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L^p-IMPROVING PROPERTIES OF X-RAY LIKE TRANSFORMS

Philip T. Gressman

1. Introduction

The purpose of this paper is to prove essentially sharp L^p-L^q estimates for nondegenerate one-dimensional averaging operators which generalize the classical X-ray transform. Let X and Y be C^∞ manifolds with $\dim X =: d_X$ and $\dim Y =: d_Y$; we assume that X and Y are equipped with measures of smooth density and that $d_Y > d_X$. Now let M be a smooth $(d_Y + 1)$-dimensional submanifold of $X \times Y$ (again equipped with a measure) such that the natural projections $\pi_X : M \to X$ and $\pi_Y : M \to Y$ have everywhere surjective differential maps. For $y \in Y$, the set $\gamma_y := \{x \in X \mid (x, y) \in M\}$ is a curve in X. As will be shown in the next section, there is an induced Radon-like operator R which averages functions of X over the curves γ_y. The focus of this paper is to study the L^p-boundedness of that operator. For simplicity, the question is posed as a bilinear one: for which p, q' does there exist a finite constant $C_{p,q'}$ such

$$\left| \int f_X(\pi_X(m)) f_Y(\pi_Y(m)) dm \right| \leq C_{p,q'} \left(\int |f_X(x)|^p dx \right)^{\frac{1}{p}} \left(\int |f_Y(y)|^{q'} dy \right)^{\frac{1}{q'}}$$

where f_X and f_Y are functions (without loss of generality, positive functions) of X and Y, respectively?

The canonical example of a problem of this type is the usual X-ray transform in $\mathbb{R}^n$: let $X := \mathbb{R}^n$, and let $Y := M_{1,n}$ be the space of all affine lines in $\mathbb{R}^n$. The space Y is equipped with a natural measure $d\lambda$, and each line $\ell \in M_{1,n}$ is, of course, also equipped with a measure $d\ell$. The X-ray transform is given simply by $Tf(\ell) := \int_\ell f$. It has long been known (see, for example, Drury [6] up to an ϵ-loss or Christ [4] for the endpoint case) that

$$\left| \iint f(p) g(\ell) d\ell(p) d\lambda(\ell) \right| \leq C ||f||_{L^{\frac{n+1}{2}}(\mathbb{R}^n)} ||g||_{L^{\frac{n+1}{n}}(M_{1,n})},$$

which corresponds to $L^{\frac{n+1}{2}} \to L^{n+1}$ boundedness of the X-ray transform T. One of the aims of this paper is to show that this particular estimate holds (at least in a restricted weak-type sense) and is optimal for any nondegenerate overdetermined Radon-like transform. This is not, however, the entire story: it will also be shown that new estimates hold true for nondegenerate operators which are "more overdetermined" than the X-ray transform.

More recently, overdetermined 1D Radon-like operators have been studied in the plane by Ricci and Travaglini [7] and in general dimension by Brandolini, Greenleaf, and Travaglini [1]. Both of these works concern themselves with the most basic type of

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nondegeneracy and rely heavily on oscillatory integrals of some form or another (using the L^2-decay of curve-carried measures and Fourier integral operators, respectively). This paper, in contrast, uses the methods of geometric combinatorics pioneered by Christ [5] to obtain optimal estimates (up to restricted weak-type) for higher versions of nondegeneracy.

Following the spirit of the earlier works of Seeger [9], Christ, Nagel, Stein, and Wainger [3], and Tao and Wright [11], a largely coordinate-independent approach will be taken throughout this paper. To that end, a series of definitions are in order. Let $\mathfrak{X}_1$ and $\mathfrak{Y}_1$ be those vector fields on M which are annihilated by $d\pi_X$ and $d\pi_Y$, respectively. Next, choose a nonvanishing representative $Y_1 \in \mathfrak{Y}_1$ (which exists by a simple dimension-counting argument) and define the map $T(V) := [V, Y_1]$ (here $[\cdot,\cdot]$ is the Lie bracket). Now let $\mathfrak{X}_j$ be the collection of all vector fields $V \in \mathfrak{X}_{j-1}$ for which $T(V) \in \mathfrak{X}_{j-1} + \mathfrak{Y}_1$. The Jacobi identity guarantees, by induction, that $\mathfrak{X}_j$ is closed under the Lie bracket. Finally, when $V \in \mathfrak{X}_1$, let $\operatorname{ord}(V) := \sup\{j > 0 \mid V \in \mathfrak{X}_j\}$. The vector fields in $\mathfrak{X}_j$ can be restricted to their values at a point m, giving a vector space $\mathfrak{X}_j|_m$ which is contained in the tangent space of M at m. With these definitions in hand, the nondegeneracy condition at the heart of this paper can be phrased as follows:

Definition 1. *The ensemble (M, X, Y, π_X, π_Y) is said to be nondegenerate through order k at the point $m \in M$ if there exist $d_X - 1$ vector fields $X_l \in \mathfrak{X}_k$ such that $\mathfrak{X}_1|_m$, $\mathfrak{Y}_1|_m$, and the commutators $\{T^k(X_l)|_m\}_{l=1,\ldots,d_X-1}$ span the tangent space of M at m.*

In the simplest case, one has nondegeneracy through order 1 if and only if the Fourier integral operator realization of the Radon-like operator has nondegenerate canonical relation (in the spirit of Brandolini, Greenleaf, and Travaglini [1]). Loosely speaking, (M, X, Y, π_X, π_Y) is nondegenerate through order k when the family curves γ_y passing through x (modulo parametrization in time) may be smoothly parametrized by the k-th order (and lower) derivatives of those curves at the point x. A prototype of this situation is the following: let $X := \mathbb{R} \times \mathbb{R}^n$, let $Y := (\mathbb{R}^n)^{k+1}$. If functions on X are written $f(t,z)$ for $t \in \mathbb{R}$ and $z \in \mathbb{R}^n$, and functions on Y are written $g(y_0,\ldots,y_k)$ for $y_0,\ldots,y_k \in \mathbb{R}^n$, let R_k be the operator given by

$$R_k f(y_0,\ldots,y_k) := \int f(t, y_0 + t y_1 + \cdots t^k y_k) dt. \tag{1}$$

The associated bilinear form is given by

$$\int R_k f(y_0,\ldots,y_k) g(y_0,\ldots,y_k) dy_0 \cdots dy_k.$$

It is easy to check that (M, X, Y, π_X, π_Y) is nondegenerate through order k at the origin in $M := \mathbb{R} \times (\mathbb{R}^n)^{k+1}$ (note that $\pi_X(t, y_0,\ldots,y_k) := (t, y_0 + t y_1 + \cdots t^k y_k)$ and $\pi_Y(t, y_0,\ldots,y_k) := (y_0,\ldots,y_k)$).

Given the nondegeneracy condition, the formulation of the main theorem goes as follows. Let $\mathcal{C}_k \subset [0,1]^2$ be the convex hull of the points $(0,1)$, $(1,0)$, and $(0,0)$ along with the special points, hereafter called $\left(\frac{1}{p_j}, \frac{1}{q_j'}\right)$, given by

$$\left\{ \left(\frac{2}{j d_X - j + 2}, 1 - \frac{2}{(j+1)(j d_X - j + 2)} \right) \right\}_{j=1,\ldots,k}.$$

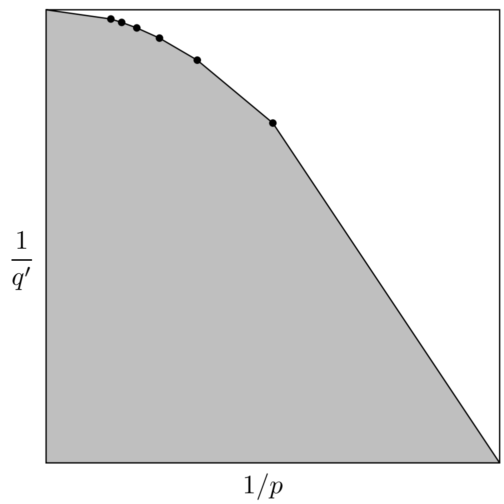

FIGURE 1. The set $\mathcal{C}_6 \subset [0,1]^2$, $d_X = 3$, with the points $\left(\frac{1}{p_j}, \frac{1}{q'_j}\right)$ marked by dots.

The shaded region in figure 1 shows a typical set $\mathcal{C}_k$ (in this case $k = 6$ and $d_X = 3$). As $k \to \infty$, the k-th special point approaches $(0, 1)$ along a curve which is roughly a parabola. Let $\mathcal{C}_k^\circ$ equal $\mathcal{C}_k$ minus the special points. One then has:

Theorem 1. *Let (M, X, Y, π_X, π_Y) be nondegenerate through order k at m. Then there exists an open set $U \subset M$ containing m and a constant $C_{p,q'} < \infty$ such that, for any positive functions f_X and f_Y on X and Y, respectively,*

$$\int_U f_X(\pi_X(m)) f_Y(\pi_Y(m)) dm \leq C ||f_X||_p ||f_Y||_{q'} \tag{2}$$

whenever $\left(\frac{1}{p}, \frac{1}{q'}\right) \in \mathcal{C}_k^\circ$. Conversely, if $p \leq p_k$ and $\left(\frac{1}{p}, \frac{1}{q'}\right) \notin \mathcal{C}_k$, then no such constant $C_{p,q'}$ exists.

The general method of proof used here was first used by Christ [5]. The idea is to consider compositions of flows on M which preserve π_X and π_Y. Unlike Christ [5] or Tao and Wright [11], the flow corresponding to π_X is multidimensional, so some extra care is needed. In particular, it is not sufficient to consider the action of the flows on M itself; they must instead be examined on the product space M^k for $k > 1$. Studying flows on M^k allows one to guarantee that the composed flows will be space filling (which, by dimensional considerations, will not happen in general for the composed flows on M) and at the same time avoids the complication of having too many free parameters in the composed flows (which happens, for example, in the work of Christ [5]). The technique of studying the composed flows on M^k is reminiscent of more general lifting arguments which appear in the works of Rothschild an! d Stein [8] and Christ, Nagel, Stein, and Wainger [3].

At this point, a brief word about the organization of this paper (and the proof of theorem 1) is in order. The next section is devoted to preliminary material including, for example, an examination of the nondegeneracy condition and the lemmas corresponding to lemma 1 of Christ [5]. Section 3 is devoted to the first major inequality, roughly analogous to a change of variables formula. In section 4, the second major inequality (based on an estimate of the size of an important Jacobian determinant) is proved and the sufficiency portion of theorem 1 is established. Finally, the necessity portion of theorem 1 is taken up in section 5 (and established via a Knapp-type example).

2. Preliminaries

It follows from the Hölder inequality, the Fubini-Tonelli theorem, and Jensen's inequality that (2) holds for for some $C_{p,q'} < \infty$ provided $\frac{1}{p} + \frac{1}{q'} \leq 1$ and U is some compact set. The sufficiency portion of theorem 1, then, will be obtained by Marcinkiewicz interpolation (see, for example, Stein and Weiss [10]) of restricted weak-type estimates which will be made for the "special points" (p_j, q'_j).

Following Tao and Wright [11], the restricted weak-type analogue of theorem 1 will be proved in its isoperimetric formulation:

Theorem 2. *Let (M, X, Y, π_X, π_Y) be nondegenerate through order k at m. Then there exists an open set $U \subset M$ containing m and a constant $C_{p,q'} < \infty$ such that, for any open set $O \subset U$*

$$|O| \geq C_{p,q'} \left(\frac{|O|}{|\pi_X(O)|} \right)^{\frac{q'}{p+q'-pq'}} \left(\frac{|O|}{|\pi_Y(O)|} \right)^{\frac{p}{p+q'-pq'}} \tag{3}$$

whenever $\left(\frac{1}{p}, \frac{1}{q'}\right) \in \mathcal{C}_k$. If $p \leq p_k$ and $\left(\frac{1}{p}, \frac{1}{q'}\right) \notin \mathcal{C}_k$, then no such constant $C_{p,q'}$ exists.

That theorem 2 is equivalent to restricted weak-type boundedness of (2) is elementary (see proposition 1.3 of Tao and Wright [11]). First note that (2) holds for f_X and f_Y characteristic functions if and only if it holds for f_X and f_Y characteristic functions of open sets. If (2) is true, then (3) follows from (2) by taking $f_X = \chi_{\pi_X(O)}$ and $f_Y = \chi_{\pi_Y(O)}$. To obtain (2) from (3), let $O = \pi_X^{-1}(O_X) \cap \pi_Y^{-1}(O_Y)$ when $f_X = \chi_{O_X}$ and $f_Y = \chi_{O_Y}$. As in the strong-type case, the inequality (3) can clearly be interpolated, so it suffices to prove (2) for the special points (p_j, q'_j).

To begin the task of proving the required restricted weak-type estimates, it is necessary to make note of several important consequences of nondegeneracy. The most elementary of these are given by the following lemma:

Lemma 1 (Facts about nondegeneracy)**.** *Suppose that (M, X, Y, π_X, π_Y) is nondegenerate through order k at m (let the distinguished vector fields be called $X_1, \ldots, X_{d_X-1}$). There is an open set $U \subset M$ containing m such that, for any $m_0 \in U$, the following are true.*

(1) *The ensemble (M, X, Y, π_X, π_Y) is nondegenerate through order j at m_0 for $j = 1, \ldots, k$.*

(2) *It must be the case that* $d_Y \geq (k+1)(d_X - 1)$*; moreover,* $\dim_{m_0} \mathfrak{X}_j = d_Y - j(d_X - 1)$ *for* $j = 1, \ldots, k+1$ *and* $\dim(\mathfrak{X}_j|_{m_0})/(\mathfrak{X}_{j+1}|_{m_0}) = d_X - 1$ *for* $j = 1, \ldots, k$.

(3) *If* $X'_1, \ldots X'_{d_X-1}$ *is a collection of vector fields in* $\mathfrak{X}_k$ *which are linearly independent at all points of* U *and have ord*$(X'_l) = k$*, then the collection* $\{T^k(X'_l)\}$ *together with* $\mathfrak{X}_1$ *and* $\mathfrak{Y}_1$ *also spans the tangent space of* M *at* m_0.

Proof. Consider, first of all, property 2. The latter portion (which depends on j) will be proved by induction on j. When $j = 1$, it follows from the implicit function theorem (because $d\pi_X$ is surjective) that $\dim_{m_0} \mathfrak{X}_1 = d_Y - d_X + 1$. To compute the dimension of the quotient space, a particular collection of $d_X - 1$ vector fields will be shown to form a basis pointwise modulo $\mathfrak{X}_2|_{m_0}$. To make this basis, observe that it follows from the definition of the spaces $\mathfrak{X}_j$ and the fact that $\mathfrak{Y}_1$ is closed under Lie brackets that $T^{k-1}(X_l) \in \mathfrak{X}_1 + \mathfrak{Y}_1$. The basis is chosen as follows: let $V_l \in \mathfrak{X}_1$ be a vector field such that $V_l - T^{k-1}(X_l) \in \mathfrak{Y}_1$ for $l = 1, \ldots, d_X - 1$.

Suppose first that the V_l were not linearly independent at m_0. Then there would exist constants c_l and a vector field $W \in \mathfrak{X}_2$ such that $\sum c_l V_l + W$ vanishes at the point m_0. Since $\dim_{m_0} \mathfrak{X}_1$ is constant near m, one may choose a collection of vector fields $V'_n \in \mathfrak{X}_1$, $n = 1, \ldots, d_Y - d_X + 1$ which are linearly independent and span $\mathfrak{X}_1|_{m_0}$ at all points near m. To accomplish this, simply take a collection which is a basis of $\mathfrak{X}_1|_m$ at m; by continuity, the chosen collection will be linearly independent at all points near m and, hence, will be a basis because it has the correct cardinality. Using the vector fields V'_n as a basis, it follows that one can express $\sum_l V_l + W$ as a linear combination $\sum_n f_n V'_n$ for some f_n which are smooth functions defined near m; furthermore, $f_n(m_0) = 0$ because $\sum c_l V_l + W$ vanishes there. Taking the Lie bracket with Y_1, it follows that $\sum_l c_l T(V_l)! = -T(W) + \sum_n f_n T(V'_n) - \sum_n (Y_1 f_n) V'_n$. Since $f_n(m_0) = 0$, the right-hand side of this inequality is in $(\mathfrak{X}_1 + \mathfrak{Y}_1)|_{m_0}$ at m_0; therefore so is $\sum_l c_l T(V_l)|_{m_0}$. Because $V_l - T^{k-1}(X_l) \in \mathfrak{Y}_1$, the same is true of $T(V_l) - T^k(X_l)$ because $\mathfrak{Y}_1$ is closed under the Lie bracket. Hence it must also be the case that $\sum_l c_l T^k(X_l)|_{m_0} \in (\mathfrak{X}_1 + \mathfrak{Y}_1)|_{m_0}$. But this cannot be the case unless $c_l = 0$ for all l (by the nondegeneracy condition, if the $T^k(X_l)$ were not linearly independent modulo $\mathfrak{X}_1 + \mathfrak{Y}_1$, the dimension of the span of $\mathfrak{X}_1|_m + \mathfrak{Y}_1|_m + \operatorname{span}\{T^k(X_l)\}$ would not equal the dimension of the tangent space of M at m).

To show that the V_l span the quotient space at m_0, let $V' \in \mathfrak{X}_1$ be any vector field. By the nondegeneracy condition, there must exist $W \in \mathfrak{X}_1$, $Y \in \mathfrak{Y}_1$ and C^∞ functions f_l such that $T(V') = W + Y + \sum_l f_l T(V)$. Hence $T(V' - \sum_l f_l V_l) = W + Y + \sum_l (Y_1 f_l) V_l$. The right-hand side of this equality is in $\mathfrak{X}_1 + \mathfrak{Y}_1$, so $V' - \sum_l f_l V_l \in \mathfrak{X}_2$ by definition of $\mathfrak{X}_2$. This means that, at all points near m, $V'|_{m_0}$ is a linear combination of the vectors $V_l|_{m_0}$ modulo $\mathfrak{X}_2|_{m_0}$ as desired.

This completes the case $j = 1$. For the induction step, assume that $j \geq 1$. It follows immediately from $\dim(\mathfrak{X}_j|_{m_0})/(\mathfrak{X}_{j+1}|_{m_0}) = d_X - 1$ and $\dim_{m_0} \mathfrak{X}_j = d_Y - j(d_X + 1)$ that $\dim_{m_0} \mathfrak{X}_{j+1} = d_Y - (j+1)(d_X - 1)$. To compute the dimension of the new quotient space, one proceeds just as before: let $V_l \in X_{j+1}$ be such that $T^{k-j-1}(X_l) - V_l \in \mathfrak{Y}_1$. If $\sum_l c_l V_l + W_{j+2}$ vanishes at m_0, for some constants c_l and some $W_{j+2} \in \mathfrak{X}_{j+2}$, it must be the case that $\sum_l c_l T(V_l)|_{m_0} \in (\mathfrak{X}_{j+1} + \mathfrak{Y}_1)|_{m_0}$. But the vector fields $T(V_l)$ are (modulo some vector fields $Y_l \in \mathfrak{Y}_1$) precisely the basis of $(\mathfrak{X}_j|_{m_0})/(\mathfrak{X}_{j+1}|_{m_0})$ constructed in the previous step. Therefore the constants c_j are zero. As for spanning the quotient space, suppose $V' \in \mathfrak{X}_{j+1}$. There exist $W \in \mathfrak{X}_{j+1}$, $Y \in \mathfrak{Y}_1$ and C^∞ functions f_l such that $T(V') = W + Y + \sum_l f_l T(V_l)$ by the fact that the vector fields $T(V_l)$ span $(\mathfrak{X}_j|_{m_0})/(\mathfrak{X}_{j+1}|_{m_0})$ near m. This means that, just as before, $V' - \sum_l f_l V_l \in \mathfrak{X}_{j+2}$.

The end result of this reasoning is that the latter portion of property 2 holds for $j = 1, \ldots, k$ (the induction argument must stop here because the method for choosing a basis of the quotient space $(\mathfrak{X}_j|_{m_0})/(\mathfrak{X}_{j+1}|_{m_0})$ is not defined for $j > k$. Note, however, that after step k, one may prove just as before that $\dim_{m_0} \mathfrak{X}_{k+1} = d_Y - (k+1)(d_X - 1)$ and, hence, $d_Y - (k+1)(d_X - 1) \geq 0$. Property 2 is therefore completely proven.

To prove property 3, notice that if f_l are C^∞ functions, then the vector field $T^k(\sum_l f_l X'_l) - \sum_l f_l T^k(X'_l)$ is in $\mathfrak{X}_1 + \mathfrak{Y}_1$. This assertion follows from the fact that $T^j(\sum_l f_l X'_l) - \sum_l f_l T^j(X'_l) \in \mathfrak{X}_{k-j+1} + \mathfrak{Y}_1$ for $j = 0, \ldots, k$, which is easily proved by induction on j using the definition of $\mathfrak{X}_{k-j+1}$ and the fact that $\mathfrak{Y}_1$ is closed under Lie brackets. It follows that the pointwise span of $T^k(X'_l)$ must be the same, modulo $(\mathfrak{X}_1 + \mathfrak{Y}_1)|_{m_0}$ as the pointwise span of $T^k(X_l)$, since the vector fields X_l may be written as linear combinations (with variable coefficients) of the X'_l.

Finally, property 1. As with property 3, let $V_l \in \mathfrak{X}_{k-j}$ be a vector field which differs from $T^j(X_l)$ by a vector field in $\mathfrak{Y}_1$. It follows that $T^{k-j}(V_l)$ is equal to $T^k(X_l)$ modulo $\mathfrak{Y}_1$, so the $T^{k-j}(V_l)$ must also span the tangent space of M near m, modulo $\mathfrak{X}_1$ and $\mathfrak{Y}_1$. □

As was the case in the work of Tao and Wright [11], vector fields lying in $\ker d\pi_X$ and $\ker d\pi_Y$ will play a central role in the proof of theorem 1. In that work, both $\mathfrak{X}_1$ and $\mathfrak{Y}_1$ were one-dimensional, hence the structure of each is more or less elementary (and, for example, there was no need to be careful about choosing representatives $X \in \mathfrak{X}_1$ and $Y \in \mathfrak{Y}_1$). In the case at hand, $\mathfrak{Y}_1$ is still simple, but now $\mathfrak{X}_1$ has dimension greater than one. It should not come as a surprise, then, that a more careful analysis of $\mathfrak{X}_1$ is in order; in particular, one must put some care in the choice of vector fields $X_{i,j} \in \mathfrak{X}_1$ whose flows will be studied. These vector fields can be taken to commute, but even more can be said. First, some notation. Given a C^∞ vector field V, the flow along V will be written $e^{tV}m$. That is, $e^{tV}m$ is the unique solution to the ODE

$$\frac{d}{dt} e^{tV} m = V|_{e^{tV}m}$$

with the initial condition that $e^{0V}m = m$. See Warner [12], for example, for the basic properties of flow maps. This same reference contains a treatment of the Frobenius theorem for C^∞ distributions of vector subspaces which is used heavily in the following lemma.

Lemma 2. *Suppose (M, X, Y, π_X, π_Y) is nondegenerate through order k at m_0. There is a neighborhood U of m_0, a smooth map $\rho_X : \pi_X(U) \to U$, and $d_Y - d_X + 1$ linearly independent vector fields vector fields $X_{i,j}$ (indexed $1 \leq i \leq k$ and $1 \leq j \leq d_X - 1$, and the rest, if any, indexed $(k+1, j)$ for $j = 1, \ldots, d_Y - (k+1)(d_X - 1)$) on U such that, for any f supported on U,*

$$\int_U f(m)dm = \int_X \int_{\mathbb{R}^{m+n-1}} f(e^{\sum_{i,j} s_{i,j} X_{i,j}} \rho_X(x)) ds dx. \tag{4}$$

The composition $\pi_X \circ \rho_X$ is the identity on $\pi_X(U)$, and the vector fields $X_{i,j}$ commute, satisfy $d\pi_X(X_{i,j}) = 0$ and have $ord(X_{i,j}) = i$ when $i \leq k$ (and $ord(X_{k+1,j}) \geq k+1$).

Proof. For $j = 1, \ldots, k+1$, there exist $j(d_X - 1) + 1$ scalar functions f_l^j, defined on a neighborhood U of m_0, with linearly independent gradients at m_0 such that $\mathfrak{X}_j$ annihilates f_l^j (this is, for example, a consequence of the Frobenius theorem because, by property 2 of lemma 1, $\mathfrak{X}_j$ is a $(d_Y - j(d_X-1))$-dimensional Frobenius distribution near m_0). Without loss of generality, let all such functions vanish at m_0. A special subcollection of all such functions, with $j = 1, \ldots, k$, is formed as follows. Let $s_{0,1}, \ldots, s_{0,d_X}$ be the functions $f_1^1, \ldots, f_{d_X}^1$. Next, take $s_{1,1}, \ldots, s_{1,d_X-1}$ to be a maximal collection of the f_l^2 such that the $s_{0,j}$ together with the $s_{1,j}$ have linearly independent gradients. Continue in this way to define $s_{i,j}$ for $0 \leq i \leq k$, $1 \leq j \leq d_X - 1$. Likewise, by Frobenius, there exists a map Φ_{k+1} defined near m_0 with values in $\mathbb{R}^{d_Y-(k+1)(d_X-1)}$ such that $m \mapsto (\Phi_{k+1}(m), f_1^{k+1}(m), \ldots, f_{(k+1)(d_X-1)+1}^{k+1}(m))$ is locally a diffeomorphism (without loss of generality, $\Phi_{k+1}(m_0) = 0$). We call the components of Φ_{k+1} by the names $s_{k+1,j}$ where $1 \leq j \leq d_Y - (k+1)(d_X - 1)$.

Now let U be a small neighborhood of m_0, and define $\Phi : U \to X \times \mathbb{R}^{d_Y - d_X + 1}$ by $\Phi(m) := (\pi_X(m), \{s_{i,j}\}_{i \geq 1})$; the functions $s_{0,j}$ are not included. The map Φ is locally a diffeomorphism. To see this, suppose $d\Phi(V) = 0$. It must be the case that $d\pi_X(V) = 0$, which means that $V \in \mathfrak{X}_1$. If $V \in \mathfrak{X}_1$ and $V(s_{1,j}) = 0$ for $j = 1, \ldots, d_X - 1$, then $V \in \mathfrak{X}_2$ (because $V \in \mathfrak{X}_1$ implies $V(s_{0,j}) = 0$ for $j = 1, \ldots, d_X$, and if $V(s_{i,j}) = 0$ whenever $i = 0, 1$, it must be the case that $V(f_l^2) = 0$ for all l). Continuing in this way, if $d\Phi(V) = 0$, it must be the case that $V \in \mathfrak{X}_{k+1}$ and $d\Phi_{k+1}(V) = 0$, meaning $V = 0$. Therefore Φ is locally a diffeomorphism. Therefore, there exists a smooth, positive function J defined near the point $(\pi_X(m_0), 0)$ for which

$$\int_M f(m) dm = \int f \circ \Phi^{-1}(x,s) J(x,s) ds dx$$

when f is supported near m_0. One can go a step further and eliminate the factor $J(x,s)$ as follows. For each pair i,j, let $s_{i,j}$ be a function of the variables $t_{i',j'}$ such that $\frac{\partial}{\partial t_{k+1,1}} s_{k+1,1}(x,t) = J(x, s(t))^{-1}$ and $s_{i,j}(x,t) = t_{i,j}$ otherwise, plus the constraint that $s(x,0) = 0$ (this assumes, of course, that there is a function $s_{k+1,1}$). By the change of variables formula,

$$\int_M f(m) dm = \int f \circ \Phi^{-1}(x, s(t,x)) dt dx.$$

Let $\rho_X(x) := \Phi^{-1}(x, s(x,0))$. This map ρ_X is a right-inverse of π_X near $\pi_X(m_0)$. Let $X_{i,j}$ be the push-forward of the vector field $\frac{\partial}{\partial t_{i,j}}$ via the map $\Phi^{-1}(x, s(x,t))$. The vector fields $X_{i,j}$ must commute because the vector fields $\frac{\partial}{\partial t_{i,j}}$ commute. Now $d\Phi(X_{k+1,1}) = J^{-1}(x,s) \frac{\partial}{\partial s_{k+1,1}}$ and $d\Phi(X_{i,j}) = \frac{\partial}{\partial s_{i,j}} + c_{i,j}(s,x) \frac{\partial}{\partial s_{k+1,1}})$ otherwise, for some function $c_{i,j}$. Just as when showing that Φ is locally a diffeomorphism, it must be the case that $X_{i,j} \in \mathfrak{X}_i$ because $d\pi_X(X_{i,j}) = 0$ and $X_{i,j}(f_n^l) = 0$ whenever $l = 1, \ldots, i$. Therefore $\mathrm{ord}(X_{i,j}) \geq i$. Furthermore, if $i < k+1$, there exists an f_n^{i+1} for some n such that $X_{i,j}(f_n^{i+1}) \neq 0$, so that $\mathrm{ord}(X_{i,j}) \leq i$ as well.

One final note: if there is not, in fact, any function $s_{k+1,1}$, one instead makes the change of variables $\frac{\partial}{\partial t_{k,1}} s_{k,1}(x,t) = J(x, s(t))^{-1}$ and $s_{i,j}(x,t) = t_{i,j}$ otherwise. Had this change been made above, one would have $X_{k+1,j} \in \mathfrak{X}_k$ but not necessarily $\mathfrak{X}_{k+1}$. If there are no vector fields $X_{k+1,j}$, this is not a problem, and the rest proceeds just as before. $\square$

As mentioned earlier, the particular vector fields $X_{i,j}$ given by lemma 2 will play a central role in the proof of theorem 1. Throughout the rest of this paper, they will be considered fixed. As for selecting a vector field $Y_1 \in \mathfrak{Y}_1$, the situation is much simpler. The analogue of lemma 2 is that there exists $Y_1 \in \mathfrak{Y}_1$ and a smooth right-inverse ρ_Y to π_Y such that

$$\int f(m)dm = \int f(e^{tY_1}\rho_Y(y))dtdy. \tag{5}$$

Throughout the rest of the paper, this choice of Y_1 will also be fixed (but the particular properties of Y_1 versus some Y_1' will not be of critical importance).

The connection between Radon-like operators and the bilinear form appearing in (2) can now be made explicit. Given (M, X, Y, π_X, π_Y), one can define operators R and R^* mapping functions on X to functions on Y and vice-versa by taking

$$Rf(y) := \int f(\pi_X(e^{tY_1}\rho_Y(y)))dt, \tag{6}$$

$$R^*g(x) := \int g(\pi_Y(e^{\sum_{i,j} s_{i,j}X_{i,j}}\rho_X(x)))ds. \tag{7}$$

It follows by (4) and (5) that

$$\int_U f(\pi_X(m))g(\pi_Y(m))dm = \int Rf(y)g(y)dy = \int f(x)R^*g(x)dx.$$

The operator R is the Radon-like operator associated to (M, X, Y, π_X, π_Y). By L^p-duality, proving boundedness of R from $L^p(X)$ to $L^q(Y)$ is equivalent to proving boundedness of (2) on $L^p(X) \times L^{q'}(Y)$ for $\frac{1}{q} + \frac{1}{q'} = 1$.

The next piece of background work which will be necessary to prove theorem 1 is the estimate of the size of an arbitrary open set $O \subset M$ restricted to the flows e^{tY_1} or $e^{\sum_{i,j} s_{i,j}X_{i,j}}$. The following proposition gives the necessary inequalities, and is the analogue of lemma 1 of Christ [5] or lemma 8.2 of Tao and Wright [11]:

Proposition 1. *Let $O \subset M$ be an open set, and let $\alpha_X := \frac{|O|}{|\pi_X(O)|}$ and $\alpha_Y := \frac{|O|}{|\pi_Y(O)|}$. Then there exists an open $O^* \subset O$ for which*

$$\chi_{O^*}(m) \int \chi_O(e^{\sum_{i,j} s_{i,j}X_{i,j}}m)dt \geq \frac{1}{4}\alpha_X \chi_{O^*}(m), \tag{8}$$

$$\chi_{O^*}(m) \int \chi_O(e^{tY_1}m)dt \geq \frac{1}{4}\alpha_Y \chi_{O^*}(m), \tag{9}$$

and $|O^*| \geq \frac{1}{4}|O|$.

Proof. Let $\widetilde{O}$ be the set of $m \in O$ for which $\int \chi_O(e^{\sum_{i,j} s_{i,j}X_{i,j}}m)ds > \frac{1}{2}\alpha_X$. By equation (4) of lemma 2, $|\widetilde{O}| = \int_{\pi_X(\widetilde{O})} \int \chi_{\widetilde{O}}(e^{\sum_{i,j} s_{i,j}X_{i,j}}\rho_X(x))dsdx$. Suppose that $x \in$

$\pi_X(\widetilde{O})$, and let $m \in \widetilde{O}$ such that $\pi_X(m) = x$. It must also hold that $e^{\sum_{i,j} s_{i,j} X_{i,j}} m \in \widetilde{O}$ for all s such that $e^{\sum_{i,j} s_{i,j} X_{i,j}} m \in O$ because

$$\int \chi_O(e^{\sum_{i,j} s'_{i,j} X_{i,j}} e^{\sum_{i,j} s_{i,j} X_{i,j}} m) ds' = \int \chi_O(e^{\sum_{i,j} (s'_{i,j} + s_{i,j}) X_{i,j}} m) ds'$$

(by virtue of the fact that the $X_{i,j}$ commute) and the right-hand side is equal to $\int \chi_O(e^{\sum_{i,j} s'_{i,j} X_{i,j}} m) ds'$ (by a linear change of variables). Therefore it must be that $\int \chi_{\widetilde{O}}(e^{\sum_{i,j} s_{i,j} X_{i,j}} \rho_X(x)) ds = \int \chi_O(e^{\sum_{i,j} s_{i,j} X_{i,j}} \rho_X(x)) ds > \frac{1}{2}\alpha_X$. Of course, if $x \in \pi_X(O)$ but $x \notin \pi_X(\widetilde{O})$, then the integral $\int \chi_O(e^{\sum_{i,j} s_{i,j} X_{i,j}} \rho_X(x)) ds \leq \frac{1}{2}\alpha_X$ and the corresponding integral $\int \chi_{\widetilde{O}}(e^{\sum_{i,j} s_{i,j} X_{i,j}} \rho_X(x)) ds$ for $\widetilde{O}$ is zero. It follows that, for all x,

$$\begin{aligned} \chi_{\pi_X(\widetilde{O})}(x) \int & \chi_{\widetilde{O}}(e^{\sum_{i,j} s_{i,j} X_{i,j}} \rho_X(x)) ds \\ & \geq \chi_{\pi_X(O)}(x) \left(\int \chi_O(e^{\sum_{i,j} s_{i,j} X_{i,j}} \rho_X(x)) ds - \tfrac{1}{2}\alpha_X \right). \end{aligned}$$

Integrating over x, it follows from (4) and the definition of α_X that $|\widetilde{O}| \geq \frac{1}{2}|O|$. By the definition of $\widetilde{O}$, any subset $O^* \subset \widetilde{O}$ satisfies (8). In that spirit, let

$$O^* := \left\{ m \in \widetilde{O} \;\middle|\; \int \chi_{\widetilde{O}}(e^{tY_1} m) dt > \frac{1}{2} \frac{|\widetilde{O}|}{|\pi_Y(\widetilde{O})|} \right\}.$$

Just as before, $|O^*| \geq \frac{1}{2}|\widetilde{O}| \geq \frac{1}{4}|O|$. But $\frac{|\widetilde{O}|}{|\pi_Y(\widetilde{O})|} \geq \frac{1}{2}\frac{|O|}{|\pi_Y(\widetilde{O})|} \geq \frac{1}{2}\frac{|O|}{|\pi_Y(O)|} = \frac{1}{2}\alpha_Y$, so (9) holds by the same reasoning as was given for (8). $\square$

The final piece of necessary preparation is to make a quantitative statement which reflects the following observation: by construction, the vector field $X_{i,j}$ behaves trivially when commuted with Y_1 up until the i-th commutator, so one expects that the composed flow $e^{tY_1} e^{\sum_{i,j} s_{i,j} X_{i,j}} m$ should also be somehow trivial up to the $(i-1)$-st order in t. The way to make this precise is to consider the projection of the flow via π_X:

Proposition 2. *Suppose (M, X, Y, π_X, π_Y) is nondegenerate through order k at m, and let $X_{i,j}$ be the vector fields as described in lemma 2. For all real numbers t sufficiently small, there exist time dependent vector fields $Y^t_{i,j}$ and $V^t_{i,j}$, defined near m, for which*

$$d\pi_X de^{tY_1}(X_{i,j}) = d\pi_X(Y^t_{i,j}) + t^i d\pi_X(V^t_{i,j}) \tag{10}$$

such that $Y^t_{i,j} \in \mathfrak{Y}_1$ and at $t = 0$, $Y^0_{i,j} = 0$ and $V^0_{i,j} = \frac{1}{i!}T^i(X_{i,j})$.

Proof. Consider the vector field $de^{tY_1}(X_{i,j})$ at a fixed point m as time t varies. By Taylor's theorem,

$$de^{tY_1}(X_{i,j}) = \sum_{l=0}^{i-1} \frac{t^l}{l!} \left(\frac{d}{dt}\right)^l \bigg|_{t=0} de^{tY_1}(X_{i,j}) + t^i V^t_{i,j}$$

where $V^0_{i,j} = \frac{1}{i!}\left(\frac{d}{dt}\right)^i \Big|_{t=0} de^{tY_1}(X_{i,j})$. Using the identity

$$\frac{d}{dt} de^{tA}(B)|_m = de^{tA}([B, A])|_m, \tag{11}$$

it follows that $V^0_{i,j} = \frac{1}{i!} T^i(X_{i,j})$. Furthermore, as $\mathrm{ord}(X_{i,j}) = i$, it must be the case that $\left(\frac{d}{dt}\right)^l \Big|_{t=0} de^{tY_1}(X_{i,j}) = T^l(X_{i,j})$ is in $\mathfrak{X}_1 + \mathfrak{Y}_1$ when $l < i$. Without loss of generality, then, let $T^l(X_{i,j}) = X^l_{i,j} + Y^l_{i,j}$ for $X^l_{i,j} \in \mathfrak{X}_1$ and $Y^l_{i,j} \in \mathfrak{Y}_1$. Letting $Y^t_{i,j} = \sum_{l=1}^{i-1} \frac{t^l}{l!} Y^l_{i,j}$ (there is no $l = 0$ term because $T^0(X_{i,j}) = X_{i,j}$) establishes the proposition, because $d\pi_X(X^l_{i,j}) = 0$. $\square$

The proof of theorem 2 (and hence theorem 1) will proceed as follows. Fix an open set $O \subset M$ contained in some sufficiently small neighborhood of m (where (M, X, Y, π_X, π_Y) is nondegenerate through order k) and let $E := \pi_X(O)$. By property 1 of lemma 1, it suffices to prove the isoperimetric inequality (3) at the special point $(\frac{1}{p_k}, \frac{1}{q'_k})$. Section 3 is devoted to a proof of the inequality

$$\int \prod_{l=1}^{k} \chi_E(\pi_X(e^{t_l Y} e^{\sum_{i,j} s_{i,j} X_{i,j}} m)) |J_k(t,s)| dt ds \leq C|E|^k \tag{12}$$

(for C independent of E) where $J_k(t,s)$ is essentially the Jacobian determinant of the map from $\mathbb{R}^{kd_X}$ to X^k given by

$$(\{t_l\}_{l=1,\ldots,k}, \{s_{i,j}\}_{i=1,\ldots,k;\ j=1,\ldots d_X-1}) \mapsto \{\pi_X(e^{t_l Y} e^{\sum_{i,j} s_{i,j} X_{i,j}} m)\}_{l=1,\ldots,k}.$$

Roughly speaking, equation (12) can only hold if this mapping has bounded multiplicity. Section 4 is devoted to the complementary inequality

$$\int \prod_{l=1}^{k} \chi_E(\pi_X(e^{t_l Y} e^{\sum_{i,j} s_{i,j} X_{i,j}} m_0)) |J_k(t,s)| dt ds \geq C \alpha_X \alpha_Y^{\frac{k(k+1)}{2}(d_X-1)+k} \tag{13}$$

(with C independent of E, α_X, and α_Y) which itself follows from proposition 1 and an estimate of the size of $|J_k(t,s)|$. Here Vandermonde polynomials necessarily enter the picture. But in this case, they present no significant difficulties in any number of dimensions. Finally, at the end of section 4, the inequalities (12) and (13) will be combined to yield an isoperimetric inequality, from which theorems 2 and 1 follow.

A word about notation: Throughout the remainder of the paper, the notation $A \lesssim B$ will mean that there exists a constant C independent of the particular open set $O \subset M$ and its projections $E := \pi_X(O)$ and $F := \pi_Y(O)$ (and, hence α_X and α_Y). The relation $A \gtrsim B$ is defined similarly, and $A \sim B$ means $A \lesssim B \lesssim A$.

3. Change of variables

As just noted, the purpose of this section is to establish (12). To this end, it is necessary to understand the behavior of the intersection of curves $\pi_X(e^{tY_1} m)$ and $\pi_X(e^{tY_1} m')$. The following lemma is an idealization of the nondegenerate case and shows that the curves behave like polynomial expressions, namely, that each curve is determined by its position at a bounded number of times t_j:

Lemma 3. *Let $U \subset \mathbb{R} \times \mathbb{R}^{lk}$ be a neighborhood of the origin, and let $\gamma : U \to \mathbb{R}^l$ be a C^∞ map for which $\gamma(0,s) = 0$ whenever $(0,s) \in U$ ($\gamma(t,s)$ may be described as* *a family of curves, parametrized by s, passing through the origin at $t = 0$). Suppose that the map $\Gamma := \mathbb{R}^{lk} \to (\mathbb{R}^l)^k$ given by*

$$\Gamma(s) := \left\{ \left(\frac{\partial^j}{\partial t^j} \gamma \right) (0, s) \right\}_{j=1,\ldots,k}$$

is invertible on a neighborhood of the origin. Then there exists an open $V \subset \mathbb{R}^{lk}$ containing the origin and a $T > 0$ such that for any $0 < t_1 < t_2 < \cdots < t_k < T$, the parameter $s \in V$ is uniquely determined by the position of the curve $\gamma(t, s)$ at times $t_1, \ldots, t_k$.

Proof. Let $\gamma^j(s) := (\frac{\partial^j}{\partial t^j}\gamma)(0, s)$ for $j = 1, \ldots, k$. By Taylor's theorem, $\gamma(t, s) = \sum_{j=1}^k \frac{t^j}{j!}\gamma^j(s) + G^{k+1}(t, s)$ for some smooth map $G^{k+1}(t, s)$. Suppose that there exist s and s' satisfying $\gamma(t_j, s) = \gamma(t_j, s')$ for $0 = t_0 < t_1 < t_2 < \cdots < t_k$, where $T := |t_k|$ is no greater than one. Given any bounded neighborhood V of the origin in $\mathbb{R}^{lk}$, there exists a constant C independent of the times t_j for which $|\Gamma(s) - \Gamma(s')| \leq CT|s - s'|$, where $|\cdot|$ is the Euclidean norm. This is proved by a repeated application of Rolle's theorem to the function $(\gamma(t, s))_i - (\gamma(t, s'))_i$ for $i = 1, \ldots, l$ as follows: since $(\gamma(t, s))_i - (\gamma(t, s'))_i$ vanishes $k+1$ times, its k-th derivative must also vanish at some time t_* with $|t_*| < T$, so $(\gamma^k(s))_i - (\gamma^k(s'))_i = \frac{\partial^k}{\partial t^k} \left((G^{k+1}(t, s))_i - (G^{k+1}(t, s'))_i\right)\big|_{t=t_*}$. The differentiability of G^{k+1} ensures $|(\gamma^k(s))_i - (\gamma^k(s'))_i| \leq CT|s - s'|$ for some C independent of T. Now step backwards: the $(k-1)$-st time derivative of $(\gamma(t, s))_i - (\gamma(t, s'))_i$ also vanishes at some time t_{**} with $|t_{**}| < T$. Now $|(\gamma^{k-1}(s))_i - (\gamma^{k-1}k(s'))_i| \leq CT|s - s'|$ by virtue of the differentiability of G^{k+1} and the estimate just made for the k-th derivative. Continuing in this way gives $|\Gamma(s) - \Gamma(s')| \leq CT|s - s'|$ as desired.

To complete the proof, observe that there exists $\epsilon > 0$ and a neighborhood $V \subset \mathbb{R}^{lk}$ containing the origin such that $\epsilon|s - s'| \leq |\Gamma(s) - \Gamma(s')|$ for $s, s' \in V$ by virtue of the invertibility of Γ. Choosing $T < C^{-1}\epsilon$, the inequality $\epsilon|s - s'| \leq CT|s - s'|$ can only be true if $s = s'$. □

To use lemma 3 in the general case, one must first fix $s_{k+1,j}$ for all j, if any such vector fields $X_{k+1,j}$ exist by lemma 2. Fix a smooth function f_0 on a neighborhood of $\pi_X(m)$ (with $f_0(\pi_X(m)) = 0$) such that $d\pi_X(Y_1)(f_0) \neq 0$; this must be possible because the nondegeneracy condition cannot be satisfied at m_0 if $Y_1|_{m_0} \in \mathfrak{X}_1|_{m_0}$ (the dimension of the span of $\mathfrak{X}_1$, $\mathfrak{Y}_1$ and the $T^k(X_l)$ would be too small at m_0). Then, near m, there exists a vector field Y_1' proportional to Y_1 such that $f_0(\pi_X(e^{tY_1'}e^{\sum_{i,j} s_{i,j}X_{i,j}}m)) = t$ for small s and t (this is merely a reparametrization of time in relation to the Radon-like operator (6)). Let $\Phi(t, s) := e^{tY_1'}e^{\sum_{i,j} s_{i,j}X_{i,j}}m$; it follows that whenever two curves $\pi_X(\Phi(t, s))$ and $\pi_X(\Phi(t', s'))$ intersect, the time parameters must be equal, since $t = f_0(\pi_X(\Phi(t, s))) = f_0(\pi_X(\Phi(t', s'))) = t'$.

Take f to be any C^∞ function which is constant along the curve $\pi_X(\Phi(t, 0))$. It follows from proposition 2 that

$$\frac{\partial}{\partial s_{i,j}} f(\pi_X(\Phi(t, s))) = d\pi_X(Y_{i,j}^t\big|_\Phi + t^i\, V_{i,j}^t\big|_\Phi)(f).$$

But when $s = 0$, $d\pi_X(Y_{i,j}^t\big|_\Phi)(f)$ is simply a multiple of $\frac{d}{dt} f(\pi_X(\Phi(t, 0)))$, and so must vanish. Differentiating i times in t and setting $t = 0$, there will be only one nonzero term, which arises from applying all time derivatives to the power t^i. Recalling that $V^0_{i,j} = \frac{1}{i!}T^i(X_{i,j})$, it follows that

$$\frac{\partial^i}{\partial t^i}\frac{\partial}{\partial s_{i,j}}\bigg|_{t=0,s=0} f(\pi_X(\Phi(t,s))) = d\pi_X(T^i(X_{i,j})\big|_m)(f).$$

Now let $f_1, \ldots, f_{d_X-1}$ be a (maximal) collection of functions which are constant along $\pi_X(\Phi(t,0))$, have linearly independent gradients, and satisfy $f_l(\pi_X(m)) = 0$. Since (M, X, Y, π_X, π_Y) is nondegenerate through order k (and the extraneous $s_{k+1,j}$ are fixed), by the previous lemma, the parameter $s_{i,j}$, $1 \leq i \leq k$, $1 \leq j \leq d_X - 1$, corresponding to the curve in $\mathbb{R}^{d_X-1}$, given by

$$(f_1(\pi_X(\Phi(t,s))), \cdots, f_{d_X-1}(\pi_X(\Phi(t,s)))),$$

is uniquely determined by the values of $f_l(\pi_X(\Phi(t,s)))$ at k distinct times $t > 0$.

Now f_0, $f_1, \cdots, f_{d_X-1}$ form a coordinate system on X. Therefore, if two curves $\pi_X(\Phi(t,s))$ and $\pi_X(\Phi(t',s'))$ intersect at k positions (other than at $\pi_X(m)$ at time 0), it must be the case that these curves reach the same points at the same times (since $t = t'$). But the uniqueness of the values of f_l for any k fixed times ensures that $s' = s$.

The end result obtained by this line of reasoning is that locally, when the appropriate $s_{k+1,j}$ are fixed, the curves $\pi_X(e^{tY_1}e^{\sum_{i,j} s_{i,j}X_{i,j}}m)$ are uniquely determined by any k-tuple of points (aside from $\pi_X(m)$) that they pass through. This means that the map $\Phi_k(t,s)$ taking $\mathbb{R}^{kd_X}$ into X^k, defined by

$$\Phi_k(t,s) := (\pi_X(e^{t_1Y_1}e^{\sum_{i,j} s_{i,j}X_{i,j}}m), \cdots, \pi_X(e^{t_kY_1}e^{\sum_{i,j} s_{i,j}X_{i,j}}m)),$$

is one-to-one for small parameters t, s. Thus there is an open set $U \subset \mathbb{R}^{kd_X}$ containing the origin such that, for any open $E \subset X$,

$$\int_U \prod_{l=1}^k \chi_E(\pi_X(e^{t_lY}e^{\sum_{i,j} s_{i,j}X_{i,j}}m))|J_k(t,s)|dtds \lesssim |E|^k$$

(for some suppressed constant independent of E), where $J_k(t,s)dtds$ is the pull-back of the integration form $dx_1 \wedge \cdots \wedge dx_k$ via the map Φ_k. This follows from the change of variables formula applied to the regions where $d\Phi_k$ is nonvanishing and the fact that the set where $d\Phi_k$ vanishes is a closed set of measure zero (to be described explicitly in the next section).

4. Jacobian factor

Now the attention is turned to inequality (13). The first and most complicated piece to put in place is to estimate the size of $|J_k(t,s)|$. It is not too difficult to see that $J_k(t,s)$ must vanish if $t_l = 0$ for any l or if $t_l = t_{l'}$ for any $l \neq l'$. By the following lemma, these are the only situations in which $J_k(s,t)$ may vanish. Also of importance is the rate of vanishing, which is also given explicitly:

Lemma 4. *If (M, X, Y, π_X, π_Y) is nondegenerate through order k at m and $s_{k+1,j}$ is fixed for all j (if any), then the Jacobian determinant $J_k(t,s)$ of the map $\Phi_k(t,s) :=$*

$(\pi_X(e^{t_l Y_1} e^{\sum_{i,j} s_{i,j} X_{i,j}} m))_{l=1,\ldots,k}$ *satisfies*

$$|J_k(t,s)| \geq C \left| \prod_{n=1}^{k} t_n \prod_{i<j} (t_j - t_i) \right|^{d_X - 1} \tag{14}$$

for some $C > 0$*, provided* t *and* s *are sufficiently small.*

Proof. To establish the claim, we will show that the Jacobian determinant can be factored by the polynomial in t suggested by (14). The factorization will be accomplished by a series of elementary matrix operations.

In a suitable coordinate system, the Jacobian factor is, up to multiplication by some nonvanishing function of s and t, equal to the absolute value of the determinant of a $(kd_X) \times (kd_X)$ matrix of a form to be described. Let $B(t)$ be a column vector of length d_X whose entries are the components of $d\pi_X(Y_1)$ (which equals $\frac{\partial}{\partial t} \pi_X(e^{tY_1} e^{\sum_{i,j} s_{i,j} X_{i,j}} m)$), and let $A_i(t)$ be a $d_X \times (d_X - 1)$ matrix whose j-th column is the component representation of $d\pi_X de^{tY_1}(X_{i,j})$ (which is $\frac{\partial}{\partial s_{i,j}} \pi_X(e^{tY_1} e^{\sum_{i,j} s_{i,j} X_{i,j}} m)$) for $j = 1, \ldots, d_X - 1$. It follows that $J_k(t,s)$ is of the form

$$\det \begin{bmatrix} B(t_1) & 0 & \cdots & 0 & A_1(t_1) & A_2(t_1) & \cdots & A_k(t_1) \\ 0 & B(t_2) & \cdots & 0 & A_1(t_2) & A_2(t_2) & \cdots & A_k(t_2) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B(t_k) & A_1(t_k) & A_2(t_k) & \cdots & A_k(t_k) \end{bmatrix}.$$

By (10), one may take $A_i(t)$ to be the components of $t^i d\pi_X(V_{i,j}^t)$ (since $d\pi_X(Y_{i,j}^t)$ is a multiple of $B(t)$). One may immediately factor $J_k(t,s)$ by $(\prod_{n=1}^k t_n)^{d_X - 1}$ as follows: multiply column n by t_n (for $n = 1, \ldots, k$) then factor each row in the the n-th group by t_n. Observe that the power is $d_X - 1$ instead of d_X. This is because one multiplied the determinant by a factor of $t_1 \cdots t_k$ in order to factor off $(t_1 \cdots t_k)^{d_X}$. Also notice that the form of the matrix is preserved, but in the place of $A_i(t)$, one now has $t^{i-1} d\pi_X(V_{i,j}^t)$.

Next, take column 1 and add to it columns 1 through k. Then subtract the rows in the first group from each of the subsequent groups. Using the identity $f(x) - f(y) = (x-y) \int_0^1 \frac{df}{dx}(\theta x + (1-\theta) y) d\theta$ and the same sort of factoring trick just used (multiplying column n by $t_n - t_1$ for $n > 1$ and factoring all rows in the n-th group by $t_n - t_1$ for $n > 1$), one can factor off an additional $(\prod_{l=2}^k (t_l - t_1))^{d_X - 1}$, and the matrix now becomes

$$\det \begin{bmatrix} B(t_1) & 0 & \cdots & 0 & A_1(t_1) & A_2(t_1) & \cdots & A_k(t_1) \\ B^{(1)}(t_2) & B(t_2) & \cdots & 0 & A_1^{(1)}(t_2) & A_2^{(1)}(t_2) & \cdots & A_k^{(1)}(t_2) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ B^{(1)}(t_k) & 0 & \cdots & B(t_k) & A_1^{(1)}(t_k) & A_2^{(1)}(t_k) & \cdots & A_k^{(1)}(t_k) \end{bmatrix},$$

where $f^{(1)}(x) := \int_0^1 \frac{df}{dx}(\theta x + (1-\theta) t_1) d\theta$. This process is repeated: add columns 3 through k to column 2, then subtract the second group of rows from each of the 3rd through k-th groups. Multiplying columns 3 through k by $(t_3 - t_2)$ through $(t_k - t_2)$ and factoring by rows gives $(\prod_{l>2} (t_l - t_2))^{d_X - 1}$. When the process is completed, one has factored $(V(t)\prod_{n=1}^k t_n)^{d_X-1}$ off the determinant, where $V(t)$ is the Vandermonde polynomial in t. The remaining determinant may be evaluated when $t=0$. Here one uses the fact that $A_i(t)$ is the component-wise representation of the vectors $t^{i-1}d\pi_X(V^t_{i,j})$ and $A(l)_i(0) = \frac{1}{l!}[\left(\frac{d}{dt}\right)^l A_i](0)$. It follows that $A_i^{(l)}(t)$ vanishes for $l < i-1$, and the remaining determinant has the form (at $t=0$)

$$\det \begin{bmatrix} B(0) & 0 & \cdots & 0 & A_1(0) & 0 & \cdots & 0 \\ * & B(0) & \cdots & 0 & * & \frac{d}{dt}A_2(0) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & B(0) & * & * & \cdots & \frac{1}{(k-1)!}\frac{d^{k-1}}{dt^{k-1}}A_k(0) \end{bmatrix}.$$

This determinant is easily evaluated (after shuffling the columns, it becomes block lower-diagonal). Modulo a constant, it is equal to

$$\prod_{i=1}^k \det(d\pi_X(Y_1), \{d\pi_X(T^i(X_{i,j}))\}_{j=1,\ldots,d_X-1}),$$

which is nonzero by virtue of the fact that (M, X, Y, π_X, π_Y) is nondegenerate through order k and property 3 of lemma 1. For small s and t, the lemma follows by continuity. □

With lemma 4 in place, one may proceed to inequality (13). Recall the situation as given at the end of section 2: let $O \subset M$ be open and contained in some prescribed open set where (14) holds with some uniform constant C, and let $E = \pi_X(O), F = \pi_Y(O)$. By proposition 1 (applied to O and then again to O^*), there exists $m_0 \in O$ such that $\int \chi_{O^*}(e^{\sum_{i,j} s_{i,j}X_{i,j}}m_0)ds \gtrsim \alpha_X$. Let this m_0 be fixed. By definition of E, it follows that

$$\prod_{l=1}^k \chi_E(\pi_X(e^{t_l Y}e^{\sum_{i,j} s_{i,j}X_{i,j}}m_0)) \geq \prod_{l=1}^k \chi_O(e^{t_l Y}e^{\sum_{i,j} s_{i,j}X_{i,j}}m_0)$$

Once again, proposition 1 is invoked: provided $e^{\sum_{i,j} s_{i,j}X_{i,j}}m_0 \in O^*$, there is a set of times t of measure $\frac{1}{2}\alpha_Y$ or greater such that $e^{tY_1}e^{\sum_{i,j} s_{i,j}X_{i,j}}m_0 \in O$. For any measurable $I \subset \mathbb{R}$, there is a constant c_n for which

$$\int_I \left| t^n + \sum_{l=0}^{n-1} c_l t^l \right| dt \geq c_n |I|^{n+1}. \tag{15}$$

One can see that this must be the case, for example, because the set of points t where $\left|t^n + \sum_{l=0}^{n-1} c_l t^l\right| \leq |I|^n$ has size $\lesssim |I|$ (see lemma 1.2 of Carbery, Christ, and Wright [2], for example). This fact, together with proposition 1 and inequality (14) show that it must be the case that

$$\int \prod_{l=1}^k \chi_E(\pi_X(e^{t_l Y}e^{\sum_{i,j} s_{i,j}X_{i,j}}m_0))|J_k(t,s)|dt$$
$$\gtrsim \alpha_Y^{\frac{k(k+1)}{2}(d_X-1)+k}\chi_{O^*}(e^{\sum_{i,j} s_{i,j}X_{i,j}}m_0)$$

by estimating the iterated integrals in $t_1, \ldots, t_k$ separately using (15). All together,

$$\int \prod_{l=1}^{k} \chi_E(\pi_X(e^{t_l Y} e^{\sum_{i,j} s_{i,j} X_{i,j}} m_0)) |J_k(t,s)| dt ds \gtrsim \alpha_X \alpha_Y^{\frac{k(k+1)}{2}(d_X - 1) + k}.$$

Now recall, from the previous section, that

$$|E|^k \gtrsim \int \prod_{l=1}^{k} \chi_E(\pi_X(e^{t_l Y} e^{\sum_{i,j} s_{i,j} X_{i,j}} m_0)) |J_k(t,s)| dt ds.$$

The conclusion is an isoperimetric inequality: given that (M, X, Y, π_X, π_Y) is nondegenerate through order k at m, there exists an open neighborhood U of m such that, for any open $O \subset U$,

$$|O|^{\frac{k(k+1)}{2}(d_X - 1) + (k+1)} \lesssim |\pi_X(O)|^{k+1} |\pi_Y(O)|^{\frac{k(k+1)}{2}(d_X-1)+k}. \tag{16}$$

This is precisely the isoperimetric inequality needed for the desired restricted weak-type estimate.

5. Necessity

Let $m \in M$ be a point where (M, X, Y, π_X, π_Y) is nondegenerate through order k, and let $Z_l := T^k(X_{k,l})$. Fix an $\epsilon > 0$ sufficiently small and let O_δ be the set

$$\left\{ e^{tY_1} e^{\sum_{i,j} s_{i,j} X_{i,j}} e^{\sum_l u_l Z_l} m \;\middle|\; |t| \leq \epsilon\delta, \; |s_{i,j}| \leq \epsilon \delta^{\max\{0, k+1-i\}}, \; |u_l| \leq \epsilon \delta^{k+1} \right\}.$$

Setting $\Phi(t,s,u) := e^{tY_1} e^{\sum_{i,j} s_{i,j} X_{i,j}} e^{\sum_l u_l Z_l}$, it follows from the linear independence of $Y_1, X_{i,j}$ and Z_l that the map Φ is locally a diffeomorphism, so it must be the case that $|O_\delta| \sim \delta^{1 + \frac{(k+1)(k+2)}{2}(d_X - 1)}$. Likewise, $\pi_Y \circ \Phi$ independent of t and locally a diffeomorphism when t is fixed. It follows that $|\pi_Y(O_\delta)| \sim \delta^{\frac{(k+1)(k+2)}{2}(d_X-1)}$.

Finally, consider the projection of O_δ under π_X. To that end, let $\Psi(\tau, s, p) := e^{\tau Y_1} e^{\sum_{i,j} s_{i,j} X_{i,j}} p$. Consider the following nonlinear ODE for $\tau(\theta)$:

$$\dot{\tau}(\theta) Y_1|_{\Psi(\tau(\theta),(1-\theta)s,p)} = \sum_{i,j} s_{i,j} Y_{i,j}^{\tau(\theta)}|_{\Psi(\tau(\theta),(1-\theta)s,p)}$$

where $Y_{i,j}^t$ is defined in proposition 2 (and is a multiple of Y_1, making the ODE for $\tau(\theta)$ well-defined). Since everything is smooth, there must be local existence of $\tau(\theta)$. Moreover, suppose $\tau(0) = t$ for $|t| \leq \delta$. If s is sufficiently small and p is in some sufficiently small neighborhood of a point m, there exists $C < \infty$ for which $|\dot{\tau}(\theta)| \leq C\delta$ since $Y_{i,j}^0 = 0$; in fact, if s is sufficiently small, the constant C may also be taken small. Thus, the solution $\tau(\theta)$ must exist until at least $\theta = 1$. Now let f be a smooth function on X, and consider the derivative with respect to θ of the function $f(\pi_X(\Psi(\tau(\theta), (1-\theta)s, p)))$. Exploiting the ODE which $\tau(\theta)$ satisfies, it follows that

$$\frac{d}{d\theta} f(\pi_X(\Psi(\tau(\theta), (1-\theta)s, p))) = -\sum_{i,j} s_{i,j} \tau(\theta)^i d\pi_X(V_{i,j}^{\tau(\theta)}|_{\Psi(\tau(\theta),(1-\theta)s,p)})(f).$$

Already it is known that $|\tau(\theta)| \leq C\delta$ when $|\tau(0)| \leq \delta$. Furthermore, if $|s_{i,j}| \leq \delta^{\max\{0,k+1-i\}}$, it must be the case that $|\frac{d}{d\theta} f(\pi_X(\Psi(\tau(\theta), (1-\theta)s, p)))| \leq C_f \delta^{k+1}$ for some C_f which depends on f, but is independent of t and s (provided they are sufficiently small) and p (provided it is sufficiently near m). Integrating the derivative from $\theta = 0$ to $\theta = 1$ gives

$$\left| f(\pi_X(e^{\tau(1)Y_1}p)) - f(\pi_X(e^{tY_1}e^{\sum_{i,j} s_{i,j}X_{i,j}}p)) \right| \leq C_f \delta^{k+1}. \tag{17}$$

If $p := e^{\sum_l u_l Z_l} m$ for $|u| \leq \delta^{k+1}$, it follows that

$$\left| f(\pi_X(e^{\tau(1)Y_1}m)) - f(\pi_X(e^{tY_1}e^{\sum_{i,j} s_{i,j}X_{i,j}}e^{\sum_l u_l Z_l}m)) \right| \leq C_f \delta^{k+1} \tag{18}$$

as well. Since $|\tau(1)| \leq C\delta$, equations (17) and (18) combined show that, if one chooses a coordinate system defined near $\pi_X(m)$, then $\pi_x(O_\delta)$ must be contained within a $C\delta^{k+1}$-neighborhood of the curve segment $\pi_X(e^{tY_1}m)$ with $|t| \leq C\delta$ for some C. Clearly, then, it must be the case that $|\pi_X(O_\delta)| \lesssim \delta^{1+(k+1)(d_X-1)}$.

If the equation $|O_\delta| \leq C|\pi_X(O_\delta)|^{\frac{1}{p}}|\pi_Y(O_\delta)|^{\frac{1}{q'}}$ is to be satisfied for some finite constant C, it must be the case that

$$1 \geq \frac{1+(k+1)(d_X-1)}{p} - \frac{(k+1)(k+2)(d_X-1)}{2q}$$

where $\frac{1}{q} + \frac{1}{q'} = 1$. Since (M, X, Y, π_X, π_Y) is also nondegenerate through order j for $j < k$ (by property 1 of lemma 1), it follows that

$$1 \geq \frac{1+(j+1)(d_X-1)}{p} - \frac{(j+1)(j+2)(d_X-1)}{2q}$$

for $0 \leq j \leq k$. These constraints give precisely the necessity portion of theorem 1.

6. Remarks

(1) If one is in the special case $d_Y = (k+1)(d_X-1)$ and (M, X, Y, π_X, π_Y) is nondegenerate through order k, then the region of boundedness indicated by theorem 1 is, in fact, valid for all $1 \leq p \leq \infty$. This can be seen by a computation, analogous to the one just completed, for which F_δ is a ball of radius δ and $O_\delta = \pi_Y^{-1}(F_\delta)$.

(2) Recall the prototype Radon-like transform (1). In proving (12), it was shown that all nondegenerate Radon-like transforms (6) behave locally like the prototype (possibly modulo a few extra parameters).

(3) The fact that theorems 1 and 2 are local results is essential. The prototype operator R_k possesses two scaling symmetries, corresponding to scalings $(t,s) \mapsto (\delta t, s)$ and $(t,s) \mapsto (t, \mu s)$ on $\mathbb{R} \times \mathbb{R}^n$, which together dictate that R_k is only of restricted weak-type (p_k, q_k) globally, where $(\frac{1}{p_k}, \frac{1}{q_k'})$ is the k-th special point of $\mathcal{C}_k$.

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Mathematics Department, Yale University, 10 Hillhouse Avenue, New Haven, CT 06520
E-mail address: `philip.gressman@yale.edu`