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MODULAR INVARIANCE, MODULAR IDENTITIES AND SUPERSINGULAR j-INVARIANTS

Antun Milas

Abstract. To every k-dimensional modular invariant vector space we associate a modular form on $SL(2,\mathbb{Z})$ of weight $2k$. We explore number theoretic properties of this form and find a sufficient condition for its vanishing which yields modular identities (e.g., Ramanujan-Watson's modular identities). Furthermore, we focus on a family of modular invariant spaces coming from suitable two-dimensional spaces via the symmetric power construction. In particular, we consider a two-dimensional space spanned by graded dimensions of certain level one modules for the affine Kac-Moody Lie algebra of type $D_4^{(1)}$. In this case, the reduction modulo prime $p = 2k+3 \geq 5$ of the modular form associated to the k-th symmetric power classifies supersingular elliptic curves in characteristic p. This construction also gives a new interpretation of certain modular forms studied by Kaneko and Zagier.

1. Introduction and notation

An especially interesting feature of every rational vertex operator algebra is the modular invariance of graded dimensions (see [27] for a precise statement). What distinguishes modular invariant spaces coming from representations of vertex operator algebras is the fact that these spaces are equipped with special spanning sets indexed by irreducible modules of the algebra, and are subject to the Verlinde formula (cf. [11]). Moreover, every irreducible graded dimension (or simply, *character*) admits a q-expansion of the form

$$q^{\bar{h}} \sum_{n=0}^{\infty} a_n q^n,$$

where $\bar{h} \in \mathbb{Q}$ and $a_n \in \mathbb{Z}_{\geq 0}$.

In [16] we showed that the internal structure of certain vertex operator algebras can be conveniently used to prove some modular identities without much use of the theory of modular forms. The key ingredient in our approach was played by certain Wronskian determinants which are intimately related to ordinary differential equations with coefficients being holomorphic modular forms. Additionally, these differential equations are closely related to certain finiteness condition on the vertex operator algebra in question. In a joint work with Mortenson and Ono [18] we studied differential equations associated to $(2, 2k+1)$ Andrews-Gordon series and observed that a suitably normalized constant coefficient in these ODEs (expressible as a quotient of two Wronskians), when restricted modulo prime $p = 2k+1$ is essentially the locus of supersingular j-invariants in characteristic p. It is an open problem to find an alternative description of the modular forms considered in [18].

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The aim of this note is to build a framework for studying modular forms expressed as a quotient of two Wronskians as in [18]. In addition we present a very elegant way of constructing supersingular polynomials by using symmetric products. Thus we are able to explicitly determine our modular forms and to relate them to known constructions in the literature (cf. [14]).

First we show how to construct a modular form of weight $2k$ from a k-dimensional modular invariant space. Let V be *modular invariant* vector space with a basis $f_1(\tau), ..., f_k(\tau)$ [1], i.e., for every i and $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2,\mathbb{Z})$, there exist constants $\gamma_{i,j}$ such that

$$f_i\left(\frac{a\tau+b}{c\tau+d}\right) = \sum_{j=1}^{k} \gamma_{i,j} f_j(\tau).$$

Let us denote the Wronskian determinant of $f_1(\tau)$,...$f_k(\tau)$ by

$$W_V = W_{(q\frac{d}{dq})}(f_1, ..., f_k),$$

where we use the Ramanujan's derivative $\left(q\frac{d}{dq}\right)$. This is an automorphic form on $SL(2,\mathbb{Z})$ (more precisely, a modular form with a character) of weight $k(k-1)$ and its properties have been recorded in the literature (cf. [1], [16], [17], [15], [19]). It is not hard to see that the sixth power of W_V is a modular form. In addition, the Wronskian of the derivatives of $f_1(\tau), ..., f_k(\tau)$,

$$W'_V = W_{(q\frac{d}{dq})}(f'_1, ..., f'_k),$$

is also an automorphic form of weight $k(k+1)$ with the same character as W_V (cf. [17]). Unlike W_V and W'_V,

$$\frac{W'_V(\tau)}{W_V(\tau)} \tag{1.1}$$

is independent of the particular choice of the basis of V. This quotient, which is the main object of our study, is a (meromorphic) modular form for $SL(2,\mathbb{Z})$ of weight $2k$ (possibly zero) [17], [18]. We also denote by $\mathcal{W}_V$ (resp. $\mathcal{W}'_V$) the normalization of W_V (resp. W'_V) (if nonzero) in which in the q-expansion the leading coefficient is one. Clearly, $\mathcal{W}_V$ and $\mathcal{W}_V$ do not depend on the basis chosen.

Alternatively, we can think of $\frac{W'_V}{W}$ as follows. There is a unique linear differential operator $\mathcal{D}_V$ of order k,

$$\mathcal{D}_V = \sum_{i=0}^{k} P_{i,V}(q)\left(q\frac{d}{dq}\right)^i,$$

which satisfies $\mathcal{D}_V y = 0$ for every $y \in V$, and $P_{k,V} = 1$. Under these conditions

$$P_{0,V}(\tau) = (-1)^k \frac{W'_V(\tau)}{W_V(\tau)},$$

so $\frac{W'_V}{W}$ is , up to a sign, just the evaluation $\mathcal{D}_V(1)$.

This paper is organized as follows. In Section 2 we obtain a sufficient condition for the vanishing of $\frac{W'_V(\tau)}{W_V(\tau)}$ (see Theorem 2.2). Then in Section 3 we focus on modular

[1]We refer to Section 2 for precise conditions on $f_i(\tau)$.

invariant spaces obtained via the symmetric power construction (cf. Theorem 3.2). In Section 4 we apply results from sections 2 and 3 to prove some modular identities, such as the Ramanujan-Watson's modular identities for the Rogers-Ramanujan's continued fraction (cf. Theorem 4.1). In Section 5 we derive a recursion formula (cf. Lemma 5.1) which can be used to give another proof of Theorem 4.1. We derive the same recursion in Section 6 in the framework of vertex operator algebras (this part can be skipped without any loss of continuity). In Section 7 we gather some results about supersingular elliptic curves and modular forms. Finally, in Section 8 we focus on supersingular congruences for modular forms obtained from $V = \mathrm{Sym}^m(U)$, where U is spanned by the graded dimensions of level one modules for $D_4^{(1)}$ (see also [18] for a related work). Our main result, Theorem 8.3, gives a nice expression for $\frac{W_V'(\tau)}{W_V(\tau)}$ as a coefficient of a certain generating series studied in [14].

Throughout the paper the Eisenstein series will be denoted by

$$G_{2k}(\tau) = \frac{-B_{2k}}{2k!} + \frac{2}{(2k-1)!}\sum_{n=1}^{\infty}\frac{n^{2k-1}q^n}{1-q^n},\ k \geq 1. \tag{1.2}$$

We will also use normalized Eisenstein series

$$E_{2k}(\tau) = (\frac{-B_{2k}}{2k!})^{-1}G_{2k}(\tau). \tag{1.3}$$

As usual, the Dedekind η-function and the discriminant are defined as

$$\begin{aligned} \eta(\tau) &= q^{1/24}\prod_{n=1}^{\infty}(1-q^n) \\ \Delta(\tau) &= E_4(\tau)^3 - E_6(\tau)^2. \end{aligned} \tag{1.4}$$

Then the j-function is defined as

$$j(\tau) = \frac{1728E_4(\tau)^3}{\Delta(\tau)}.$$

A holomorphic modular form for $SL(2,\mathbb{Z})$ is assumed to be holomorphic in $\mathbb{H}$ (the upper half-plane) with a possible pole at the infinity. The order of vanishing at the infinity of f will be denoted by $\mathrm{ord}_{i\infty}(f)$. The graded ring of holomorphic modular forms including at the infinity will be denoted by $M = \mathbb{C}[E_4, E_6]$. Its graded components will be denoted by M_k. Every $f(\tau) \in M_k$ can be written uniquely as

$$f(\tau) = \Delta^t(\tau)E_4^{\delta}(\tau)E_6^{\epsilon}(\tau)\tilde{F}(f, j(\tau)), \tag{1.5}$$

where $\tilde{F}(f, j)$ is a polynomial of degree $\leq t$ and

$$k = 12t + 4\delta + 6\epsilon,$$

where $0 \leq \delta \leq 2$ and $0 \leq \epsilon \leq 1$.

2. On the vanishing of $\frac{W_V'(\tau)}{W_V(\tau)}$

The goal of this section is to obtain a sufficient condition for the vanishing of $\frac{W_V'(\tau)}{W_V(\tau)}$. All our modular invariant spaces are assumed to have a basis $\{f_1(\tau), ..., f_k(\tau)\}$ of holomorphic functions in $\mathbb{H}$ with the q-expansion of the form

$$f_i(\tau) = q^{h_i} \sum_{n=0}^{\infty} a_n^{(i)} q^n, \tag{2.6}$$

where $h_i \in \mathbb{Q}$. The rationale for this assumption rests on the general form of irreducible characters of rational vertex operator algebras [6] (see also [1]). In fact, in all our applications $a_n^{(i)}$ are nonnegative integers. We start with an auxiliary result:

Lemma 2.1. *Let $\{f_1(\tau), ..., f_k(\tau)\}$ be a basis of V with q-expansions as in (2.6). Then we can find a (new) basis of V*

$$\bar{f}_i(\tau) = q^{\bar{h}_i} \sum_{n=0}^{\infty} \bar{a}_n^{(i)} q^n, \quad \bar{a}_0^{(i)} \neq 0, \; i = 1, ..., k, \tag{2.7}$$

where

$$\bar{h}_1 < \bar{h}_2 < \cdots < \bar{h}_k, \tag{2.8}$$

so that

$$\mathrm{ord}_{i\infty} W_V(\tau) = \sum_{i=1}^{k} \bar{h}_i.$$

The numbers $\bar{h}_i$ are uniquely determined.

Proof: The uniqueness of $\bar{h}_i$ follows easily by the induction on k. Clearly, $W_{(q\frac{d}{dq})}(\bar{f}_1, ..., \bar{f}_k)$ is a nonzero multiple of $W_{(q\frac{d}{dq})}(f_1, ..., f_k)$. Now, the leading coefficient in the q-expansion of W_V is (up to a sign) the Vandermonde determinant $V(\bar{h}_1, ..., \bar{h}_k) = \prod_{i<j}(\bar{h}_i - \bar{h}_j) \neq 0$, and the leading power of q is $\sum_{i=1}^{k} \bar{h}_i$. □

The vanishing of W'_V simply means that there is a linear relation

$$\sum_{i=1}^{k} \lambda_i f_i(\tau) = C \neq 0. \tag{2.9}$$

The following result gives a sufficient condition for the vanishing of W'_V.

Theorem 2.2. *Let $\{f_1(\tau), ..., f_k(\tau)\}$ be a basis of a modular invariant space V satisfying (2.7) and (2.8) such that*

- (i) $\frac{W'_V(\tau)}{W_V(\tau)}$ *is holomorphic (i.e., $W_V(\tau)$ is nonvanishing in $\mathbb{H}$).*
- (ii) *There exists $r \geq \lfloor \frac{k}{6} \rfloor$ and $f_{i_0}(\tau), ..., f_{i_r}(\tau)$ with*

$$\mathrm{ord}_{i\infty} f_{i_j}(\tau) = j, \quad \text{for } j = 0, ..., r. \tag{2.10}$$

Then $\frac{W'_V(\tau)}{W_V(\tau)}$ is identically zero. If in addition $f_{i_0}(\tau)$,...,$f_{i_r}(\tau)$ are the only f_i with positive integer powers of q, then there exist constants λ_i, C, such that

$$\sum_{j=0}^{r} \lambda_i f_{i_j}(\tau) = C \neq 0.$$

Proof: Clearly, $\bar{h}_{i_j} = j$ for $j = 0, ..., r$. Because of $\mathrm{ord}_{i\infty} f'_{i_0}(\tau) \geq 1$ and $\mathrm{ord}_{i\infty} f'_{i_j}(\tau) = j$ for $j \geq 1$, we can find constants λ_j such that

$$\mathrm{ord}_{i\infty}(\sum_{j=0}^{r} \lambda_j f'_{i_j}(\tau)) \geq r + 1. \tag{2.11}$$

We claim that $\frac{W'_V(\tau)}{W_V(\tau)}$ is zero. Suppose that $\frac{W'_V(\tau)}{W_V(\tau)} \neq 0$. By Lemma 2.1, we have $\mathrm{ord}_{i\infty} W_V(\tau) = \sum_{i=1}^{k} \bar{h}_i$ and $\mathrm{ord}_{i\infty} W'_V(\tau) \geq r + 1 + \sum_{i=1}^{k} \bar{h}_i$ (keep in mind that W'_V is just the Wronskian of $f'_1(\tau), ..., f'_k(\tau)$). Thus,

$$\mathrm{ord}_{i\infty} \frac{W'_V(\tau)}{W_V(\tau)} \geq r + 1 > \lfloor \frac{k}{6} \rfloor.$$

It is known that the order of vanishing at the infinity of a nonzero holomorphic modular form of weight $2k$ is at most $\lfloor \frac{k}{6} \rfloor$. The first claim holds.

Suppose now that $f_{i_0}(\tau)$,...,$f_{i_r}(\tau)$ are the only f_i with positive integer powers of q. Because of (2.7) q-powers of f_i are integral if and only if $\bar{h}_i$ is an integer. If $\sum_{j=0}^{r} \lambda_j f'_{i_j}(\tau)$ is nonzero, then $\mathrm{ord}_{i\infty} \sum_{j=0}^{r} \lambda_j f'_{i_j}(\tau)$ is finite and therefore $\mathrm{ord}_{i\infty} W'_V$ is also finite and W'_V is nonzero. We have a contradiction. □

3. Wronskians and symmetric powers

Definition 3.1. Let U be a modular invariant space and m a positive integer. The modular invariant space spanned by

$$\{f_1 \cdots f_m : \ f_i \in U\},$$

is called the m-th symmetric power [2] of U and is denoted by $\mathrm{Sym}^m(U)$.

For $\dim(U) \geq 3$ it is a nontrivial task to find even the dimension of $V = \mathrm{Sym}^m(U)$, let alone to extract any information regarding $\frac{W'_V(\tau)}{W_V(\tau)}$. However, if $\dim(U) = 2$, the situation is much better and we have the following result.

Theorem 3.2. *Let U be a two-dimensional modular invariant space, then $V = \mathrm{Sym}^m(U)$ is $(m+1)$-dimensional and*

$$\mathcal{W}_V(\tau) = \mathcal{W}_U(\tau)^{\frac{m(m+1)}{2}}.$$

If in addition $W_U(\tau)$ is nonvanishing, then $\frac{W'_V(\tau)}{W_V(\tau)}$ is a holomorphic modular form of weight $2m + 2$ and

$$\mathcal{W}_V(\tau) = \eta(\tau)^{2m(m+1)}. \tag{3.12}$$

Proof: Let f_1 and f_2 form a basis of U. Then the set

$$\{f_1^i f_2^{m-i} : \ i = 0, ..., m\}, \tag{3.13}$$

is linearly independent (otherwise f_1/f_2 would be a constant), so it gives a basis of V. Now, by using basic properties of the Wronskian, we have

[2]This terminology will be explained later in Section 5.

$$\begin{aligned}
W_V(\tau) &= W_{(q\frac{d}{dq})}(f_1^m, f_1^{m-1}f_2, ..., f_1 f_2^{m-1}, f_2^m) \\
&= (f_1^m)^{m+1} W_{(q\frac{d}{dq})}(1, (f_2/f_1), ..., (f_2/f_1)^m) \\
&= f_1^{m(m+1)} W_{(q\frac{d}{dq})}((f_2/f_1)', ..., ((f_2/f_1)^m)') \\
&= f_1^{m(m+1)} ((f_2/f_1)')^m W_{(q\frac{d}{dq})}(1, (f_2/f_1), ..., (m-1)(f_2/f_1)^{m-1}) \\
&= m! f_1^{m(m+1)} ((f_2/f_1)')^m W_{(q\frac{d}{dq})}(1, (f_2/f_1), ..., (f_2/f_1)^{m-1}) \\
&= \left(\prod_{k=1}^m k!\right) f_1^{m(m+1)} ((f_2/f_1)')^{m(m+1)/2} \\
&= \left(\prod_{k=1}^m k!\right) (f_2' f_1 - f_1' f_2)^{m(m+1)/2} \\
&= \left(\prod_{k=1}^m k!\right) W_U(\tau)^{m(m+1)/2}.
\end{aligned}$$

If W_U is nonvanishing then $\mathcal{W}_U(\tau) = \eta(\tau)^4$ (cf. [17]) and (3.12) follows. □

4. Ramanujan-Watson's modular identities

The Rogers-Ramanujan continued fraction [2] is defined as

$$R(q) := \cfrac{q^{1/5}}{1 + \cfrac{q}{1 + \cfrac{q}{1 + \cdots}}}.$$

In one of his notebooks Ramanujan stated that $R(e^{-\pi\sqrt{r}})$ can be exactly found for every positive rational number r. The main identities that support Ramanujan's claim are the Rogers-Ramanujan identities (cf. [2]) and a pair of modular identities recorded by Ramanujan [21]:

Theorem 4.1. *We have*

$$\frac{1}{R(q)} - 1 - R(q) = \frac{\eta(\tau/5)}{\eta(5\tau)}, \tag{4.14}$$

$$\frac{1}{R(q)^5} - 11 - R(q)^5 = \left(\frac{\eta(\tau)}{\eta(5\tau)}\right)^6. \tag{4.15}$$

The first proof of Theorem 4.1 was obtained by Watson [25]. There are other proofs in the literature that use methods similar to those available to Ramanujan (see [3] for another proof and [4] for a discussion on this subject). More analytic proofs use nontrivial facts such as explicit forms of Hauptmodules for certain modular curves (see [8] for a nice review). We will prove (4.15) by using Theorem 2.2.

Firstly, we will need the following well-known fact (we refer the reader to [5] for a discussion in the context of the two-dimensional conformal field theory):

Lemma 4.2. *Let U be the vector space spanned by*

$$ch_1(\tau) := q^{11/60} \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+2})(1-q^{5n+3})}, \tag{4.16}$$

$$ch_2(\tau) := q^{-1/60} \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+1})(1-q^{5n+4})}. \tag{4.17}$$

The modular transformation

$$\tau \mapsto \frac{-1}{\tau},$$

induces an endomorphism of U, which in the basis $\{ch_1, ch_2\}$ is represented by the matrix

$$S = \frac{2}{\sqrt{5}} \begin{bmatrix} -sin\left(\frac{2\pi}{5}\right) & sin\left(\frac{4\pi}{5}\right) \\ sin\left(\frac{4\pi}{5}\right) & sin\left(\frac{2\pi}{5}\right) \end{bmatrix}.$$

Proof: By using Jacobi Triple Product Identity we first rewrite (4.16)-(4.17) as quotients of two theta constants

$$ch_1(\tau) = \frac{q^{9/40} \sum_{n\in\mathbb{Z}} (-1)^n q^{\frac{5n^2+3n}{2}}}{\eta(\tau)}, \tag{4.18}$$

$$ch_2(\tau) = \frac{q^{1/40} \sum_{n\in\mathbb{Z}} (-1)^n q^{\frac{5n^2+n}{2}}}{\eta(\tau)}. \tag{4.19}$$

Now, apply the formula

$$\eta(-1/\tau) = \sqrt{-i\tau}\,\eta(\tau). \tag{4.20}$$

and the modular transformation formulas for the two theta constants in the numerators of (4.18)-(4.19), under $\tau \mapsto \frac{-1}{\tau}$. For an explicit computations in this case see, for instance, [8]. □

Proof of (4.14): Observe first that

$$ch_1(\tau) \cdot ch_2(\tau) = \frac{\eta(5\tau)}{\eta(\tau)}.$$

From (4.20) we have

$$ch_1(-1/\tau)ch_2(-1/\tau) = \frac{\sqrt{-i\tau/5}\,\eta(\tau/5)}{\sqrt{-i\tau}\,\eta(\tau)} = \frac{1}{\sqrt{5}} \frac{\eta(\tau/5)}{\eta(\tau)}. \tag{4.21}$$

On the other hand because of the lemma and a few trigonometric identities for $sin\left(\frac{2\pi}{5}\right)$ and $sin\left(\frac{4\pi}{5}\right)$

$$ch_1(-1/\tau)ch_2(-1/\tau) = \frac{1}{\sqrt{5}} \left(-ch_1(\tau)ch_2(\tau) - ch_1(\tau)^2 + ch_2(\tau)^2\right). \tag{4.22}$$

Now, after we equate the right-hand sides of (4.21) and (4.22), cancel the factor $\frac{1}{\sqrt{5}}$ and multiply both sides by

$$\frac{1}{ch_1(\tau)ch_2(\tau)} = \frac{\eta(\tau)}{\eta(5\tau)},$$

we get

$$-1 - \frac{ch_1(\tau)}{ch_2(\tau)} + \frac{ch_2(\tau)}{ch_1(\tau)} = \frac{\eta(\tau/5)}{\eta(5\tau)}. \tag{4.23}$$

Finally, we recall the Rogers-Ramanujan identities [2]:

$$R(q) = q^{1/5} \frac{\prod_{n=0}^{\infty}(1-q^{5n+1})(1-q^{5n+4})}{\prod_{n=0}^{\infty}(1-q^{5n+2})(1-q^{5n+3})}$$

and observe that $R(\tau) = \frac{ch_1(\tau)}{ch_2(\tau)}$. □

Proof of (4.15): We will prove the following equivalent statement:

$$\frac{ch_2(\tau)^5}{ch_1(\tau)^5} - 11 - \frac{ch_1(\tau)^5}{ch_2(\tau)^5} = \left(\frac{1}{ch_1(\tau)ch_2(\tau)}\right)^6,$$

which can be rewritten as

$$ch_2(\tau)^{11}ch_1(\tau) - 11ch_1(\tau)^6 ch_2(\tau)^6 - ch_1(\tau)^{11}ch_2(\tau) = 1. \tag{4.24}$$

Consider the 12-th symmetric power of U with a basis

$$\{ch_1^i(\tau)ch_2^{12-i}(\tau), \quad 0 \leq i \leq 12\}.$$

If we let $ch_1^i(\tau)ch_2^{12-j}(\tau) = q^{\bar{h}_j} + \cdots$, then the exponents $\bar{h}_j$ satisfy $\bar{h}_0 < \bar{h}_1 < \cdots < \bar{h}_{12}$. The crucial observation here is that

$$ch_1^i(\tau)ch_2^{12-i}(\tau) \in \mathbb{Q}[[q]],$$

if and only if $i = 1$, $i = 6$ or $i = 11$. For all other i the powers are nonintegral. More precisely,

$$\begin{aligned} ch_1(\tau)ch_2^{11}(\tau) &= 1 + 11q + 67q^2 \cdots, \\ ch_1^6(\tau)ch_2^6(\tau) &= q + 6q^2 + \cdots, \\ ch_1^{11}(\tau)ch_2(\tau) &= q^2 + \cdots. \end{aligned} \tag{4.25}$$

Observe that

$$\mathrm{ord}_{i\infty}((ch_1(\tau)ch_2^{11}(\tau))' - 11(ch_1^6(\tau)ch_2^6(\tau))' - (ch_1^{11}(\tau)ch_2(\tau))') \geq 3. \tag{4.26}$$

Now, we are ready to apply Theorem 2.2. Here $k = 13$, $r = 2$, and $\mathcal{W}_{\mathrm{Sym}^{12}(U)}(\tau) = \Delta(\tau)^{13}$ is nonvanishing (cf. [16] and Theorem 3.2). Now, (4.26) and Theorem 2.2 imply

$$ch_2(\tau)^{11}ch_1(\tau) - 11ch_1(\tau)^6 ch_2(\tau)^6 - ch_1(\tau)^{11}ch_2(\tau) = C \neq 0. \tag{4.27}$$

The constant C is clearly 1. □

Remark 1. It is possible to prove (4.15) without referring to Theorem 2.2 and [16]. Notice that (4.26) implies that $\mathrm{ord}_{i\infty}(W'_V) \geq 16$. But there is no modular form of weight $13 \cdot 14 = 182$ with this behavior at the cusp.

Our Theorem 2.2 can be applied in a variety of situations as long as the degree of the symmetric power is not too big. Here we apply our method in the case of the vector space spanned by

$$f_1(\tau) = \frac{\sum_{n\in\mathbb{Z}} q^{n^2}}{\eta(\tau)},$$
$$f_2(\tau) = \frac{\sum_{n\in\mathbb{Z}} q^{(n+1/2)^2}}{\eta(\tau)}.$$

Those familiar with the theory of affine Kac-Moody Lie algebras will recognize these series as modified graded dimensions of two distinguished irreducible representations of the affine Lie algebra of type $A_1^{(1)}$ [12]. It is not hard to see that the vector space spanned by $f_1(\tau)$ and $f_2(\tau)$ is modular invariant [5], [12]. Then we have the following analogue of Theorem 4.1:

Proposition 4.3. *We have*

$$2\frac{f_1(\tau)}{f_2(\tau)} - 2\frac{f_2(\tau)}{f_1(\tau)} = \left(\frac{\eta(\tau/2)}{\eta(2\tau)}\right)^4, \tag{4.28}$$

and

$$f_1(\tau)^5 f_2(\tau) - f_2(\tau)^5 f_1(\tau) = 2. \tag{4.29}$$

The identity (4.29) is equivalent to the following classical identity for Weber modular functions:

$$\prod_{n=1}^{\infty}(1+q^{2n-1})^8 - 16q\prod_{n=1}^{\infty}(1+q^{2n})^8 = \prod_{n=1}^{\infty}(1-q^{2n-1})^8. \tag{4.30}$$

Proof: Firstly, we apply the Jacobi Triple Product Identity [2] so that

$$f_1(\tau) = \frac{\prod_{n=1}^{\infty}(1-q^{2n})(1+q^{2n-1})^2}{\eta(\tau)}, \tag{4.31}$$

$$f_2(\tau) = \frac{2q^{1/4}\prod_{n=1}^{\infty}(1-q^{2n})(1+q^{2n})^2}{\eta(\tau)}. \tag{4.32}$$

For (4.28), notice that

$$f_1(\tau)f_2(\tau) = 2\left(\frac{\eta(2\tau)}{\eta(\tau)}\right)^4.$$

Now apply $\tau \mapsto \frac{-1}{\tau}$ and proceed as in the proof of (4.14).

Similarly, (4.29) follows from analysis of $f_1^i(\tau)f_2^{6-i}(\tau)$, by following the steps as in the proof of (4.15). The identity (4.30) is now a consequence of (4.29), (4.31) and (4.32). □

5. Vanishing of $\frac{\mathcal{W}'_V(\tau)}{\mathcal{W}_V(\tau)}$: ODE approach

We have seen that $\frac{W'_V(\tau)}{W_V(\tau)} = 0$ can be deduced by a careful analysis of the order of vanishing of $\frac{W'_V(\tau)}{W_V(\tau)}$ (or W'_V) at the infinity. In this section we deduce a related result by using elementary theory of ordinary differential equations.

To every second order homogenous ODE of the form

$$\left(q\frac{d}{dq}\right)^2 y + P(q)\left(q\frac{d}{dq}\right) y + Q(q)y = 0, \tag{5.33}$$

we associate its m-th symmetric power ODE which is by the definition a homogeneous ODE of minimal order with a fundamental system of solutions

$$\{f^i g^{m-i} \ i = 0, ..., m\},$$

where $\{f, g\}$ is a fundamental system of solutions of (5.33). See [24] for more about symmetric powers of ODEs in general. Now, unlike symmetric powers for equations of the order three, the m-th symmetric power of (5.33) is always of order $m+1$ (cf. Section 3) and is given by

$$(-1)^{m+1}\frac{W_{(q\frac{d}{dq})}(y, f^m, f^{m-1}g, ..., fg^{m-1}, g^m)}{W_{(q\frac{d}{dq})}(f^m, f^{m-1}g, ..., fg^{m-1}, g^m)} = 0.$$

By expanding the determinant in the numerator we obtain

$$\left(q\frac{d}{dq}\right)^{m+1} y + \sum_{i=0}^{m} Q_{m,i}(q)\left(q\frac{d}{dq}\right)^i y = 0. \tag{5.34}$$

Clearly, the "constant" coefficient $Q_{m,0}(q)$ is equal to

$$(-1)^{m+1}\frac{W_{(q\frac{d}{dq})}((f^m)', (f^{m-1}g)', ..., (fg^{m-1})', (g^m)')}{W_{(q\frac{d}{dq})}(f^m, f^{m-1}g, ..., fg^{m-1}, g^m)}. \tag{5.35}$$

We will show that it is possible to compute $Q_{m,0}(q)$ via a certain recursion formula. Let

$$\Theta_h := \left(q\frac{d}{dq}\right) + hG_2(q). \tag{5.36}$$

Then $\Theta_h : L_k \longrightarrow L_{k+2}$, where L_k stands for any modular invariant space of weight k for $SL(2,\mathbb{Z})$, in particular, the vector space of holomorphic modular forms of weight h. We will use notation

$$\Theta^k := \Theta_{2k} \circ \cdots \circ \Theta_2 \circ \Theta_0.$$

From now on we will focus on the following ODE:

$$\Theta^2 y + Q(q)y = 0, \tag{5.37}$$

where $Q(q)$ is a (meromorphic) modular form of weight 4.

Lemma 5.1. *Fix $m \geq 2$. Let*

$$\begin{aligned} R_1 &= mQ, \\ R_2 &= m\Theta Q, \\ R_{i+1} &= \Theta R_i + (i+1)(m-i)QR_{i-1}, \quad i = 2, ..., m-1. \end{aligned} \tag{5.38}$$

Then

$$Q_{m,0} = R_m.$$

Proof: Fix a positive integer m. The m-th symmetric power of (5.37) is given by

$$D_{m+1}y = 0,$$

where D_{m+1} is obtained recursively from

$$\begin{aligned} D_0 &= 1, \\ D_1 &= \Theta \\ D_{i+1} &= \Theta D_i + i(m-i+1)Q(q)D_{i-1}, \quad 0 < i \le m. \end{aligned} \tag{5.39}$$

This recursion formula can be proven by induction and seems to be known in the literature (see for instance Theorem 5.9 in [7]). For example, the second symmetric power ($m = 2$) of (5.37) is given by

$$D_3 y = 0, \quad \text{where} \quad D_3 = \Theta^3 + 4Q\Theta + 2\Theta(Q).$$

Since every differential operator D_i (which depends on m) admits an expansion

$$D_i = \sum_{j=0}^{i} R_{j,i}(q)\Theta^i,$$

if we let now

$$R_{j-1} = R_{j,0}, \quad j \ge 2,$$

then from the formula (5.39) we have

$$\begin{aligned} R_1 &= R_{2,0} = mQ, \\ R_2 &= R_{3,0} = m\Theta Q, \\ R_{i+1} &= R_{i,0} = \Theta R_i + (i+1)(m-i)QR_{i-1}, \quad 2 \le i \le m-1. \end{aligned}$$

□

Now, we specialize everything to an ODE of type

$$\Theta^2 y + \lambda G_4(\tau) y = 0, \quad \lambda \in \mathbb{C}. \tag{5.40}$$

Lemma 5.2. *Let $m = 12$ and $Q = \lambda G_4(\tau)$. Then*

$$R_{12} = 0, \quad \textit{if and only if} \quad \lambda \in \{-\frac{11}{5}, -\frac{25}{4}, -15, -40, 0\}. \tag{5.41}$$

Proof: Follows after some computation by using Lemma 5.1 and the formulas

$$\begin{aligned} \Theta G_4 &= 14 G_6, \\ \Theta G_6 &= \frac{60 G_4^2}{7}, \end{aligned} \tag{5.42}$$

known to Ramanujan. □

The following proposition is from [15] (it was also proven in [16]):

Proposition 5.3. *The series $ch_1(\tau)$ and $ch_2(\tau)$ form a fundamental system of solutions of*

$$\Theta^2 y - \frac{11}{5} G_4(\tau) y = 0. \tag{5.43}$$

Proof of (4.15): The Proposition 5.3 and the vanishing of $Q_{12,0}(q)$ in Lemma 5.2 for $\lambda = -\frac{11}{5}$ implies the vanishing of $\frac{W_V'(\tau)}{W_V(\tau)}$. The proof now follows. □

6. The recursion (5.38) via vertex operator algebras

In [16] we obtained a representation theoretic proof of a pair of Ramanujan's identities based on an internal structure of certain irreducible representations of the Virasoro algebra. The same framework can be used to prove the formula (4.15).

In this section we will use the notation from [16]. Let U be as in Section 4. Notice that $\mathrm{Sym}^{12}(U)$ is just the vector space spanned by graded dimensions of irreducible modules of the tensor product vertex operator algebra $L(-22/5,0)^{\otimes^{12}}$ [10], where $L(-22/5,0)$ is the vertex operator algebra associated to $\mathcal{M}(2,5)$ Virasoro minimal models [9] [16]. Let $L(n)$ and $L[n]$ be two sets of generators of the Virasoro algebra as in [27], [16]. We showed in [16] that

$$\mathrm{tr}|_W o(L[-2]^2\mathbf{1})q^{L(0)+11/60} = 0, \tag{6.44}$$

for every $L(-22/5,0)$–module W. For $0 \leq i \leq 12$, let

$$v_i = i! S(L[-2]\mathbf{1} \otimes \cdots \otimes L[-2]\mathbf{1} \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1}) \in L(-22/5,0)^{\otimes^{12}},$$

where in the first i tensor slots we have the vector $L[-2]\mathbf{1}$, and on the remaining $(12-i)$ tensor slots the vector $\mathbf{1}$, and S denotes the symmetrization (e.g., $S(L[-2] \otimes L[-2] \otimes \mathbf{1}) = L[-2] \otimes L[-2] \otimes \mathbf{1} + L[-2] \otimes \mathbf{1} \otimes L[-2] + \mathbf{1} \otimes L[-2] \otimes L[-2]$). It is known (see for instance [27], [16]), that for every vertex operator algebra V, a V-module M, and a homogeneous vector w the following identity holds

$$\begin{aligned}\mathrm{tr}|_M o(L[-2]w)q^{L(0)-c/24} \quad &= \quad \left(\left(q\frac{d}{dq}\right) + \deg(w)G_2(\tau)\right)\mathrm{tr}|_M o(w)q^{L(0)-c/24} \\ &\quad + \sum_{i=1}^{\infty} G_{2i+2}(\tau)\mathrm{tr}|_M o(L[2i]w)q^{L(0)-c/24}.\end{aligned}$$

From (6.44) and the previous formula applied for $V = L(-22/5,0)^{\otimes^{12}}$ and $v = v_i$, we get

$$\mathrm{tr}|_M o(v_{i+1})q^{L^{tot}(0)+11/5} = \mathrm{tr}|_M o(L^{tot}[-2]v_i)q^{L^{tot}(0)+11/5}$$

$$= \Theta\left(\mathrm{tr}|_M o(v_i)q^{L^{tot}(0)+\frac{11}{5}}\right) + i(12-i+1)(-\frac{11}{5}G_4(\tau))\mathrm{tr}|_M o(v_{i-1})q^{L^{tot}(0)+11/5},$$

which is equivalent to the formula (5.39). Furthermore,

$$\begin{aligned}&\frac{1}{12!}\mathrm{tr}|_M o(L^{tot}[-2]\cdot v_{12})q^{L^{tot}(0)+\frac{11}{5}} = \mathrm{tr}|_M o(L[-2]^2\mathbf{1} \otimes \cdots \otimes L[-2]\mathbf{1})q^{L^{tot}(0)+\frac{11}{5}} + \cdots \\ &+\mathrm{tr}|_M o(L[-2]\mathbf{1} \otimes \cdots \otimes L[-2]^2\mathbf{1})q^{L^{tot}(0)+\frac{11}{5}} = 0,\end{aligned}$$

because of (6.44). Here $L^{tot}[-2]$ is a Virasoro generator acting on the tensor product vertex operator algebra via comultiplication. Now we can proceed as in Lemma 5.1, and we get $R_{12} = 0$.

6.1. On $L(c_{2,5},0)^{\otimes^{12}}$ and $L(c_{2,27},0)$. In this section we give a combinatorial interpretation of (4.15) in terms of colored partitions and discuss some related work.

Let us recall [2] that $q^{-11/60}\mathrm{ch}_1(q)$ (resp. $q^{1/60}\mathrm{ch}_2(q)$) is actually the generating series for the number of partitions in parts congruent to ± 2 mod 5, (resp. ± 1 mod 5). Let $P_{j_1,j_2,j_3,j_4,j_5}(n)$ denotes the number of colored partitions of n where every part of size i mod 5 can be colored in at most j_i colors. Then we have

Proposition 6.1. *For every $n \geq 2$,*

$$P_{11,1,1,11,0}(n) = 11P_{6,6,6,6,0}(n-1) + P_{1,11,11,1,0}(n-2).$$

Proof: It is known that the generating functions of colored partitions in which every part of size j can be colored with at most c_j colors is given by

$$\prod_{j=1}^{\infty} \frac{1}{(1-q^j)^{c_j}}.$$

The statement now follows from (4.15). □

Remark 2. Modular forms $\frac{\mathcal{W}'_V(\tau)}{\mathcal{W}_V(\tau)}$ associated to irreducible characters of $\mathcal{M}(2,2k+1)$ Virasoro minimal models (essentially Andrews-Gordon series [9]) have recently been studied in [18] in connection with supersingular j-invariants. We proved that the quotient $\frac{\mathcal{W}'_V(\tau)}{\mathcal{W}_V(\tau)}$ is trivial if and only if $k = 6s^2 - 6s + 1$, $s \geq 2$, which is equivalent to a family of q-series identities among irreducible characters. For $s = 2$, $(k = 13)$ the vanishing is equivalent to the following three term combinatorial identity:

$$P_{27,12}(n) = P_{27,6}(n-1) + P_{27,3}(n-2),$$

where $P_{a,b}(n)$ denotes the number of partitions of n into parts which are not congruent to $0, \pm b$ mod a. This identity, compared with Proposition 6.1, indicates that vertex operator algebra $L(-22/5,0)^{\otimes^{12}}$ shares some similarities with $L(c_{2,27},0)$. For example, both vertex operator algebras have exactly 13 linearly independent irreducible characters.

Remark 3. Four constants $-\frac{11}{5}$, $\frac{-25}{4}$, -15 and -40, appearing in Lemma 5.2 all give rise to two-dimensional modular invariant spaces coming from irreducible characters of certain integrable lowest weight representations of Kac-Moody Lie algebras (e.g., in the $\lambda = -\frac{25}{4}$ case, for a fundamental system of solutions of $\Theta^2 y - \frac{25}{4}G_4 y = 0$ we can take f_1 and f_2 as in Proposition 4.3). Similarly for $\lambda = -40$ (see Section 8) and $\lambda = -15$ (cf. [19]). Only the $\lambda = 0$ case has no interpretation in terms of graded dimensions, in which case for a fundamental system of solutions we can take

$$g_1(\tau) = \int_{\tau}^{i\infty} \eta(s)^4 ds \quad \text{and} \quad g_2(\tau) = 1.$$

7. Supersingular j-invariants

In this section we closely follow [20]. Let $f(\tau) \in M_k$ and $\tilde{F}(f,x)$ as in (1.5). Also, let

$$h_k(x) := \begin{cases} 1 \text{ if } k \equiv 0 \pmod{12}, \\ x^2(x-1728) \text{ if } k \equiv 2 \pmod{12}, \\ x \text{ if } k \equiv 4 \pmod{12}, \\ x-1728 \text{ if } k \equiv 6 \pmod{12}, \\ x^2 \text{ if } k \equiv 8 \pmod{12}, \\ x(x-1728) \text{ if } k \equiv 10 \pmod{12}. \end{cases}$$

Then, we define the *divisor polynomial* $F(f,x)$ by

$$F(f,x) := h_k(x)\tilde{F}(f,x). \tag{7.45}$$

Let us recall a few known results about supersingular j-invariants. We say that an elliptic curve over a field K of characteristic $p>0$ is *supersingular* if the group $E(\bar{K})$ has no torsion [23]. It is known that there are only finitely many supersingular curves over $\bar{\mathbb{F}}_p$. If $p \geq 5$ is prime, then the supersingular loci $S_p(x)$ and $\widetilde{S}_p(x)$ are defined in $\mathbb{F}_p[x]$ by the following products over isomorphism classes of supersingular elliptic curves:

$$S_p(x) := \prod_{E/\overline{\mathbb{F}}_p \text{ supersingular}} (x - j(E)),$$

$$\widetilde{S}_p(x) := \prod_{\substack{E/\overline{\mathbb{F}}_p \text{ supersingular} \\ j(E) \notin \{0,1728\}}} (x - j(E)). \tag{7.46}$$

It is known that the polynomial $S_p(x)$ splits completely in $\mathbb{F}_{p^2}$ [23]. Define $\epsilon_\omega(p)$ and $\epsilon_i(p)$ by

$$\epsilon_\omega(p) := \begin{cases} 0 & \text{if } p \equiv 1 \pmod{3}, \\ 1 & \text{if } p \equiv 2 \pmod{3}, \end{cases}$$

$$\epsilon_i(p) := \begin{cases} 0 & \text{if } p \equiv 1 \pmod{4}, \\ 1 & \text{if } p \equiv 3 \pmod{4}, \end{cases}$$

The following proposition relates $S_p(x)$ to $\tilde{S}_p(x)$ [23].

Proposition 7.1. *If $p \geq 5$ is prime, then*

$$\begin{aligned} S_p(x) &= x^{\epsilon_\omega(p)}(x-1728)^{\epsilon_i(p)} \cdot \prod_{\alpha \in \mathfrak{S}_p} (x-\alpha) \cdot \prod_{g \in \mathfrak{M}_p} g(x) \\ &= x^{\epsilon_\omega(p)}(x-1728)^{\epsilon_i(p)} \tilde{S}_p(x). \end{aligned}$$

Deligne found the following congruence (see [22]).

Theorem 7.2. *If $p \geq 5$ is prime, then*

$$F(E_{p-1}, x) \equiv S_p(x) \pmod{p}.$$

Remark 4. The Von-Staudt congruences imply for primes p, that $\frac{2(p-1)}{B_{p-1}} \equiv 0 \pmod{p}$, where B_n denotes the usual nth Bernoulli number. It follows that

$$E_{p-1}(\tau) \equiv 1 \pmod{p}.$$

If $p \geq 5$ is prime, then Theorem 7.2 combined with the definition of divisor polynomials, implies that if $f(\tau) \in M_{p-1}$ and $f(\tau) \equiv 1 \pmod{p}$, then

$$F(f, j(\tau)) \equiv S_p(j(\tau)) \pmod{p}.$$

8. Symmetric powers associated to level one representations of $D_4^{(1)}$

In this section we focus on a particular family of modular forms which give supersingular j-invariants in prime characteristics. In what follows $p \geq 5$ is prime and

$$p = 2m + 3. \tag{8.47}$$

It is known [12] (see also [19]) that the graded dimensions of level one highest weight modules for $D_4^{(1)}$ span a two-dimensional vector space U with a basis consisting of eighth powers of two Weber modular functions:

$$\mathfrak{f}^8 = q^{-1/6} \prod_{n=1}^{\infty} (1 + q^{n-1/2})^8$$

$$\mathfrak{f}_2^8 = q^{1/3} \prod_{n=1}^{\infty} (1 + q^n)^8.$$

Notice that $\frac{1}{3} - \frac{1}{6} = \frac{1}{6}$, so $\mathcal{W}_U(\tau) = \eta(\tau)^4$ by [17]. Thus, (cf. [17] or [15]):

Lemma 8.1. *The infinite products $\mathfrak{f}^8$ and $\mathfrak{f}_2^8$ form a fundamental set of solutions of the ODE (5.40) with $\lambda = -40$.*

We will focus on the m-th symmetric power of U. As we already mentioned $\mathrm{Sym}^m(U)$ is $(m+1)$-dimensional. In what follows we will use a result from [14]. In that paper, among other things, Kaneko and Zagier studied the generating series of the form

$$G_\alpha(x) = (1 - 3E_4x^4 + 2E_6x^6)^\alpha,$$

for some special $\alpha \in \mathbb{Q}$. For $l \in \mathbb{N}$ and $\alpha \in \mathbb{Q}$ let us define

$$G_{l,\alpha} = \mathrm{Coeff}_{x^{2l}}(1 - 3E_4x^4 + 2E_6x^6)^\alpha \in \mathbb{Q}[E_4, E_6].$$

The following result is from [14].

Proposition 8.2. *For every prime $p \geq 5$*

$$G_{\frac{p-1}{2}, \frac{p-3}{6}} \equiv 12^{\frac{p-1}{2}} \pmod{p}.$$

The main idea behind the proof of Proposition 8.2 is the congruence

$$(1 - 3E_4x^4 + 2E_6x^6)^{(p-3)/6} \equiv (1 - 3E_4x^4 + 2E_6x^6)^{-1/2} \pmod{p}, \tag{8.48}$$

the Von-Staudt congruences (cf. Remark 4) and a parametrization of the elliptic curve E_τ by using the Weierstrass $\wp$-function (see [14] for details).

Let us recall again that the graded vector space $M = \mathbb{C}[E_4, E_6]$ admits a graded map Θ (5.36) from M_k to M_{k+2}, which can be written as [26]

$$\Theta = -\frac{E_6}{3}\frac{\partial}{\partial E_4} - \frac{E_4^2}{2}\frac{\partial}{\partial E_6}. \tag{8.49}$$

The goal of this section is to prove the following result.

Theorem 8.3. *Let $V = \mathrm{Sym}^m(U)$. Then*

(i) *For every $m \geq 1$,*

$$\frac{W_V'(\tau)}{W_V(\tau)} = (-1)^{m+1}\frac{(m+1)!}{6^{m+1}}\mathrm{Coeff}_{x^{2m+2}}(1 - 3E_4x^4 + 2E_6x^6)^{\frac{m}{3}}.$$

(ii) *For m and p as in (8.47)*

$$\frac{W_V'(\tau)}{W_V(\tau)} \equiv (-1)^{(p-1)/2}\left(\frac{2}{p}\right)\left(\frac{p-1}{2}\right)! \pmod p,$$

where $\left(\frac{\cdot}{p}\right)$ is the Legendre symbol.

(iii) *For m and p as in (8.47)*

$$F\left(\frac{\mathcal{W}_V'(\tau)}{\mathcal{W}_V(\tau)}, j(\tau)\right) \equiv S_p(j(\tau)) \pmod p.$$

Proof: Firstly,

$$(1 + x + y)^{m/3} = \sum_{r=0,s=0}^{\infty}\frac{m/3(m/3-1)\cdots(m/3-r-s+1)}{r!s!}x^ry^s, \tag{8.50}$$

gives

$$\begin{aligned}(1 - 3E_4x^4 + 2E_6x^6)^{m/3} \\ = \sum_{l=0}^{\infty}\left(\sum_{r,s\geq 0, 2r+3s=l}\frac{m/3(m/3-1)\cdots(m/3-r-s+1)}{r!s!}(-3E_4)^r(2E_6)^s\right)x^{2l}.\end{aligned} \tag{8.51}$$

Clearly,

$$\begin{aligned} G_{l,m/3} &= \mathrm{Coeff}_{x^{2l}}(1 - 3E_4x^4 + 2E_6x^6)^{m/3} \\ &= \sum_{r,s\geq 0, 2r+3s=l}\frac{m/3(m/3-1)\cdots(m/3-r-s+1)}{r!s!}(-3E_4)^r(2E_6)^s. \end{aligned} \tag{8.52}$$

Now, let

$$\bar{G}_{l,m/3} = \frac{l!}{2^l3^l}G_{l,m/3}.$$

Claim: We have

$$\bar{G}_{2,m/3} = \frac{-mE_4}{18}, \quad \bar{G}_{3,m/3} = \frac{mE_6}{54},$$

and for $l = 2r + 3s \geq 4$,

$$\bar{G}_{l,m/3} = \Theta\bar{G}_{l-1,m/3} + (l-1)(m-l+2)\frac{-E_4}{18}\bar{G}_{l-2,m/3}. \tag{8.53}$$

To prove the claim it is enough to consider the coefficient of $E_4^r E_6^s$ on both sides of (8.53) and check the initial conditions. The coefficient of $E_4^r E_6^s$ on the left-hand side of (8.53) is equal to

$$\frac{(2r+3s)!m/3(m/3-1)\cdots(m/3-r-s+1)(-1)^r}{r!s!2^{2r+2s}2^{r+3s}}. \tag{8.54}$$

The coefficient of $E_4^r E_6^s$ of the right-hand side of (8.53) is

$$\begin{aligned}\frac{(2r+3s-1)!m/3(m/3-1)\cdots(m/3-r-s+1)(-1)^r}{r!(s-1)!2^{2r+3s}3^{r+3s-1}} \\ + \frac{(2r+3s-1)!m/3(m/3-1)\cdots(m/3-r-s+2)(-1)^{r+1}}{(r-2)!s!2^{2r+2s-1}3^{r+3s+1}} \\ + \frac{(2r+3s-2)!}{(r-1)!s!2^{2r+2s-1}3^{r+3s+1}} \\ \cdot (m/3)(m/3-1)\cdots(m/3-r-s+2)\cdot(-1)^r(2r+3s-1)(m-2r-3s+2),\end{aligned} \tag{8.55}$$

where for $r=0$ (resp. $r=1$) the second and third (resp. second) term drops. From the identity

$$\frac{3s(\frac{m}{3}-r-s+1)}{2r+3s} - \frac{2r(r-1)}{3(2r+3s)} + \frac{2r(m-2r-3s+2)}{3(2r+3s)} = \frac{m}{3} - r - s + 1, \tag{8.56}$$

it follows that (8.54) is equal to (8.55). Thus, the recursion holds. It is easy to see that

$$\bar{G}_{2,m/3} = \frac{-mE_4}{18} = (-40)mG_4, \quad \bar{G}_{3,m/3} = \frac{mE_6}{54} = (-40)m\Theta G_4. \tag{8.57}$$

Now, equations (8.53) and (8.57), together with Lemma 5.1 and (5.35) imply that $\bar{G}_{l,m/3}$ satisfy the same recursion and the same initial conditions as R_{l-1} in Lemma 5.1, for $Q = -40G_4$. Thus the formula (i) holds. The part (ii) now follows from Proposition 8.2 and $\frac{((p-1)/2)!}{6^{(p-1)/2}} 12^{(p-1)/2} \equiv ((p-1)/2)! \left(\frac{2}{p}\right) \pmod p$. Finally, the equation (iii) follows from (ii) and Remark 4. □

Remark 5. Notice that our proof provides also a description of $G_{l,m/3}$ for *every* $l \le m$ via the recursion in Lemma 5.1. Also, for $m \equiv 0 \pmod 3$ the vanishing of $\frac{W_V'(\tau)}{W_V(\tau)}$ is equivalent to (4.30).

Remark 6. It would be nice to have a purely representation theoretic proof of Theorem 8.3 via certain differential equations of order two studied in [14] and [13], by using techniques from [16].

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Department of Mathematics and Statistics, SUNY Albany, Albany, New York 12222
E-mail address: amilas@math.albany.edu