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L^p REGULARITY FOR KOHN'S OPERATOR

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Abstract. In [6], Kohn constructed an example of a sum of squares of complex vector fields satisfying Hörmander's condition that lost derivatives, but was nevertheless hypoelliptic. He also demonstrated optimal L^2 regularity. In this note, we announce the corresponding L^p regularity ($1 < p < \infty$), and demonstrate the method in a simpler case. The result follows from the construction of a parametrix for these operators using NIS operators.

1. Introduction

Define the operators $L = \partial_z + i\bar{z}\partial_t$ and $\bar{L} = \partial_{\bar{z}} - iz\partial_t$ (here $z \in \mathbb{C}$ and $t \in \mathbb{R}$). Let $k > 0$ be an integer and let $f : \mathbb{R} \to \mathbb{R}$ be a C^∞ function that vanishes at 0 with exactly order k and suppose $f(x) > 0$ for $x > 0$ (a useful example is $f(x) = x^k$). Define

$$\mathcal{A} = L\bar{L} + \bar{L}f(|z|^2)L$$

We announce the following theorem:

Theorem 1.1. *Suppose* $1 < p < \infty$ *and let* u *be a distribution. Suppose* $\mathcal{A}u \in L^p_s$ *near a point* (z, t)*. Then* $u \in L^p_{s-k+1}$ *near* (z, t)*.*

Here L^p_s denotes the s-L^p Sobolev space. This may be restated as "$\mathcal{A}$ loses $k - 1$ derivatives in L^p Sobolev spaces." The case $p = 2$ was handled in [6]. Since then, more general results concerning L^2 regularity and analytic hypoellipticity have been found (see [2, 9, 10, 13] and references therein). It is well known that $\mathcal{A}$ gains derivatives in L^p Sobolev spaces away from $z = 0$ (by the results of [11]) so Theorem 1.1 is really only about $z = 0$. It is achieved by a construction of a parametrix using the NIS operators of [3, 5, 7]. One can see from the explicit construction of the parametrix that all of the loss actually happens in the t variable, much like the results in [2]. Also, that this loss is optimal follows just as in [2] (see Remark 5.2).

For the purposes of this note, we shall exhibit the proof in a simpler case. Namely, define $L = \partial_x + ix\partial_t$ and $\bar{L} = \partial_x - ix\partial_t$ (here $x, t \in \mathbb{R}$). We denote

$$A_1 = L\bar{L}$$
$$A_2 = \bar{L}x^{2k}L$$
$$\mathcal{A} = A_1 + A_2$$

The operator $\mathcal{A}$ was studied in [4]. We shall prove

Theorem 1.2. *Suppose* $1 < p < \infty$ *and let* u *be a distribution. Suppose* $\mathcal{A}u \in L^p_s$ *near a point* (x, t)*. Then* $u \in L^p_{s-k+1}$ *near* (x, t)*.*

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We will construct a parametrix using the pseudodifferential operators of [8], which are closely related to NIS operators in this case. We will see here, as well, that all of the loss happens in the t variable–ie, all of the loss in the parametrix comes from differentiation in t. Our proof of Theorem 1.2 is related to the general framework of [9] (see also [10] where the case $p = 2$ of Theorem 1.2 is established for a larger class of operators). Their methods do not seem to lead directly to L^p estimates, since they use the wider class of pseudodifferential operators of [1], which do not seem to yield L^p estimates. The proof of the harder case (Theorem 1.1) is also related to their methods, but less so, since the kernel of $L\bar{L}$ is much larger in this case (see Remark 4.2). Their methods do lead to the optimal L^2 estimates, but require conjugation by a Fourier integral operator.

Kohn's original proof [6] relied on a priori L^2 estimates. The main difficulty in that method of proof for an operator with a loss of derivatives comes from localizing the estimates. Ie, the general commutators that arise lead to errors that are worse than what one is trying to bound. To deal with this difficulty, Kohn used microlocalization techniques to reduce the problem to estimates involving operators that commuted well. We will see below that using NIS operators allows us to sweep some of that difficulty under the rug, since NIS operators are already known to be pseudolocal.

These results will be part of my doctoral dissertation at Princeton University. They were done under the supervision of Elias Stein, who I wish to thank for his constant support and encouragement and for originally suggesting the problem.

2. Some background

In this section, we review the S_ρ pseudodifferential operators from [8], and some other results contained in [8]. For more details, and proofs, we refer the reader there. If we let (ξ, τ) be dual to (x, t), then we will be using S_ρ operators associated to the metric:

$$\rho(x, t, \xi, \tau) = \left((\xi^2 + x^2\tau^2)^2 + \tau^2\right)^{\frac{1}{4}}$$

For convenience sake, for the rest of the section, we will denote $z = (x, t)$ and $\zeta = (\xi, \tau)$.

We now define (for $m \in \mathbb{R}$) a preliminary class $\hat{S}^m_\rho$ defined by, $a(z, \zeta) \in C^\infty(\mathbb{R}^4)$ is in $\hat{S}^m_\rho$ provided for all $\eta_j \in \mathbb{R}^2$, $|\eta_j| \geq 1$ we have (for $|\zeta| \geq 1$):

$$|(\eta_1, \partial_\zeta) \cdots (\eta_k, \partial_\zeta) a(z, \zeta)|$$
$$\leq C_k \rho(z, \zeta)^m \prod_{j=1}^{k} \left[\left(\frac{\rho(z, \eta_j)}{\rho(z, \zeta)} \right) + \left(\frac{\rho(z, \eta_j)}{\rho(z, \zeta)} \right)^2 \right]$$

and for $|\zeta| < 1$ and any multi-index α, we have:

$$|\partial^\alpha_\zeta a(z, \zeta)| \leq C_\alpha$$

S^m_ρ is defined to be the largest class of symbols $a(z, \zeta) \in \hat{S}^m_\rho$ such that:

$$\partial_x a(z, \zeta) = a_1(z, \zeta)\xi + a_2(z, \zeta)\tau + a_0(z, \zeta)$$
$$\partial_t a(z, \zeta) = a'_1(z, \zeta)\xi + a'_2(z, \zeta)\tau + a'_0(z, \zeta)$$

where $a_1, a_2, a'_1, a'_2 \in S^{m-1}_\rho$ and $a_0, a'_0 \in S^m_\rho$. For the most part, we will only be using operators in S^m_ρ, where $m \leq 0$. For $a(z, \zeta) \in S^m_\rho$, we will denote by $a(z, D)$ the operator corresponding to that symbol in the usual way, and $(a \circ b)(z, \zeta)$ will denote the symbol of $a(z, D)b(z, D)$, and we will denote by $Op(S^m_\rho)$ the set of operators corresponding to symbols in S^m_ρ. We shall only use a few facts about S_ρ. Namely,

- If $a(z, \zeta) \in S^{m_1}_\rho$ and $b(z, \zeta) \in S^{m_2}_\rho$, then $a \circ b \in S^{m_1+m_2}_\rho$.
- If $a(z, \zeta) \in S^m_\rho$, then $[x, a(z, D)] \in Op(S^{m-1}_\rho)$.
- Operators in $Op(S^m_\rho)$ are pseudolocal.
- Operators in $Op(S^0_\rho)$ are continuous $L^p_s \to L^p_{s,loc}$ for all $s \in \mathbb{R}$ and all $1 < p < \infty$.
- If $a(z, \zeta) \in S^m_\rho$, then $L \circ a(z, D) \in Op(S^{m+1}_\rho)$, $\bar{L} \circ a(z, D) \in Op(S^{m+1}_\rho)$, and $D_t \circ a(z, D) \in Op(S^{m+2}_\rho)$. Similarly for $a(z, D) \circ L$, $a(z, D) \circ \bar{L}$, and $a(z, D) \circ D_t$.
- If $a(z, \zeta) \in S^m_\rho$, then $a(z, D)^* \in Op(S^m_\rho)$, where $a(z, D)^*$ denotes the L^2 adjoint of $a(z, D)$.

Remark 2.1. Strictly speaking, symbols in S_ρ must have compact support in the z variable, however that will not be true for some of our operators (in particular, the ones introduced in the later part of this section). This is not an essential difference. One can either extend the results we need to these particular operators (with exactly the same proofs as in [8]), or one can use NIS operators, and all of our proofs would be the same.

We will now state some results from an example in [8] that will be useful to us. These can be found on pages $118 - 121$ of [8]. For $\alpha \in \mathbb{C}$, define

$$\mathcal{L}_\alpha = \partial_x^2 + x^2\partial_t^2 - i\alpha\partial_t = L\bar{L} + (1 - \alpha)i\partial_t$$

notice that here we have replaced α by $-\alpha$ in [8].

For $\alpha \neq \pm 1, \pm 3, \pm 5, \ldots$, there exists $p_\alpha(x, \xi, \tau) \in S^{-2}_\rho$ (independent of t but not of τ) such that

$$p_\alpha(x, D)\mathcal{L}_\alpha = \mathcal{L}_\alpha p_\alpha(x, D) = I$$

The map $\alpha \mapsto p_\alpha$ is actually holomorphic with a simple pole at $\pm$ the odd integers.

Note that $A_1 = \mathcal{L}_1$. Let Q be the projection onto the L^2 kernel of A_1. Then $Q \in Op(S^0_\rho)$, and there exists $P \in Op(S^{-2}_\rho)$ such that $A_1 P = P A_1 = 1 - Q$.

Proposition 2.2. *Suppose $q(x, \xi, \tau) \in S^0_\rho$ is the symbol of Q. Then $\tau^{-m} q \in S^{-2m}_\rho$, for any integer m.*

Proof. The result is obvious for m negative, so we turn to m positive. Consider

$$\begin{aligned}\mathcal{L}_0 Q &= (L\bar{L} + \bar{L}L)Q \\ &= (\bar{L}L - L\bar{L})Q \\ &= 2D_t Q\end{aligned}$$

And so the symbol of $p_0(x, D)^m Q$ is equal to $2^{-m}\tau^{-m}q(x, \xi, \tau)$, and therefore

$$\tau^{-m}q(x, \xi, \tau) \in S^{-2m}_\rho$$

□

Corollary 2.3. $\tau^{-a} q \in S^{-2a}_\rho$ *for any real number a.*

Proof. Note, $\tau^{-a}q = \tau^{-a+m}\tau^{-m}q$. We already know that $\tau^{-m}q \in S_\rho^{-2m}$. For any finite number of derivatives in τ, we may take m so large that τ^{-a+m} remains smooth under that many differentiations. Then the bounds for S_ρ classes follow easily. $\square$

Remark 2.4. Corollary 2.3 can be seen just as easily by looking at the symbol of Q, which equals

$$ce^{-ix\xi}e^{-\frac{\tau^2x^2+\xi^2}{2\tau}}$$

when $\tau > 0$ and 0 on $\tau \leq 0$.

3. Operators that vanish at 0

The purpose of this section is to establish the following result:

Theorem 3.1. *Let $a(x,t) \in C_0^\infty(\mathbb{R}^2)$ then, there exists an operator $T_a \in Op(S_\rho^{-2})$ such that $\mathcal{A}T_a = a(x,t)x^{2k} + r^{(-1)}(x,t,D)$, where $r^{(-1)} \in S_\rho^{-1}$.*

To see this, consider:

$$\begin{aligned}
\mathcal{A} &= L\bar{L} + \bar{L}x^{2k}L \\
&= L\bar{L} + 2kx^{2k-1}L + x^{2k}\bar{L}L \\
&= (1+x^{2k})\big(L\bar{L} + \frac{x^{2k}}{1+x^{2k}}2i\partial_t\big) + 2kx^{2k-1}L \\
&= (1+x^{2k})\mathcal{L}_{\beta(x)} + 2kx^{2k-1}L
\end{aligned}$$

where $\beta(x) = \frac{1-x^{2k}}{1+x^{2k}}$. So, we see to prove Theorem 3.1, it suffices to prove it for $\mathcal{L}_{\beta(x)}$, since if we have T_a satisfying the result for $\mathcal{L}_{\beta(x)}$, $T_a \circ \frac{1}{1+x^{2k}}$ satisfies the result for $\mathcal{A}$. Hence, Theorem 3.1 follows from the following lemma:

Lemma 3.2. *Let $a(x,t) \in C_0^\infty(\mathbb{R}^2)$. Then there exists an operator $T_a \in Op(S_\rho^{-2})$ such that $\mathcal{L}_{\beta(x)}T_a = a(x,t)x^{2k} + r^{(-1)}(x,t,D)$, where $r^{(-1)} \in S_\rho^{-1}$.*

Proof. Recall $p_\alpha \in S_\rho^{-2}$ such that $\mathcal{L}_\alpha p_\alpha(x,D) = I$ for $\alpha \neq \pm1, \pm3, \pm5, \ldots$. p_α depends holomorphicly on α with a simple pole at 1. Indeed, $(1-\alpha)p_\alpha(x,\xi,\tau)$ is a C^∞ map $\mathbb{C}\backslash\{-1,\pm3,\pm5,\ldots\} \to S_\rho^{-2}$. This can be seen either directly, or by using Cauchy's theorem and the fact that p_α is uniformly in S_ρ^{-2} on compact sets of $\mathbb{C}\backslash\{\pm1,\pm3,\pm5,\ldots\}$ (all of which can be easily checked from the results in [8]). In summary, $(1-\alpha)p_\alpha(x,\xi,\tau)$ is jointly C^∞ in (α,x,ξ,τ) on the appropriate set and moreover $\partial_\alpha^n(1-\alpha)p_\alpha \in S_\rho^{-2}$ for all n.

Given $a(x,t) \in C_0^\infty$, we define $s(x,\xi,\tau) = (1-\beta(x))a(x,t)p_{\beta(x)}(x,\xi,\tau) \in S_\rho^{-2}$. Now consider

$$\mathcal{L}_{\beta(x)}s(x,D) = \big(L\bar{L} + (1-\beta(x))i\partial_t\big)(1-\beta(x))a(x,t)p_{\beta(x)}(x,\xi,\tau)$$

If any of the vector fields land on the $\beta(x)$ of $(1-\beta(x))p_{\beta(x)}$ we've seen that we are left with a term in S_ρ^{-1} and therefore part of our error term (this is since $\alpha \mapsto (1-\alpha)p_\alpha$ is a C^∞ map to S_ρ^{-2}). Similarly, if any vector field lands on $a(x,t)$ we also get an error term. Therefore, modulo such error terms, we can imagine $\beta(x)$ is fixed, in which case we know $\mathcal{L}_\alpha(1-\alpha)p_\alpha(x,D) = (1-\alpha)$ and so we have:

$$\mathcal{L}_{\beta(x)}s(x,D) = (1-\beta(x))a(x,t) + r^{(-1)}(x,t,D)$$

where $r^{(-1)} \in S_\rho^{-1}$. Since $1-\beta(x) = x^{2k}f(x)$ where $f(x)$ is a never vanishing C^∞ function, we may take $T_a = s(x,D) \circ \frac{b(x,t)}{f(x)}$, where $b(x,t) \in C_0^\infty$ is 1 on a large ball containing the support of a. Then, T_a satisfies the conclusion of the lemma. $\square$

Remark 3.3. The reader familiar with [11] will note the relationship between their work and the operator in Lemma 3.2. Indeed, if one were to "lift" the vector fields L and $\bar{L}$ to the Heisenberg group (as in [11]), then one could approximate $(1-\alpha)p_\alpha(x,D)$ ($\alpha \in [0,1]$) by using a convolution operator on the Heisenberg group (see [12]). Then, to achieve Lemma 3.2, one would merely need to make α depend on x which is done in general in [11]. Therefore, the operator constructed in Lemma 3.2 is an operator of type 2 in the sense of [11].

4. Computation of some operators

In this section, we need to preform some explicit computations of some operators. To do so, we shall conjugate our operators with the partial Fourier transform $t \to \tau$. Since all of our operators are translation invariant in t, their symbols depend only on x, ξ, and τ. Thus, fixing τ, we may (formally) consider each of our operators as a pseudodifferential operator in just the x variable (ie as a function of x and ξ). If T were the original operator, denote this new family of operators (one for each fixed τ) by $T^{(\tau)}$. For example, the symbol for $Q^{(\tau)}$ would be $q(x,\xi,\tau)$ but just considered as a function of x and ξ.

Remark 4.1. In this section, we will preform computations of such operators as $QAxBQ$, where A and B are S_ρ pseudodifferential operators. The multiplication by x may worry the reader, since it is not in $Op(S_\rho^0)$. As it turns out (Proposition 4.5) xQ is really an S_ρ pseudodifferential operator. Thus, we may view $QAxBQ$ as $QA[x,B]Q + QABxQ$ where everything involved is an S_ρ pseudodifferential operator. We are therefore justified in computing our operators on Schwartz space and extending by continuity.

We know $\bar{L}^{(\tau)} = \partial_x + x\tau$, and so has kernel spanned by $e^{-\frac{\tau}{2}x^2}$. This is in $L^2(\mathbb{R})$ only for $\tau > 0$, and therefore $Q^{(\tau)}$ is zero for $\tau \le 0$ and is projection onto this one dimensional space for $\tau > 0$. In this section, we will only be concerned with operators of the form QTQ, where T is translation invariant in t. Thus we may restrict ourselves to computing for a fixed $\tau > 0$.

Remark 4.2. At this point, we are in a position to explain one of the main differences between the proof of Theorem 1.1 and Theorem 1.2. Indeed, the corresponding operator $Q^{(\tau)}$ in the proof of Theorem 1.1 is projection onto an infinite dimensional space, instead of a one dimensional space.

Let U_τ be the unitary change of variables given by:

$$U_\tau f(x) = \tau^{-\frac{1}{4}} f(\tau^{-\frac{1}{2}}x)$$

Then, $U_\tau A_2^{(\tau)} U_\tau^{-1} = \tau^{1-k} A_2^{(1)}$. By the considerations in the preceding paragraph, it is a simple change of variables to see that $U_\tau Q^{(\tau)} U_\tau^{-1} = Q^{(1)}$.

Proposition 4.3. *There is a nonzero constant c such that $cD_t^{k-1} Q A_2 Q = Q$.*

Proof. If we conjugate $Q^{(\tau)} A_2^{(\tau)} Q^{(\tau)}$ by U_τ, we get $\tau^{1-k} Q^{(1)} A_2^{(1)} Q^{(1)}$. Since $Q^{(1)}$ is projection onto a one dimensional space, we know $Q^{(1)} A_2^{(1)} Q^{(1)} = c_0 Q^{(1)}$. To complete the proof, it remains to see that $c_0 \neq 0$, as the result would then follow by conjugating by U_τ^{-1} and then taking the inverse Fourier transform $\tau \to t$.

But note,

$$\begin{aligned}\left(Q^{(1)} A_2^{(1)} Q^{(1)} f, f\right)_{L^2(\mathbb{R})} &= -\left(x L^{(1)} Q^{(1)} f, x L^{(1)} Q^{(1)} f\right)_{L^2(\mathbb{R})} \\ &= -\|x L^{(1)} Q^{(1)} f\|_{L^2(\mathbb{R})}^2\end{aligned}$$

Hence, to show $c_0 \neq 0$, we need only find an f in the kernel of $\bar{L}^{(1)}$ but not in the kernel of $L^{(1)}$. Such an f clearly exists. □

Lemma 4.4. $QxQ \in Op(S_\rho^{-1})$

Proof. As in Proposition 4.3, when we conjugate by U_τ, we are left with

$$c\tau^{-\frac{1}{2}} Q^{(1)}$$

Upon conjugating by U_τ^{-1} we are then left with $c\tau^{-\frac{1}{2}} Q^{(\tau)}$. Corollary 2.3 now applies and yields the lemma. □

Proposition 4.5. *$x^n Q \in Op(S_\rho^{-n})$.*

Proof. Consider, letting $S^{(b)}$ denote an arbitrary operator in $Op(S_\rho^b)$,

$$\begin{aligned}x^n Q &= x^{n-1} QxQ + x^{n-1}[x, Q]Q \\ &= x^{n-1} QxQ + x^{n-1} S^{(-1)} Q \\ &= x^{n-1} QxQ + \sum_{j=0}^{n-1} S^{(-1-j)} x^{n-1-j} Q \\ &= (x^{n-1} Q)(QxQ) + \sum_{j=0}^{n-1} S^{(-1-j)} x^{n-1-j} Q\end{aligned}$$

and induction and Lemma 4.4 complete the proof. □

Remark 4.6. Proposition 4.5 can be easily seen by looking directly at the symbol for q, but we find this approach less messy, and more analogous to the techniques involved in the proof of Theorem 1.1.

Corollary 4.7. $D_t^{k-1} A_2 Q \in Op(S_\rho^0)$

Proof. Consider, letting $S^{(b)}$ denote an arbitrary operator in $Op(S^b_\rho)$,

$$\begin{aligned} D_t^{k-1} A_2 Q &= D_t^{k-1} \bar{L} x^{2k} L Q \\ &= -2k D_t^{k-1} \bar{L} x^{2k-1} Q + D_t^{k-1} \bar{L} L x^{2k} Q \\ &= D_t^{k-1} \bar{L} S^{(-2k+1)} + D_t^{k-1} \bar{L} L S^{(-2k)} \end{aligned}$$

and the result follows. □

5. The parametrix

When considering $\mathcal{A}u = f$, we may (without loss of generality) imagine u has support in a fixed compact set. Thus, by pseudolocality, we may imagine that there is no cut off function in Theorem 3.1. Ie, we may work as if we have $T \in Op(S_\rho^{-2})$ such that $\mathcal{A}T = x^{2k}$ modulo $Op(S_\rho^{-1})$.

In this section, we will be constructing a right parametrix. There is really no difference in constructing this and a left parametrix.

Proposition 5.1. *There exists an operator $B \in Op(S_\rho^{-2})$ such that $\mathcal{A}B = 1 - Q + S^{(-1)}$, where $S^{(-1)} \in Op(S_\rho^{-1})$.*

Proof. Consider, $\mathcal{A}P = (A_1 + A_2)P = 1 - Q + A_2 P$. But,

$$A_2 P = \bar{L} x^{2k} L P = 2k x^{2k-1} L P + x^{2k} \bar{L} L P$$

Now, $2kx^{2k-1}LP \in Op(S_\rho^{-1})$ and so may become part of our error term. To complete the proof, merely take $B = P - T\bar{L}LP$, where T is the operator discussed above from Theorem 3.1. □

Let B be the operator from Proposition 5.1, and let c be the constant from Proposition 4.3. If we denote by $\equiv$ equality modulo operators in $Op(S_\rho^{-1})$, we have:

$$\begin{aligned} (A_1 + A_2)(B + cD_t^{k-1}Q) &\equiv (1 - Q) + cD_t^{k-1} A_2 Q \\ &= (1 - Q) + cD_t^{k-1} Q A_2 Q + c(1 - Q) D_t^{k-1} A_2 Q \\ &= 1 + c(1 - Q) D_t^{k-1} A_2 Q \end{aligned} \tag{1}$$

But, by Corollary 4.7, $D_t^{k-1} A_2 Q = S^{(0)} \in Op(S^0_\rho)$, and therefore,

$$(A_1 + A_2)(B + cD_t^{k-1}Q - cBS^{(0)}) \equiv 1$$

Hence, there exists an operator $U^{(-2)} \in Op(S_\rho^{-2})$ such that

$$\mathcal{A}(cD_t^{k-1}Q + U^{(-2)}) \equiv 1$$

Since operators in S^0_ρ are continuous $L^p_s \to L^p_{s,loc}$ for all s, and are pseudolocal, Theorem 1.2 follows immediately (at least it would from a left parametrix; merely take the adjoint of this right parametrix to get a left parametrix).

Remark 5.2. In fact, the loss in Theorems 1.1 and 1.2 cannot be improved. This follows from the same proof as in [2], which applies just as well to L^p estimates. To modify their method to prove the optimality for Theorem 1.2, merely use the function

$$v_\lambda(x, t) = e^{-\lambda(x^2 - 2it - (x^2 - 2it)^2)}$$

6. Theorem 1.1

In this section, we wish to make a few remarks concerning the differences between the proofs of Theorem 1.1 and Theorem 1.2. Theorem 1.1 is more complicated in a few ways. Most notably, as pointed out in Remark 4.2, $Q^{(1)}$ becomes projection onto an infinite dimensional space, as opposed to a one dimensional space. Also the analog of Proposition 4.5 does not hold (see Remark 6.3), and so some effort is required to avoid its use. For the remainder of this section, we will discuss the proof of Theorem 1.1 in the special case $f(x) = x^k$. The full theorem follows from the same ideas, after expanding f into its Taylor series centered at 0.

The crux of the argument remains in Theorem 3.1. Which takes the following form:

Theorem 6.1. *Let $a(z,t) \in C_0^\infty(\mathbb{C} \times \mathbb{R})$ and $N > 0$ be an integer. Then, there exists an NIS operator $T_{a,N}$, smoothing of order 2, such that*

$$\mathcal{A}T_{a,N} = a(z,t)|z|^{2kN} + R^{(N)}$$

where $R^{(N)}$ is an NIS operator, smoothing of order N.

Using Theorem 6.1, one needs only invert $\mathcal{A}$ modulo operators of the form

$$|z|^{2kN}T^{(-N)}$$

where $T^{(-N)}$ is an NIS operator smoothing of order $-N$, and then a full parametrix would follow from Theorem 6.1.

Remark 6.2. Here "an NIS operator smoothing of order m" is the analog of an operator in $Op(S_\rho^{-m})$.

Letting Q be the orthogonal projection onto the L^2 kernel of $L\bar{L}$, we next compute

$$QA_2Q = Q\mathcal{M}$$

as before. In the proof of Theorem 1.2, $\mathcal{M}^{-1}$ is (up to a constant) equal to D_t^{k-1}. However, in the case of Theorem 1.1, $\mathcal{M}^{-1} = M(D_\theta)D_t^{k-1}$, where $M(D_\theta)$ is an operator that is computed in the proof of Theorem 1.1 (here we have written $z = re^{i\theta}$). It turns out that $QM(D_\theta)$ is pseudolocal and bounded on all L^p Sobolev spaces. In the end, the loss of derivatives comes in the same place as in Theorem 1.2, namely in the inversion of the Toeplitz operator QA_2Q.

The proof does not end there, though. In Theorem 1.2, we had Proposition 4.5 (see Remark 6.3), and no such proposition holds for Theorem 1.1. Thus, in the computation (1), we cannot take care of the term $(1-Q)A_2Q\mathcal{M}^{-1}$ as before. We must therefore continue the series to get:

$$(A_1 + A_2)(B + Q\mathcal{M}^{-1} - PA_2Q\mathcal{M}^{-1}) \equiv 1 - A_2PA_2Q\mathcal{M}^{-1} \tag{2}$$

We separate the error term of (2) into two parts. The first part

$$QA_2PA_2Q\mathcal{M}^{-1}$$

turns out to be essentially an NIS operator, smoothing of order 0, that vanishes at $z = 0$ like $|z|^{2k}$ and can therefore be taken into our error term using Theorem 6.1. The other term is of the form:

$$(1-Q)A_2PA_2Q\mathcal{M}^{-1}$$

and we repeat the process, by subtracting off

$$PA_2PA_2Q\mathcal{M}^{-1}$$

from our partial parametrix. Each time we do this process, we add a power of A_2P to our error term. A_2P is an operator of order 0 that vanishes at 0, though, so if we have enough powers of it, we may use Theorem 6.1 to complete the construction of the parametrix.

Remark 6.3. After this paper was written, the author realized that while Proposition 4.5 does not hold in the case of Theorem 1.1, one can prove an analog of Corollary 4.7 which makes the last paragraph above superfluous. However, the method outlined in that paragraph is still useful for the more general case when $f(x)$ is a more complicated function than just x^k.

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