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CLOSURE RELATIONS FOR TOTALLY NONNEGATIVE CELLS IN G/P.

K. Rietsch

Abstract. The totally nonnegative part of a partial flag variety G/P has been shown in [11, 10] to be a union of algbraic cells. We show that the closure of a cell is a union of cells and give a combinatorial description of the closure relations. The totally nonnegative cells are defined by intersecting $(G/P)_{\geq 0}$ with a certain stratification of G/P defined by Lusztig [7]. We also verify the same closure relations for these strata.

1. Introduction

For a reductive algebraic group over $\mathbb{C}$ split over $\mathbb{R}$ with fixed choice of Chevalley generators in the Lie algebra, there is a well defined notion of positive, or $\mathbb{R}_{>0}$-valued, points due to Lusztig [5]. In the case of GL_n with the standard choices the resulting "$GL_n(\mathbb{R}_{>0})$" recovers the classical notion of totally positive matrices, that is matrices all of whose minors are in $\mathbb{R}_{>0}$. For general G the set $G(\mathbb{R}_{>0})$, or $G_{>0}$ as we will denote it, is therefore called the totally positive part of G. The closure $G_{\geq 0}$ of $G_{>0}$ (in the real topology) is called the totally nonnegative part of G.

These notions extend in a natural way to flag varieties G/P, [5, 6]. That is, one has a notion of $(G/P)_{>0}$ – this is a semi-algebraic subset of the real points in G/P – and of $(G/P)_{\geq 0}$, the closure of $(G/P)_{>0}$. Now recall that G/B has a decomposition into smooth strata $\mathcal{R}_{v,w}$, obtained as intersections of opposed Bruhat cells and indexed by pairs v, w in the Weyl group with $v \leq w$. In [7] Lusztig defined an analogous decomposition of G/P into smooth strata $\mathcal{P}_{x,u,w}$. These decompositions intersected with the $(G/P)_{\geq 0}$ give cell decompositions of the totally nonnegative parts of the G/P, [11, 10]. We call the components of this decomposition of $(G/P)_{\geq 0}$ the totally nonnegative cells in G/P. Note that there is one open totally nonnegative cell in G/P, namely $(G/P)_{>0}$ itself.

It was proved in [6] that $(G/B)_{\geq 0}$ is contractible, and the same holds for $(G/P)_{\geq 0}$ by the same proof. Also Fomin and Shapiro [1] studied links of totally nonnegative cells inside a big cell of SL_n/B, in particular showing them to be contractible. Beyond these special cases, however, not much is known about the closures of the individual cells or how the cells are glued together.

In this paper we prove that the closure of a totally nonnegative cell is a union of totally nonnegative cells and describe the closure relations in terms of the Weyl group. In the full flag variety case we show that $\mathcal{R}^{>0}_{v',w'} \subseteq \overline{\mathcal{R}^{>0}_{v,w}}$, whenever $v \leq v' \leq w' \leq w$,

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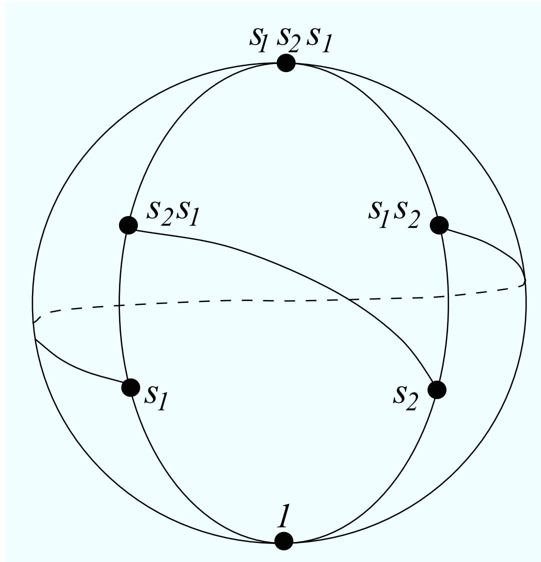

FIGURE 1. $(SL_3/B)_{\geq 0}$ with its cell decomposition

see Theorem 4.1. This theorem is then generalized to G/P (see Theorem 6.1), with the combinatorial description of the cells and their closure relations given in Section 6.

A main difficulty in proving such results lies in how to find a totally nonnegative cell $\mathcal{R}_{v,w}^{>0}$ inside the closure of another. While much detailed information is available about the individual cells, such as parameterizations, explicit defining equalities/inequalities (see [9]), none of these results readily extend to the closures of the cells. The central idea which allows us to relate this problem to an easier special case is contained in Lemma 4.3.

The combinatorial properties of our poset describing the closure relations between totally nonnegative cells were recently investigated by Williams [14]. Her results suggest, and she conjectures, that $(G/P)_{\geq 0}$ is a regular CW complex homeomorphic to a closed ball. For an illustration of the totally nonnegative part with its cell decomposition in the case of SL_3/B see Figure 1.

In the last section we verify the same closure relations as among the totally nonnegative cells, for the strata $\mathcal{P}_{x,u,w}$ of G/P. In this case G/P is either taken again over $\mathbb{R}$, or over an algebraically closed field $\mathbb{K}$ (and Zariski topology). The closure relations among the totally nonnegative cells could in retrospect be viewed as an $\mathbb{R}_{>0}$-valued analogue of Proposition 7.2, although the results involving positivity are more difficult to prove.

Lusztig's stratification of G/P (over $\mathbb{C}$) has recently been reinterpreted by Goodearl and Yakimov [2] in a Poisson geometric setting. Namely the strata arise as torus-orbits of symplectic leaves for a certain natural Poisson structure on G/P. Their paper also independently gives closure relations among these strata [2, Theorem 1.8] which look quite different from our Proposition 7.2. The combinatorial equivalence of their description of the poset structure with ours was recently proved by Xuhua He [13] using arguments from [3].

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2. Preliminaries

2.1. We recall some basic notation and results from algebraic groups, see e.g. [12]. Suppose $\mathbb{K}$ is an algebraically closed field, $\mathbb{K} = \overline{\mathbb{K}}$, or $\mathbb{K} = \mathbb{R}$. Let G be a semisimple linear algebraic group over $\overline{\mathbb{K}}$ split over $\mathbb{K}$. We identify G and any related spaces with their $\mathbb{K}$-valued points. If $\mathbb{K} = \mathbb{R}$ we consider them with their real topology (as real manifolds or subsets thereof), otherwise we consider their Zariski topology.

Let T be a split torus and B^+ and B^- opposite Borel subgroups containing T. The unipotent radicals of B^+ and B^- are denoted U^+ and U^-, respectively. Let $\{\alpha_i \mid i \in I\}$ be the set of simple roots associated to B^+ and $\{\alpha_i^\vee \mid i \in I\}$ the corresponding coroots. Then we have the simple root subgroups $U^+_{\alpha_i} \subseteq U^+$ and $U^-_{\alpha_i} \subseteq U^-$. Furthermore assume we are given homomorphisms

$$\phi_i : SL_2(\mathbb{K}) \to G, \qquad i \in I,$$

such that

$$\phi_i\left(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}\right) = \alpha_i^\vee(t), \qquad t \in \mathbb{K}^*,$$

and such that

$$\phi_i\left(\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}\right) := x_i(m), \quad \phi_i\left(\begin{pmatrix} 1 & 0 \\ m & 1 \end{pmatrix}\right) := y_i(m),$$

define isomorphisms $x_i : \mathbb{K} \to U^+_{\alpha_i}$ and $y_i : \mathbb{K} \to U^-_{\alpha_i}$.

Let $W = N_G(T)/T$ be the Weyl group of G. For $i \in I$ the elements

$$\dot{s}_i = x_i(-1)y_i(1)x_i(-1)$$

represent the simple reflections $s_i \in W$. If $w = s_{i_1} \dots s_{i_m}$ is a reduced expression for w then we write $\ell(w) = m$ for the length of w. It is also known that the representative

$$\dot{w} = \dot{s}_{i_1} \dots \dot{s}_{i_m}$$

of w is well defined, independent of the choice of reduced expression. Inside W we have a longest element which is denoted by w_0.

2.2. Let $J \subseteq I$. The parabolic subgroup $W_J \subseteq W$ is the subgroup generated by all of the s_j with $j \in J$. Let w_J denote the longest element in W_J. We also consider the set W^J of minimal coset representatives for W/W_J, and the set $W^J_{max} = W^J w_J$ of maximal coset representatives.

The parabolic subgroup W_J of W corresponds to a parabolic subgroup P_J in G containing B^+. Namely, P_J is the subgroup of G generated by B^+ and the elements $\dot{w}$ for $w \in W_J$. Let $\mathcal{P}^J$ be the set of parabolic subgroups P conjugate to P_J. This is a homogeneous space for the conjugation action of G and can be identified with the partial flag variety G/P_J via

$$G/P_J \xrightarrow{\sim} \mathcal{P}^J \ : \ gP_J \mapsto gP_Jg^{-1}.$$

In the case $J = \emptyset$ we are identifying the full flag variety G/B^+ with the variety $\mathcal{B}$ of Borel subgroups in G. We have the usual projection from the full flag variety to any partial flag variety which takes the form $\pi = \pi^J : \mathcal{B} \to \mathcal{P}^J$, where $\pi(B)$ is the unique parabolic subgroup of type J containing B.

The conjugate of a parabolic subgroup P by an element $g \in G$ will be denoted by $g \cdot P := gPg^{-1}$.

2.3. Recall the Bruhat decomposition for the full flag variety,

$$\mathcal{B} = \bigsqcup_{w \in W} B^+ \dot{w} \cdot B^+,$$

and the Bruhat order $\leq$ on W. The Bruhat cell $B^+\dot{w} \cdot B^+$ is isomorphic to $\mathbb{K}^{\ell(w)}$. And the Bruhat order has the property

$$v \leq w \quad \Longleftrightarrow \quad B^+\dot{v} \cdot B^+ \subseteq \overline{B^+\dot{w} \cdot B^+},$$

for $v, w \in W$.

It is a well known consequence of Bruhat decomposition that $\mathcal{B} \times \mathcal{B}$ is the union of the G-orbits $\mathcal{O}(w) = G \cdot (B^+, \dot{w} \cdot B^+)$, with G acting diagonally. Therefore to any pair (B_1, B_2) of Borel subgroups one can associate a unique $w \in W$ such that

$$(B_1, B_2) = (g \cdot B^+, g\dot{w} \cdot B^+)$$

for some $g \in G$. We write

$$B_1 \xrightarrow{w} B_2$$

in this case and call w the relative position of B_1 and B_2.

2.4. Finally, let us consider the two opposite Bruhat decompositions

$$\mathcal{B} = \bigsqcup_{w \in W} B^+ \dot{w} \cdot B^+ = \bigsqcup_{v \in W} B^- \dot{v} \cdot B^+.$$

Note that $B^-\dot{v} \cdot B^+ \cong \mathbb{K}^{\ell(w_0) - \ell(v)}$. The closure relations for these opposite Bruhat cells are given by $B^-\dot{v}' \cdot B^+ \subset \overline{B^-\dot{v} \cdot B^+}$ if and only if $v \leq v'$. We define

$$\mathcal{R}_{v,w} := B^+\dot{w} \cdot B^+ \cap B^-\dot{v} \cdot B^+,$$

the intersection of opposed Bruhat cells. This intersection is empty unless $v \leq w$, in which case it is smooth of dimension $\ell(w) - \ell(v)$, see [4, 7].

3. Total Positivity for G and $\mathcal{B}$

Let $\mathbb{K} = \mathbb{R}$. The totally nonnegative part $G_{\geq 0}$ of G is defined by Lusztig [5] to be the semigroup inside G generated by the sets

$$\begin{aligned} &\{x_i(t) \mid t \in \mathbb{R}_{>0}, i \in I\}, \\ &\{y_i(t) \mid t \in \mathbb{R}_{>0}, i \in I\}, \text{ and} \\ &T_{>0} := \{t \in T \mid \chi(t) > 0 \text{ all } \chi \in X^*(T)\}. \end{aligned}$$

When $G = SL_n(\mathbb{R})$ then by a Theorem of A. Whitney's this definition agrees with the classical notion of totally nonnegative matrices inside $SL_n(\mathbb{R})$, that is those matrices all of whose minors are nonnegative.

3.1. We recall some basic facts about total positivity for G from [5]. Let $U^+_{\geq 0} := G_{\geq 0} \cap U^+$ and $U^-_{\geq 0} := G_{\geq 0} \cap U^-$. For $w \in W$ and $s_{i_1} \dots s_{i_m} = w$ a reduced expression define

$$\begin{aligned} U^+(w) &:= \{x_{i_1}(t_1)x_{i_2}(t_2)\dots x_{i_m}(t_m) \mid t_i \in \mathbb{R}_{>0}\}, \\ U^-(w) &:= \{y_{i_1}(t_1)y_{i_2}(t_2)\dots y_{i_m}(t_m) \mid t_i \in \mathbb{R}_{>0}\}. \end{aligned}$$

These sets are independent of the chosen reduced expression and give

$$\begin{aligned} U^+(w) &= U^+_{\geq 0} \cap B^- \dot{w} B^-, \\ U^-(w) &= U^-_{\geq 0} \cap B^+ \dot{w} B^+. \end{aligned}$$

In particular $U^+_{\geq 0} = \bigsqcup_{w \in W} U^+(w)$ and $U^-_{\geq 0} = \bigsqcup_{w \in W} U^-(w)$. Moreover $U^+(w)$ and $U^-(w)$ are isomorphic to $\mathbb{R}^{\ell(w)}_{>0}$ using the t_i as coordinates.

Suppose $v \leq w$ then $U^+(v)$ can be obtained from $U^+(w)$ by letting certain of the t_i coordinates tend to zero. So $U^+(v)$ lies in the closure of $U^+(w)$. Moreover the condition $v \leq w$ is necessary by the analogous property of the Bruhat decomposition. The same goes for the $U^-(w)$. Since $U^+_{\geq 0}$ and $U^-_{\geq 0}$ are closed in G by [5, Proposition 4.2], we have

$$\overline{U^+(w)} = \bigsqcup_{v \leq w} U^+(v), \qquad \overline{U^-(w)} = \bigsqcup_{v \leq w} U^-(w).$$

Note that in particular $\overline{U^+(w_0)} = U^+_{\geq 0}$ and $\overline{U^-(w_0)} = U^-_{\geq 0}$. The totally positive parts for U^+ and U^- are defined by

$$U^+_{>0} := U^+(w_0), \qquad U^-_{>0} := U^-(w_0).$$

3.2. The totally positive and totally nonnegative parts of the flag variety $\mathcal{B}$ are defined by

$$\begin{aligned} \mathcal{B}_{>0} &:= \{y \cdot B^+ \mid y \in U^-_{>0}\}, \\ \mathcal{B}_{\geq 0} &:= \overline{\mathcal{B}_{>0}}. \end{aligned}$$

By [5, Theorem 8.7] $\mathcal{B}_{>0}$ can be described in a symmetric way as

$$\mathcal{B}_{>0} = \{x \cdot B^- \mid x \in U^+_{>0}\}.$$

In other words $\mathcal{B}_{>0}$ is invariant under the automorphism of G (and hence $\mathcal{B}$) which swaps the $x_i(t)$ and the $y_i(t)$.

3.3. The set $\mathcal{B}_{\geq 0}$ again has a cell decomposition, which was conjectured in [5] and proved in [11]. This result was also proved again in [9] and with explicit descriptions of the cells given. We recall the construction from [9] below.

Let $v \leq w$ and let $\mathbf{w} = (i_1, \dots, i_m)$ encode a reduced expression $s_{i_1} \dots s_{i_m}$ for w. Then there exists a unique subexpression $s_{i_{j_1}} \dots s_{i_{j_k}}$ for v in $\mathbf{w}$ with the property that, for $l = 1, \dots, k$,

$$s_{i_{j_1}} \dots s_{i_{j_l}} s_{i_r} > s_{i_{j_1}} \dots s_{i_{j_l}} \quad \text{whenever } j_l < r \leq j_{l+1},$$

where $j_{k+1} := m$. It is the rightmost reduced subexpression for v in $\mathbf{w}$ and we denote it by $\mathbf{v} = (j_1, \dots, j_k)$.

Then we define

$$\mathcal{R}_{v,w}^{>0} := \left\{ g_1 \dots g_m \cdot B^+ \;\middle|\; \text{where } g_r = \begin{cases} \dot{s}_{i_r}, & \text{if } r \in \{j_1, \dots, j_k\}, \\ y_{i_r}(t_r),\ t_r \in \mathbb{R}_{>0} & \text{otherwise.} \end{cases} \right\}$$

By [9, Theorem 11.3] we have that this definition is independent of the reduced expression for w, and

$$\mathcal{R}_{v,w}^{>0} = \mathcal{R}_{v,w} \cap \mathcal{B}_{\geq 0}.$$

Moreover the $\mathcal{R}_{v,w}^{>0}$ are isomorphic to $\mathbb{R}_{>0}^{\ell(w)-\ell(v)}$. So this gives an explicit decomposition of $\mathcal{B}_{\geq 0}$ into cells.

We also write $(B^+\dot{w} \cdot B^+)_{\geq 0}$ for the intersection of the Bruhat cell with $\mathcal{B}_{\geq 0}$. Then of course

$$(B^+\dot{w} \cdot B^+)_{\geq 0} = \bigsqcup_{v;\ v \leq w} \mathcal{R}_{v,w}^{>0}.$$

If $v = 1$ then we have $\mathcal{R}_{1,w} = U^-(w) \cdot B^+$, and the above decomposition of $\mathcal{B}_{\geq 0}$ extends Lusztig's cell decomposition of $U^-_{\geq 0} \cong U^-_{\geq 0} \cdot B^+$.

3.4. We can apply the symmetry from Section 3.2 to get an alternate description for the $\mathcal{R}_{v,w}^{>0}$. Namely let

$$\tilde{\mathcal{R}}_{v,w}^{>0} := \left\{ g_1 \dots g_m \cdot B^- \;\middle|\; \text{where } g_r = \begin{cases} \dot{s}_{i_r}^{-1}, & \text{if } r \in \{j_1, \dots, j_k\}, \\ x_{i_r}(t_r),\ t_r \in \mathbb{R}_{>0} & \text{otherwise.} \end{cases} \right\}$$

with the same notation as in Section 3.3. Then it follows that

$$\tilde{\mathcal{R}}_{v,w}^{>0} = \left(B^+\dot{v} \cdot B^- \cap B^-\dot{w} \cdot B^-\right) \cap \mathcal{B}_{\geq 0} = \mathcal{R}_{ww_0,vw_0}^{>0}.$$

3.5. Suppose $w, w_1, w_2 \in W$ with $w = w_1 w_2$ and such that the lengths add, $\ell(w) = \ell(w_1) + \ell(w_2)$. Then there is a well defined map

$$\begin{array}{rcl} \pi_{w_1}^w : B^+\dot{w} \cdot B^+ & \to & B^+\dot{w}_1 \cdot B^+ \\ B = z\dot{w} \cdot B^+ & \mapsto & z\dot{w}_1 \cdot B^+ \end{array}$$

where $z \in U^+$. We call $\pi_{w_1}^w$ a reduction map. The element $\pi_{w_1}^w(B)$ is uniquely determined by the property

$$B^+ \xrightarrow{w_1} \pi_{w_1}^w(B) \xrightarrow{w_2} B.$$

It was proved in [11] that $\pi_{w_1}^w$ preserves total positivity. That is, if $B = z\dot{w} \cdot B^+ \in \mathcal{B}_{\geq 0}$ then $\pi_{w_1}^w(B)$ lies in $\mathcal{B}_{\geq 0}$.

Now let $B = g_1 \dots g_m \cdot B^+ \in \mathcal{R}_{v,w}^{>0}$ where the factors g_i and all the notation are as in Section 3.3. And additionally suppose the reduced expression for w is a product of reduced expressions for w_1 and w_2, so $w_1 = s_{i_1} \dots s_{i_{m'}}$ and $w_2 = s_{i_{m'+1}} \dots s_{i_m}$. Then we have explicitly

$$\pi_{w_1}^w(g_1 g_2 \dots g_m \cdot B^+) = g_1 \dots g_{m'} \cdot B^+.$$

In particular the reduction map restricts to a map $\pi_{w_1}^w : \mathcal{R}_{v,w}^{>0} \to \mathcal{R}_{v_{(m')},w_1}^{>0}$ where $v_{(m')} = s_{i_{j_1}} \dots s_{i_{j_p}}$ for $j_p \leq m' < j_{p+1}$.

4. Closure relations for the cells in $\mathcal{B}_{\geq 0}$

The aim of this section is to prove the following theorem.

Theorem 4.1. *Let $v, w \in W$. Then*

$$\overline{\mathcal{R}^{>0}_{v,w}} = \bigsqcup_{v \leq v' \leq w' \leq w} \mathcal{R}^{>0}_{v',w'}.$$

We begin with an easy special case.

Lemma 4.2. *For $v \in W$,*

$$\overline{\mathcal{R}^{>0}_{v,w_0}} \cap B^+ \dot{w}_0 \cdot B^+ = \bigsqcup_{v' \geq v} \mathcal{R}^{>0}_{v',w_0}.$$

Proof. By Section 3.4 we can rewrite

$$\mathcal{R}^{>0}_{v',w_0} = \tilde{\mathcal{R}}^{>0}_{1,v'w_0} = U^+(v'w_0)\dot{w}_0 \cdot B^+.$$

Now $v' \geq v$ implies $v'w_0^{-1} \leq vw_0$. Hence by Section 3.1 we have the inclusion $U^+(v'w_0) \subseteq \overline{U^+(vw_0)}$, and therefore $\mathcal{R}^{>0}_{v',w_0} \subseteq \overline{\mathcal{R}^{>0}_{v,w_0}}$. The opposite inclusion follows from the closure relations of Bruhat decomposition. □

Lemma 4.3. *There is a homeomorphism*

$$\phi = \phi_w : U^-(w_0 w^{-1}) \times (B^+ \dot{w} \cdot B^+)_{\geq 0} \xrightarrow{\sim} \bigsqcup_{v \leq w} \mathcal{R}^{>0}_{v,w_0}$$

defined by $\phi(u, B) := u \cdot B$. Moreover

$$\phi_w(U^-(w_0 w^{-1}) \times \mathcal{R}^{>0}_{v,w}) = \mathcal{R}^{>0}_{v,w_0}.$$

Proof. Choose a reduced expression $\mathbf{w}_0 = (i_1, \ldots, i_N)$ for w_0 such that $(i_1, \ldots i_r)$ is a reduced expression for $w_0 w^{-1}$. Then using the parameterizations of $U^-(w_0 w^{-1})$ and $\mathcal{R}^{>0}_{v,w_0}$ described in Sections 3.1 and 3.3, respectively, we see that

$$\begin{array}{rcl} U^-(w_0 w^{-1}) \times \mathcal{R}^{>0}_{v,w} & \xrightarrow{\sim} & \mathcal{R}^{>0}_{v,w_0}, \\ (u, B) & \mapsto & u \cdot B. \end{array}$$

is an isomorphism. Explicitly we have

$$\mathcal{R}_{v,w_0} = \left\{ y_{i_1}(t_1) \ldots y_{i_r}(t_r) g_{r+1} \ldots g_N \cdot B^+ \;\middle|\; g_{r+1} \ldots g_N \cdot B^+ \in \mathcal{R}^{>0}_{v,w} \right\}$$

Now applying the reduction map $\pi^{w_0}_{w_0 w^{-1}}$ from Section 3.5 we get

$$\begin{array}{rcl} \mathcal{R}^{>0}_{v,w_0} & \longrightarrow & \mathcal{R}^{>0}_{1,w_0 w^{-1}} \\ y_{i_1}(t_1) \ldots y_{i_r}(t_r) g_{r+1} \ldots g_N \cdot B^+ & \mapsto & y_{i_1}(t_1) \ldots y_{i_r}(t_r) \cdot B^+ \end{array}$$

Note that $\pi^{w_0}_{w_0 w^{-1}}$ is defined on the whole Bruhat cell $B^+ \dot{w}_0 \cdot B^+$. We can therefore combine these maps for varying v and compose with the isomorphism $U^- \cdot B^+ \xrightarrow{\sim} U^-$, to get

$$p_1 : \bigsqcup_{v \leq w} \mathcal{R}^{>0}_{v,w_0} \to U^-(w_0 w^{-1}).$$

The inverse to ϕ is now given by

$$\begin{aligned}\psi \ : \ \bigsqcup_{v\le w}\mathcal{R}^{>0}_{v,w_0} &\ \to\ U^-(w_0w^{-1})\times(B^+\dot w\cdot B^+)_{\ge 0}\\ B &\ \mapsto\ \big(p_1(B), p_1(B)^{-1}\cdot B\big).\end{aligned}$$

□

Lemma 4.4. *Let $v\le v'\le w\in W$. Then*

$$\mathcal{R}^{>0}_{v',w}\subseteq\overline{\mathcal{R}^{>0}_{v,w}}.$$

Proof. Using Lemma 4.2 we see that

$$\mathcal{R}^{>0}_{v',w_0}\subset\overline{\mathcal{R}^{>0}_{v,w_0}}\cap\bigsqcup_{x\le w}\mathcal{R}^{>0}_{x,w_0}.$$

Applying $\psi:=\phi^{-1}$ from Lemma 4.3 to both sides of this inclusion gives

$$\psi(\mathcal{R}^{>0}_{v',w_0})\subset\psi(\overline{\mathcal{R}^{>0}_{v,w_0}}\cap\bigsqcup_{x\le w}\mathcal{R}^{>0}_{x,w_0})\subset\overline{\psi(\mathcal{R}^{>0}_{v,w_0})}.$$

Therefore

$$U^-(w_0w^{-1})\times\mathcal{R}^{>0}_{v',w}=\psi(\mathcal{R}^{>0}_{v',w_0})\subset\overline{\psi(\mathcal{R}^{>0}_{v,w_0})}=\overline{U^-(w_0w^{-1})\times\mathcal{R}^{>0}_{v,w}},$$

where we may take the closure on the right hand side to be the closure inside the domain of ϕ, that is inside $U^-(w_0w^{-1})\times(B^+\dot w\cdot B^+)_{\ge 0}$. It follows that $\mathcal{R}^{>0}_{v',w}\subseteq\overline{\mathcal{R}^{>0}_{v,w}}$. □

Proof of Theorem 4.1. By [5] $\mathcal{B}_{\ge 0}$ is symmetric with respect to interchanging B^+ and B^-, see Section 3.2. Therefore it follows that Lemma 4.4 also holds for the $\tilde{\mathcal{R}}^{>0}_{v,w}$ defined in Section 3.4. That is, $\tilde{\mathcal{R}}^{>0}_{v',w}\subseteq\overline{\tilde{\mathcal{R}}^{>0}_{v,w}}$ whenever $v\le v'\le w$. Now if $v\le v'\le w'\le w$, then

$$\mathcal{R}^{>0}_{v',w'}\subseteq\overline{\mathcal{R}^{>0}_{v',w}}=\overline{\tilde{\mathcal{R}}^{>0}_{ww_0,v'w_0}}\subseteq\overline{\tilde{\mathcal{R}}^{>0}_{ww_0,vw_0}}=\overline{\mathcal{R}^{>0}_{v,w}}.$$

This shows the inclusion $\supseteq$ in the statement of Theorem 4.1.

The other inclusion is clear from the closure relations of Bruhat decomposition. $\mathcal{R}^{>0}_{v',w'}\cap\overline{\mathcal{R}^{>0}_{v,w}}\neq\emptyset$ implies on the one hand $\mathcal{R}^{>0}_{v',w'}\cap\overline{B^-\dot v\cdot B^+}\neq\emptyset$ and on the other hand $\mathcal{R}^{>0}_{v',w'}\cap\overline{B^+\dot w\cdot B^+}\neq\emptyset$. So we must have $v\le v'$ and $w'\le w$. □

We note that Theorem 4.1 implies $\overline{\mathcal{R}_{v,w}}\cap\mathcal{B}_{\ge 0}=\overline{\mathcal{R}^{>0}_{v,w}}$.

5. Lusztig's decomposition of $\mathcal{P}^J$

The stratification of $\mathcal{B}$ into smooth pieces $\mathcal{R}_{v,w}$ has an analogue for partial flag varieties introduced by Lusztig in [7].

Consider a triple of Weyl group elements $x,u,w\in W$ with $x\in W^J_{max}$, $w\in W^J$ and $u\in W_J$. Then $\mathcal{P}^J_{x,u,w}\subset\mathcal{P}^J$ is defined as the set of all $P\in\mathcal{P}^J$ such that there exist Borel subgroups B_L and B_R inside P satisfying

$$B^+\xrightarrow{w}B_L\xrightarrow{u}B_R\xrightarrow{x^{-1}w_0}B^-.$$

An equivalent characterization of $\mathcal{P}^J_{x,u,w}$ is

$$\mathcal{P}^J_{x,u,w} = \pi^J(\mathcal{R}_{x,wu}) = \pi^J(\mathcal{R}_{xu^{-1},w}).$$

It is not hard to see that B_L and B_R are uniquely determined as the Borel subgroups in P 'closest to' B^+ respectively B^- with regard to their relative position, and the projection maps $\mathcal{R}_{x,wu} \to \mathcal{P}^J_{x,u,v}$ and $\mathcal{R}_{xu^{-1},w} \to \mathcal{P}^J_{x,u,v}$ are isomorphisms. In particular $\mathcal{P}^J_{x,u,w}$ is nonempty if and only if $x \leq wu$, in which case it is smooth of dimension $\ell(w) + \ell(u) - \ell(x)$.

Let us denote the indexing set for this decomposition of $\mathcal{P}^J$ by Q^J. So

$$Q^J := \{(x,u,w) \in W^J_{max} \times W_J \times W^J \mid x \leq wu\}.$$

We define the following partial oder on Q^J.

Definition 5.1. Let (x',u',w') and (x,u,w) in Q^J. Then define

$$(x',u',w') \leq (x,u,w)$$

if and only if there exist $u'_1, u'_2 \in W_J$ satisfying $u'_1 u'_2 = u'$ with $\ell(u'_1) + \ell(u'_2) = \ell(u')$, and such that

$$xu^{-1} \leq x' {u'_2}^{-1} \leq w'u'_1 \leq w. \tag{5.1}$$

6. Totally nonnegative cells in $\mathcal{P}^J$ and their closure relations

The totally positive and nonnegative parts of $\mathcal{P}^J$ are defined in [6] by

$$\begin{aligned} \mathcal{P}^J_{>0} &= \pi^J(\mathcal{B}_{>0}), \\ \mathcal{P}^J_{\geq 0} &= \pi^J(\mathcal{B}_{\geq 0}). \end{aligned}$$

Since π^J is closed it follows that $\mathcal{P}^J_{\geq 0} = \overline{\mathcal{P}^J_{>0}}$.

We decompose $\mathcal{P}^J_{\geq 0}$ by intersecting it with the strata $\mathcal{P}^J_{x,u,w}$ from Section 5. From the definitions and the fact that reduction preserves total positivity it follows that ([10, Lemma 3.2])

$$\mathcal{P}^J_{x,u,w;>0} := \mathcal{P}^J_{x,u,w} \cap \mathcal{P}^J_{\geq 0} = \pi^J(\mathcal{R}^{>0}_{x,wu}) = \pi^J(\mathcal{R}^{>0}_{xu^{-1},w}).$$

Keeping in mind that $\pi^J : \mathcal{R}_{x,wu} \to \mathcal{P}^J_{x,u,w}$, say, is an isomorphism, we see that

$$\mathcal{P}^J_{x,u,w;>0} \cong \mathcal{R}^{>0}_{x,wu} \cong \mathbb{R}^{\ell(w)+\ell(u)-\ell(x)}_{>0},$$

for any triple $(x,u,w) \in Q^J$.

We will prove the following theorem describing the closure relations between the strata $\mathcal{P}^J_{x,u,w;>0}$ in $\mathcal{P}^J_{\geq 0}$.

Theorem 6.1. *Let $(x,u,w) \in Q^J$ with partial order as in Definition 5.1, then*

$$\overline{\mathcal{P}^J_{x,u,w;>0}} = \bigsqcup_{(x',u',w') \leq (x,u,w)} \mathcal{P}^J_{x',u',w';>0}.$$

Proof. Note first that we have $\overline{\mathcal{P}^J_{x,u,w;>0}} = \pi^J(\overline{\mathcal{R}^{>0}_{xu^{-1},w}}) = \pi^J(\overline{\mathcal{R}^{>0}_{x,wu}})$.

Now suppose $(x',u',w') \le (x,u,w)$. So we have u_1' and u_2' as in Definition 5.1. Let $P' \in \mathcal{P}^{>0}_{x',u',w'}$ with its associated Borel subgroups $B_L' \in \mathcal{R}^{>0}_{x'u'^{-1},w'}$ and $B_R' \in \mathcal{R}^{>0}_{x',w'u'}$ such that $B_L' \subset P'$ and $B_R' \subset P'$. In particular

$$B^+ \xrightarrow{w'} B_L' \xrightarrow{u'} B_R' \xrightarrow{x'^{-1}w_0} B^-.$$

Let $B' := \pi^{w'u'}_{w'u_1'}(B_R')$ using the reduction map from Section 3.5. Then we have

$$B^+ \xrightarrow{w'} B_L' \xrightarrow{u_1'} B' \xrightarrow{u_2'} B_R' \xrightarrow{x'^{-1}w_0} B^-.$$

Since reduction preserves total positivity we have $B' \in \mathcal{R}^{>0}_{x'u_2'^{-1},w'u_1'}$. Also $\pi^J(B_L') = P'$ and the fact that (B_L', B') have relative position $u_1' \in W_J$ implies that $\pi(B') = P'$. Now by (5.1) together with Theorem 4.1 it follows that $\mathcal{R}^{>0}_{x'u_2'^{-1},w'u_1'} \subset \overline{\mathcal{R}^{>0}_{xu^{-1},w}}$. Therefore $P' = \pi(B') \in \pi(\overline{\mathcal{R}^{>0}_{xu^{-1},w}}) = \overline{\mathcal{P}^J_{x,u,w;>0}}$. This proves the inclusion $\supseteq$.

For the opposite inclusion suppose that $\mathcal{P}_{x',u',w';>0} \cap \overline{\mathcal{P}_{x,u,w;>0}} \neq \emptyset$. So let P' be an element of this intersection. Then there exists a $\tilde{B}' \in \overline{\mathcal{R}^{>0}_{xu^{-1},w}}$ such that $\pi(\tilde{B}') = P'$. Since $\tilde{B}' \subset P'$ we have $u_1, u_2 \in W_J$ such that

$$\begin{array}{ccccccc} B^+ & \xrightarrow{w'} & B_L' & & \xrightarrow{u'} & & B_R' & \xrightarrow{(x')^{-1}w_0} & B^- \\ & & u_1 & \searrow & & \nearrow & u_2 & & \\ & & & & \tilde{B}' & & & & \end{array}.$$

If it happens to be the case that $\ell(u_1u_2) = \ell(u_1) + \ell(u_2)$ then $u' = u_1u_2$. So we can set $u_1' = u_1$ and $u_2' = u_2$ and are done. Otherwise there exists a simple reflection $s = s_{i_1} \in W_J$ such that $u_1 s \le u_1$ and $su_2 \le u_2$. We have

$$\begin{array}{ccccccccc} B_L' & & & & \xrightarrow{u'} & & & & B_R' \\ u_1 s & \searrow & & & & & & \nearrow & su_2 \\ & & B_{L,1}' & & \xrightarrow{s \text{ or } 1} & & B_{R,1}' & & \\ & & s & \searrow & & \nearrow & s & & \\ & & & & \tilde{B}' & & & & \end{array}.$$

Here $B_{L,1}'$ is obtained from $\tilde{B}'$ by reduction, $B_{L,1}' = \pi^{wu_1}_{wu_1s}(\tilde{B}')$. This implies $B_{L,1}' \in \mathcal{B}_{\ge 0}$ (see Section 3.5). Using the inequalities

$$xu^{-1} \le x'u_2^{-1} \le x'(su_2)^{-1}, \qquad w'u_1s \le w'u_1 \le w,$$

and Theorem 4.1 it follows that

$$B_{L,1}' \in \mathcal{R}^{>0}_{x'u_2^{-1},w'u_1s} \sqcup \mathcal{R}^{>0}_{x'(su_2)^{-1},w'u_1s} \subseteq \overline{\mathcal{R}^{>0}_{xu^{-1},w}}.$$

Therefore we are now in the analogous situation as before

$$\begin{array}{ccccccc} B^+ & \xrightarrow{w'} & B_L' & & \xrightarrow{u'} & & B_R' & \xrightarrow{(x')^{-1}w_0} & B^- \\ & & u_1^{(1)} & \searrow & & \nearrow & u_2^{(1)} & & \\ & & & & B_{L,1}' & & & & \end{array}.$$

with totally nonnegative $B_{L,1}'$, but where $u_1^{(1)} = u_1 s < u_1$. If again $\ell(u_1^{(1)}u_2^{(1)}) \neq \ell(u_1^{(1)}) + \ell(u_2^{(1)})$ then we can repeat the argument above. So we replace $u_1^{(1)}$ with a shorter $u_1^{(2)} = u_1^{(1)} s_{i_2}$, and $B'_{L,1}$ with $B'_{L,2}$ also in $\overline{\mathcal{R}^{>0}_{xu^{-1},w}}$. Iterating this process we must eventually arrive at a case where $\ell(u_1^{(k)} u_2^{(k)}) = \ell(u_1^{(k)}) + \ell(u_2^{(k)})$ (after at most $k = \ell(u_1)$ steps), at which point we set $u_1^{(k)} = u_1'$ and $u_2^{(k)} = u_2'$, and $B'_{L,k} \in \overline{\mathcal{R}^{>0}_{xu^{-1},w}}$ implies the inequalities (5.1). □

7. Closure relations for the strata $\mathcal{P}^J_{x,u,w}$ in $\mathcal{P}^J$

Let $\mathbb{K}$ be again as in Section 2. Then we have a decomposition

$$\mathcal{P}^J = \bigsqcup_{(x,u,w)\in Q^J} \mathcal{P}^J_{x,u,w}$$

defined as in Section 6. We can now deduce the analogue of Theorem 6.1 for the strata $\mathcal{P}^J_{x,u,w}$ in $\mathcal{P}^J$.

Again the full flag variety case needs to be treated first. After that the proof proceeds in the same way as for Theorem 6.1.

Proposition 7.1. *Let $v, w \in W$ with $v \le w$. Then*

$$\overline{\mathcal{R}_{v,w}} = \bigsqcup_{v \le v' \le w' \le w} \mathcal{R}_{v',w'}. \tag{7.1}$$

The above result does not seem to appear in the literature, so we include a quick proof. It is however well-known to experts, [8].

Proof. The inclusion $\subseteq$ follows from the closure relations for Bruhat decomposition.

Let us prove the inclusion $\supseteq$. It suffices by B^+-B^- symmetry to show that $\mathcal{R}_{v',w} \subseteq \overline{\mathcal{R}_{v,w}}$ whenever $v \le v'$ (as in the proof of Theorem 4.1). Consider the map

$$\begin{array}{rccc} \gamma & : B^+\dot{w}\cdot B^+ \times (U^- \cap \dot{w}U^-\dot{w}^{-1}) & \longrightarrow & \dot{w}U^- \cdot B^+, \\ & (B, y) & \mapsto & y \cdot B. \end{array}$$

It is an isomorphism and by restriction gives rise to isomorphisms

$$\gamma_v \ : \mathcal{R}_{v,w} \times (U^- \cap \dot{w}U^-\dot{w}^{-1}) \longrightarrow B^-\dot{v}\cdot B^+ \ \cap \ \dot{w}U^- \cdot B^+,$$

as in [6, Section 1.4]. Now if $v' \ge v$ then we have

$$B^-\dot{v}' \cdot B^+ \cap \dot{w}U^- \cdot B^+ \subset \overline{B^-\dot{v}\cdot B^+} \cap \dot{w}U^- \cdot B^+.$$

Applying γ^{-1} to this inclusion we see that $\mathcal{R}_{v',w} \subset \overline{\mathcal{R}_{v,w}}$ and the proposition follows. □

Now consider the partial flag variety case. Since π^J is proper we have that $\overline{\mathcal{P}^J_{x,u,w}} = \pi^J(\overline{\mathcal{R}_{x,wu}}) = \pi^J(\overline{\mathcal{R}_{xu^{-1},w}})$. The following proposition follows from the same proof as Theorem 6.1, only using Proposition 7.1 in place of Theorem 4.1 and leaving out the positivity considerations.

Proposition 7.2. *Let $(x, u, w) \in Q^J$ with partial order as in Definition 5.1, then*

$$\overline{\mathcal{P}^J_{x,u,w}} = \bigsqcup_{(x',u',w') \le (x,u,w)} \mathcal{P}^J_{x',u',w'}.$$

□

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Department of Mathematics, King's College London, Strand, London WC2R 2LS

Department of Pure Mathematics, University of Waterloo, Canada
E-mail address: rietsch@mth.kcl.ac.uk