ for all i. Examining the diagonal entries of $\phi(B_i)A_iB_{i+1}^{-1}$, we see that we must have $e \geq pa_i - a_{i+1} \geq 0$ for all i.

Lemma 2.6. *We have $e/(p-1) \geq a_i \geq 0$ for all i. Furthermore, if any $a_i = 0$ then all $a_i = 0$; and if any $a_i = e/(p-1)$, then all $a_i = e/(p-1)$.*

Proof. Suppose that $a_j \leq 0$. Then $pa_j \geq a_{j+1}$, so $a_{j+1} \leq 0$. Thus $a_i \leq 0$ for all i. But adding the inequalities gives $(p-1)(a_1 + \cdots + a_r) \geq 0$, so in fact $a_1 = \cdots = a_r = 0$. The other half of the lemma follows in a similar fashion. □

Note that the ordinary objects are precisely those with all $a_i = 0$ or all $a_i = e/(p-1)$. We now show that there is a chain of rational curves linking any non-ordinary point to the point corresponding to $C \cdot \mathfrak{M}_{\mathbb{F}}$, where $C_i = \left(\begin{smallmatrix} u^{-1} & 0 \\ 0 & u \end{smallmatrix}\right)$.

Choose a non-ordinary point $D \cdot \mathfrak{M}_{\mathbb{F}}$, $D_i = \left(\begin{smallmatrix} u^{-b_i} & w_i \\ 0 & u^{b_i} \end{smallmatrix}\right)$. We claim that there is a chain of rational curves linking this to the point $D' \cdot \mathfrak{M}_{\mathbb{F}}$, $D'_i = \left(\begin{smallmatrix} u^{-b_i} & 0 \\ 0 & u^{b_i} \end{smallmatrix}\right)$. Clearly, it suffices to demonstrate that there is a rational curve from $D \cdot \mathfrak{M}_{\mathbb{F}}$ to the point $D^j \cdot \mathfrak{M}_{\mathbb{F}}$, where

$$D^j_i = \begin{cases} D_i, & i \neq j \\ \begin{pmatrix} u^{-b_j} & 0 \\ 0 & u^{b_j} \end{pmatrix}, & i = j. \end{cases}$$

But this is easy; just apply Lemma 2.4 with $N = (N_i)$,

$$N_i = \begin{cases} 0, & i \neq j \\ \begin{pmatrix} 0 & -w_j u^{-b_j} \\ 0 & 0 \end{pmatrix}, & i = j. \end{cases}$$

It now suffices to show that there is a chain of rational curves linking $D' \cdot \mathfrak{M}_{\mathbb{F}}$ to $C \cdot \mathfrak{M}_{\mathbb{F}}$. Suppose that $D'' \cdot \mathfrak{M}_{\mathbb{F}}$ also corresponds to a point of $\mathcal{GR}^{\mathbf{v}}_{V_{\mathbb{F}},0}(\mathbb{F})$, where for some j we have

$$D''_i = \begin{cases} D'_i, & i \neq j \\ \begin{pmatrix} u^{1-b_j} & 0 \\ 0 & u^{b_j-1} \end{pmatrix}, & i = j. \end{cases}$$

Then we claim that there is a rational curve linking $D' \cdot \mathfrak{M}_{\mathbb{F}}$ and $D'' \cdot \mathfrak{M}_{\mathbb{F}}$. Note that $D'' = ED'$, where

$$E_i = \begin{cases} 1, & i \neq j \\ \left(\begin{smallmatrix} u^{-1} & 0 \\ 0 & u \end{smallmatrix}\right), & i = j. \end{cases}$$

Since $\begin{pmatrix} u^{-1} & 0 \\ 0 & u \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 2u \end{pmatrix} \begin{pmatrix} 2 & -u \\ u^{-1} & 0 \end{pmatrix}$, and $\begin{pmatrix} 0 & 1 \\ -1 & 2u \end{pmatrix} \in \mathrm{GL}_2(\mathbb{F}[[u]])$, we can apply Lemma 2.4 with

$$N_i = \begin{cases} 0, & i \neq j \\ \begin{pmatrix} 1 & -u \\ u^{-1} & -1 \end{pmatrix}, & i = j. \end{cases}$$

Proposition 2.3 now follows from:

Lemma 2.7. *If $e/(p-1) > a_i > 0$ and $e \geq pa_i - a_{i+1} \geq 0$ for all i, and not all the a_i are equal to 1, then for some j we can define*

$$a_i' = \begin{cases} a_i, & i \neq j \\ a_j - 1, & i = j \end{cases}$$

and we have $e \geq pa_i' - a_{i+1}' \geq 0$ for all i.

Proof. Suppose not. Then for each i, either $pa_{i-1} - (a_i - 1) > e$, or $p(a_i - 1) - a_{i+1} < 0$; that is, either $pa_{i-1} - a_i = e$, or $p - 1 \geq pa_i - a_{i+1} \geq 0$. Comparing the statements for i, $i+1$, we see that either $pa_i - a_{i+1} = e$ for all i, or $p - 1 \geq pa_i - a_{i+1} \geq 0$ for all i. In the former case we have $a_i = e/(p-1)$ for all i, a contradiction. In the latter case, summing the inequalities gives $r(p-1) \geq (p-1)(a_1 + \cdots + a_r) \geq (r+1)(p-1)$, a contradiction. □

□

3. Modularity lifting theorems

The results of section 2 can easily be combined with the machinery of [Kis04] to yield modularity lifting theorems. For example, we have the following:

Theorem 3.1. *Let $p > 2$, let F be a totally real field, and let E be a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}$. Let $\rho : G_F \to \mathrm{GL}_2(\mathcal{O})$ be a continuous representation unramified outside of a finite set of primes, with determinant a finite order character times the p-adic cyclotomic character. Suppose that*

1. *ρ is potentially Barsotti-Tate at each $v|p$.*
2. *There exists a Hilbert modular form f of parallel weight 2 over F such that $\overline{\rho}_f \sim \overline{\rho}$, and for each $v|p$, ρ is potentially ordinary at v if and only if ρ_f is.*
3. *$\overline{\rho}|_{F(\zeta_p)}$ is absolutely irreducible, and if $p > 3$ then $[F(\zeta_p) : F] > 3$.*

Then ρ is modular.

Proof. The proof of this theorem is almost identical to the proof of Theorem 3.5.5 of [Kis04]. Indeed, the only changes needed are to replace property *(iii)* of the field F' chosen there by "*(iii)* If $v|p$ then $\overline{\rho}|G_{F_v}$ is trivial, and the residue field at v is not $\mathbb{F}_p$", and to note that Theorem 3.4.11 of [Kis04] is still valid in the context in which we need it, by Proposition 2.3. □

For the applications to mod p Hilbert modular forms in [Gee06] it is important not to have to assume that ρ is potentially ordinary at v if and only if ρ_f is. Fortunately, in [Gee06] it is only necessary to work with totally real fields in which p is unramified, and in that case we are able, following [Kis05], to remove this assumption.

Theorem 3.2. *Let $p > 2$, let F be a totally real field in which p is unramified, and let E be a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}$. Let $\rho : G_F \to \mathrm{GL}_2(\mathcal{O})$ be a continuous representation unramified outside of a finite set of primes, with determinant a finite order character times the p-adic cyclotomic character. Suppose that*

(1) *ρ is potentially Barsotti-Tate at each $v|p$.*
(2) *$\overline{\rho}$ is modular.*
(3) *$\overline{\rho}|_{F(\zeta_p)}$ is absolutely irreducible.*

Then ρ is modular.

Proof. Firstly, note that by a standard result (see e.g. [BDJ05]) we have $\overline{\rho} \sim \overline{\rho}_f$, where f is a form of parallel weight 2. We now follow the proof of Corollary 2.13 of [Kis05]. Let $\mathcal{S}'$ denote the set of $v|p$ such that $\rho|_{G_v}$ is potentially ordinary. After applying Lemma 3.3 below, we may assume that $\overline{\rho} \sim \overline{\rho}_f$, where f is a form of parallel weight 2, and ρ_f is potentially ordinary and potentially Barsotti-Tate for all $v \in \mathcal{S}'$.

We may now make a solvable base change so that the hypotheses on F in Theorem 3.1 are still satisfied, and in addition $[F : \mathbb{Q}]$ is even, and at every place $v|p$ f is either unramified or special of conductor 1. By our choice of f, $\rho_f|_{G_v}$ is Barsotti-Tate and ordinary at each place $v \in \mathcal{S}'$. In order to apply Theorem 3.1, we need to check that we can replace f by a form f' such that $\overline{\rho} \sim \overline{\rho}_{f'}$, and $\rho_{f'}$ is Barsotti-Tate at all $v|p$ and is ordinary if and only if ρ is. That is, we wish to choose f' so that $\rho_{f'}$ is Barsotti-Tate and ordinary at all places $v \in \mathcal{S}'$, and is Barsotti-Tate and non-ordinary at all other places dividing p. The existence of such an f' follows at once from the proof of Theorem 3.5.7 of [Kis04]. The theorem then follows from Theorem 3.1. □

Lemma 3.3. *Let F be a totally real field in which p is unramified, and $\mathcal{S}'$ a set of places of F dividing p. Let f be a Hilbert modular cusp form over F of parallel weight 2, with $\overline{\rho}_f$ absolutely irreducible, and suppose that for $v \in \mathcal{S}'$ $\overline{\rho}_f|_{G_{F_v}}$ is the reduction of a potentially Barsotti-Tate representation of G_{F_v} which is also potentially ordinary.*

Then there is a Hilbert modular cusp form f' over F of parallel weight 2 with $\overline{\rho}_{f'} \sim \overline{\rho}_f$, and such that for all $v \in \mathcal{S}'$, $\rho_{f'}$ becomes ordinary and Barsotti-Tate over some finite extension of F_v.

Proof. We follow the proof of Lemma 2.14 of [Kis05], indicating only the modifications that need to be made. Replacing the appeal to [CDT99] with one to Proposition 1.1 of [Dia05], the proof of Lemma 2.14 of [Kis05] shows that we can find f' such that $\overline{\rho}_{f'} \sim \overline{\rho}_f$, and such that for all $v \in \mathcal{S}'$, $\rho_{f'}$ becomes Barsotti-Tate over $F_v(\zeta_{q_v})$, where q_v is the degree of the residue field of F at v. Furthermore, we can assume that the type of $\rho_{f'}|_{G_{F_v}}$ is $\widetilde{\omega}_1 \oplus \widetilde{\omega}_2$, where $\overline{\rho}_{f'}|_{G_{F_v}} \sim \left(\begin{smallmatrix} \omega_1\chi & * \\ 0 & \omega_2 \end{smallmatrix}\right)$, where χ is the cyclotomic character, and a tilde denotes the Teichmuller lift. Let $\mathcal{G}$ denote the p-divisible group over $\mathcal{O}_{F_v(\zeta_{q_v})}$ corresponding to $\rho_{f'}|_{F_v(\zeta_{q_v})}$ Then by a scheme-theoretic closure argument, $\mathcal{G}[p]$ fits into a short exact sequence

$$0 \to \mathcal{G}_1 \to \mathcal{G}[p] \to \mathcal{G}_2 \to 0.$$

The information about the type then determines the descent data on the Breuil modules corresponding to $\mathcal{G}_1$ and $\mathcal{G}_2$. We will be done if we can show that $\mathcal{G}_1$ is multiplicative and $\mathcal{G}_2$ is étale. However, by the hypothesis on $\mathcal{S}'$ we can write down a multiplicative group scheme $\mathcal{G}_1'$ with the same descent data and generic fibre as $\mathcal{G}_1$.

Then Lemma 3.4 below shows that $\mathcal{G}_1$ is indeed multiplicative. The same argument shows that $\mathcal{G}_2$ is étale. □

Lemma 3.4. *Let k be a finite field of characteristic p, and let $L = W(k)[1/p]$. Fix $\pi = (-p)^{1/(p^d-1)}$ where $d = [k : \mathbb{F}_p]$, and let $K = L(\pi)$. Let E be a finite field containing k. Let $\mathcal{G}$ and $\mathcal{G}'$ be finite flat rank one E-module schemes over $\mathcal{O}_K$ with generic fibre descent data to L. Suppose that the generic fibres of $\mathcal{G}$ and $\mathcal{G}'$ are isomorphic as G_L-representations, and that $\mathcal{G}$ and $\mathcal{G}'$ have the same descent data. Then $\mathcal{G}$ and $\mathcal{G}'$ are isomorphic.*

Proof. This follows from a direct computation using Breuil modules with descent data. Specifically, it follows at once from Example A.3.3 of [Sav06], which computes the generic fibre of any finite flat rank one E-module scheme over $\mathcal{O}_K$ with generic fibre descent data to L. □

References

[BDJ05] K. Buzzard, F. Diamond, and F. Jarvis, *On Serre's conjecture for mod l Galois representations over totally real fields*, in preparation, 2005.

[CDT99] B. Conrad, F. Diamond, and R. Taylor, *Modularity of certain potentially Barsotti-Tate Galois representations*, J. Amer. Math. Soc. **12** (1999), no. 2, 521–567.

[Dia05] F. Diamond, *A correspondence between representations of local Galois groups and Lie-type groups*, to appear in L-functions and Galois representations (Durham 2004), 2005.

[Gee06] Toby Gee, *On the weights of mod p Hilbert modular forms.* Preprint 2006.

[Kis04] Mark Kisin, *Moduli of finite flat group schemes, and modularity.* Preprint 2004.

[Kis05] ———, *Modularity for some geometric Galois representations*, to appear in L-functions and Galois representations (Durham 2004), 2005.

[Sav06] David Savitt, *Breuil modules for Raynaud schemes*, appendix to [Gee06], 2006.

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