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THE NOETHER INEQUALITY FOR SMOOTH MINIMAL 3-FOLDS

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Abstract. Let X be a smooth projective minimal 3-fold of general type. We prove the sharp inequality

$$K_X^3 \geq \frac{2}{3}(2p_g(X) - 5),$$

an analogue of the classical Noether inequality for algebraic surfaces of general type.

1. Introduction

In the 1980's M. Reid, observing the importance of the Noether inequality: $K^2 \geq 2p_g - 4$ for surfaces of general type, asked the following question

Question 1.1. (M. Reid) What is the 3-dimensional version of Noether's inequality?

Question 1.1 is obviously a very important aspect of threefold geography, just like the well known Miyaoka-Yau inequality. There have been already several works dedicated to the above question:

• M. Kobayashi (1992, [7]) studied Gorenstein minimal 3-folds of general type and found an infinite number of examples (Proposition 3.2 in [7]) satisfying the equality:

(1.1) $$K^3 = \tfrac{4}{3}p_g - \tfrac{10}{3}.$$

• M. Chen (2004, [3]) gave effective Noether type inequalities for arbitrary minimal 3-folds of general type.

• M. Chen (2004, [2]) answered Question 1.1 under the assumption that the 3-fold X is smooth with an ample canonical line bundle, proving the sharp inequality: $K^3 \geq \frac{4}{3}p_g - \frac{10}{3}$.

[In the above three items, $K^3 := K_X^3$ is the canonical volume and $p_g := p_g(X)$ is the geometric genus of X.]

In this paper, we will generalize the main theorem of [2]. The aim is to answer Question 1.1 under a weaker condition:

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Theorem 1.2. *Let X be a smooth projective minimal 3-fold of general type. Then the sharp Noether inequality:*

$$K_X^3 \geq \frac{2}{3}(2p_g(X) - 5)$$

holds.

Remark 1.3. The inequality in Theorem 1.2 is sharp because of M. Kobayashi's interesting examples (cf. Equation (1.1)).

As an application of our results, we present the following corollary which gives a classification of 3-folds of general type with small "slope" K^3/p_g:

Corollary 1.4. *Let X be a projective minimal (i.e., K_X is nef) Gorenstein 3-fold of general type with canonical singularities. Assume $K_X^3 < \frac{7}{5}p_g(X) - 2$. Then X is canonically fibred by curves of genus 2.*

The assumption in Corollary 1.4 is not empty again because of M. Kobayashi's examples.

1.5. The set up. Let X be a projective minimal Gorenstein 3-fold of general type with canonical singularities. According to the work of M. Reid [9] and Y. Kawamata (Lemma 5.1 of) [6], there is a minimal model Y with a birational morphism $\nu : Y \longrightarrow X$ such that $K_Y = \nu^*(K_X)$ and such that Y is factorial with at worst terminal singularities. Thus we may always assume that X is factorial with only (necessarily finitely many) terminal singularities. Observing that $K_X^3 \geq 2$ (see 2.1 below), the inequality in Theorem 1.2 is automatically true whenever $p_g(X) \leq 4$. So the essential argumentation takes place when $p_g(X)$ is bigger and we are led to study the canonical map $\Phi := \Phi_{|K_X|}$ as in the two dimensional case.

Take a birational modification $\pi : X' \longrightarrow X$, which exists by Hironaka's big theorem, such that:

(1) X' is smooth;

(2) the movable part of $|K_{X'}|$ is base point free;

(3) $\pi^*(K_X)$ is supported by a normal crossing divisor (so that we are in a position to apply the Kawamata-Viehweg vanishing theorem [5, 11]).

We will fix some notation below. Denote by g the composition $\Phi \circ \pi$. So $g : X' \longrightarrow W' \subseteq \mathbb{P}^N$ is a morphism. Let $g : X' \xrightarrow{f} B \xrightarrow{s} W'$ be the Stein factorization of g (thus B is normal and f has connected fibers). We can write:

$$K_{X'} = \pi^*(K_X) + E_\pi = M + Z',$$

where M is the movable part of $|K_{X'}|$, Z' the fixed part and E_π an effective divisor which is a linear combination of distinct exceptional divisors. We may also write:

$$\pi^*(K_X) = M + E',$$

where $E' = Z' - E_\pi$ is an effective divisor. On X, one may write $K_X \sim N + Z$ where N is the movable part and Z the fixed part. So

$$\pi^*(N) = M + \sum_{i=1}^{s} d_i E_i$$

with $d_i > 0$ for all i. The above sum runs over all those exceptional divisors of π that lie over the base locus of M. On the other hand, one may write $E_\pi = \sum_{j=1}^{t} e_j E_j$ where the sum runs over *all* exceptional divisors of π. One has $e_j > 0$ for all $1 \le j \le t$ because X is terminal. Apparently, one has $t \ge s$.

Set $d := \dim(B)$. We say that X *is canonically fibred by surfaces* if $d = 1$. Under this situation, we have an induced fibration $f : X' \longrightarrow B$ onto a smooth curve B. Denote by $b := g(B)$ the geometric genus of B.

Notations

K^3	the canonical volume of a 3-fold in question
$p_g = h^0(\mathcal{O}(K))$	the geometric genus
$q(V) = h^1(\mathcal{O}_V)$	the irregularity of V
$h^2(\mathcal{O}_V)$	the second irregularity of a 3-fold V
$\chi(\mathcal{O}_V)$	the Euler Poincare characteristic of V
(K^2, p_g)	invariants of a minimal surface of general type
$g(B)$	the genus of a curve B
$\equiv$	numerical equivalence
$\sim$	linear equivalence
$\ulcorner . \urcorner$	the round up of $\cdot$ ($\ulcorner x \urcorner := min\{n \in \mathbb{Z} \vert n \ge x\}$)
$D\vert_S$	the restriction of the divisor D to S
$D \cdot C$	the intersection number of a divisor D with a curve C

2. Reduction to the surface case and the lower bound of K^3

2.1. K^3 **is even.** Suppose that D is any divisor on a smooth 3-fold V. The Riemann-Roch theorem (cf. appendix in Hartshorne's book [4]) gives:

$$\chi(\mathcal{O}_V(D)) = \frac{D^3}{6} - \frac{K_V \cdot D^2}{4} + \frac{D \cdot (K_V^2 + c_2(V))}{12} + \chi(\mathcal{O}_V).$$

A direct calculation shows that

$$\chi(\mathcal{O}_V(D)) + \chi(\mathcal{O}_V(-D)) = \frac{-K_V \cdot D^2}{2} + 2\chi(\mathcal{O}_V) \in \mathbb{Z}.$$

Therefore, $K_V \cdot D^2$ is an even number.

Now let X be a projective minimal Gorenstein 3-fold of general type. Denote by $\nu : V \longrightarrow X$ a smooth birational modification. Let D be any divisor on X. Then $K_X \cdot D^2 = K_V \cdot (\nu^* D)^2$ is even. Especially, K_X^3 is even and positive.

2.2. Known results. Let X be a projective minimal factorial 3-fold of general type with terminal singularities. The following Noether type inequalities have already been established, where $d = \dim \Phi_{|K_X|}(X)$.

- if $d = 3$, then $K_X^3 \ge 2p_g(X) - 6$ (cf. M. Kobayashi's Main Theorem in [7]);
- if $d = 2$, then $K_X^3 \ge \ulcorner \frac{2}{3}(g-1) \urcorner (p_g(X) - 2)$ (cf. Chen's Theorem 4.1(ii) in [3]), where g is the genus of a general fiber of the induced fibration $f : X' \longrightarrow B$; if furthermore X is smooth, then $K_X^3 \ge \frac{2}{3}(2p_g(X) - 5)$ (cf. Chen's Theorem 4.3 in [3]);

• if $d = 1$ and the general fiber S of the induced fibration $f : X' \longrightarrow B$ is not a surface of type $(K^2, p_g) = (1, 2)$, then $K_X^3 \geq 2p_g(X) - 4$ (cf. Chen's Theorem 4.1(iii) in [3]).

In order to prove Theorem 1.2, we have to treat the remaining case (in the above third item) where X is canonically fibred by surfaces of type $(1, 2)$. Note that Theorem 1.2 was proved in [2] only under the stronger assumption of K_X being ample. Assuming only the nefness of K_X, we can see that the method in [2] is no longer effective and the situation could be more complicated. It is the aim of this paper to overcome this obstacle and prove our Theorem 1.2.

The rest of this section is devoted to deducing several key inequalities through the $\mathbb{Q}$-divisor method.

2.3. Key inequalities. Keep the same notation as in 1.5 and assume that K_X is nef and big. Suppose, from now on, $d = 1$ and $p_g(X) \geq 3$. We have an induced fibration $f : X' \longrightarrow B$. Denote by S a general fiber of f. Let $\sigma : S \longrightarrow S_0$ be the contraction onto the minimal model. Suppose $(K_{S_0}^2, p_g(S_0)) = (1, 2)$.

By Lemma 4.5 of [3], we have two cases exactly:

$$q(X) = b = 1 \quad \text{and} \quad h^2(\mathcal{O}_X) = 0,$$

$$q(X) = b = 0 \quad \text{and} \quad h^2(\mathcal{O}_X) \leq 1.$$

One may write $M = \sum_{i=1}^{a} S_i$ as a disjoint union of distinct smooth fibers of f, where $a = p_g(X) - 1$ if $b = 0$, or $a = p_g(X)$ otherwise. Noting that $\pi^*(K_X)_{|S} \leq K_S$ is a nef and big Cartier divisor and that $\sigma^*(K_{S_0})$ is the positive part of the Zariski decomposition of K_S, so $\pi^*(K_X)_{|S}^2 = \sigma^*(K_{S_0})^2 = 1$, and $\pi^*(K_X)_{|S} \sim \sigma^*(K_{S_0})$ by the uniqueness of the Zariski decomposition. According to the construction of π, we know that $E'_{|S} \sim \pi^*(K_X)_{|S}$ is a normal crossing divisor for a general fiber S.

Now let us assume $\alpha_3 \in (0, 1)$ be a real number such that

$$h^0(S, K_S + \ulcorner \alpha E'_{|S} \urcorner) \geq 3$$

for all $\alpha > \alpha_3$. We may now write a as $a = m_2 + m_3 + 1$, where m_2, m_3 are non-negative integers and

$$\frac{a - m_3}{a} > \alpha_3.$$

Such integers exist: for instance, one may take $m_3 = 0$ and $m_2 = a - 1$. What we will show in next sections is that we can find a nontrivial decomposition of a, i.e., with $m_3 > 0$.

Once we have the above setting, we may deduce an interesting inequality as follows. Write

$$M \sim S_0 + \sum_{i=1}^{m_2} S_{2,i} + \sum_{j=1}^{m_3} S_{3,j}.$$

Since

$$\pi^*(K_X) - \sum_{j=1}^{m_3} S_{3,j} - \frac{m_3}{a} E' \equiv (1 - \frac{m_3}{a}) \pi^*(K_X)$$

is nef and big and has normal crossings, the Kawamata-Viehweg vanishing theorem ([5, 11]) yields

$$H^1(X', K_{X'} + \ulcorner \pi^*(K_X) - \sum_{j=1}^{m_3} S_{3,j} - \frac{m_3}{a} E' \urcorner) = 0$$

and hence the exact sequence:

$$\begin{aligned} 0 &\longrightarrow H^0(X', K_{X'} + \ulcorner \pi^*(K_X) - \sum_{j=1}^{m_3} S_{3,j} - \frac{m_3}{a} E' \urcorner) \\ &\longrightarrow H^0(X', K_{X'} + \ulcorner \pi^*(K_X) - \frac{m_3}{a} E' \urcorner) \\ &\longrightarrow \oplus_{j=1}^{m_3} H^0(S_{3,j}, K_{S_{3,j}} + \ulcorner (1 - \frac{m_3}{a}) E' \urcorner_{|S_{3,j}}) \longrightarrow 0. \end{aligned}$$

In the above sequence, we obviously have

$$\ulcorner (1 - \frac{m_3}{a}) E' \urcorner_{|S_{3,j}} \geq \ulcorner (1 - \frac{m_3}{a}) E'_{|S_{3,j}} \urcorner$$

and

$$(1 - \frac{m_3}{a}) E'_{|S_{3,j}} \equiv \frac{a - m_3}{a} \pi^*(K_X)_{|S_{3,j}}.$$

So one has

$$h^0(S_{3,j}, K_{S_{3,j}} + \ulcorner (1 - \frac{m_3}{a}) E'_{|S_{3,j}} \urcorner) \geq 3$$

for sufficiently general $S_{3,j}$ as a fiber of f by our definition of α_3. The above sequence then gives the inequality

$$\begin{aligned} &h^0(X', K_{X'} + \ulcorner \pi^*(K_X) - \frac{m_3}{a} E' \urcorner) \\ &\geq h^0(K_{X'} + \ulcorner \pi^*(K_X) - \sum_{j=1}^{m_3} S_{3,j} - \frac{m_3}{a} E' \urcorner) + 3m_3. \end{aligned} \tag{2.1}$$

It is obvious that one has

$$\begin{aligned} &h^0(K_{X'} + \ulcorner \pi^*(K_X) - \sum_{j=1}^{m_3} S_{3,j} - \frac{m_3}{a} E' \urcorner) \\ &\geq h^0(K_{X'} + \ulcorner \pi^*(K_X) - \sum_{j=1}^{m_3} S_{3,j} - \frac{m_2 + m_3}{a} E' \urcorner). \end{aligned}$$

Similarly, because

$$\pi^*(K_X) - \sum_{i=1}^{m_2} S_{2,i} - \sum_{j=1}^{m_3} S_{3,j} - \frac{m_2 + m_3}{a} E' \equiv \frac{1}{a} \pi^*(K_X)$$

is nef and big and with normal crossings, the vanishing theorem gives

$$H^1(K_{X'} + \ulcorner \pi^*(K_X) - \sum_{i=1}^{m_2} S_{2,i} - \sum_{j=1}^{m_3} S_{3,j} - \frac{m_2 + m_3}{a} E' \urcorner) = 0.$$

So we have the following exact sequence:

$$\begin{aligned}0 &\longrightarrow H^0(X', K_{X'} + \ulcorner \pi^*(K_X) - \sum_{i=1}^{m_2} S_{2,i} - \sum_{j=1}^{m_3} S_{3,j} - \frac{m_2+m_3}{a} E' \urcorner) \\ &\longrightarrow H^0(X', K_{X'} + \ulcorner \pi^*(K_X) - \sum_{j=1}^{m_3} S_{3,j} - \frac{m_2+m_3}{a} E' \urcorner) \longrightarrow \\ &\oplus_{i=1}^{m_2} H^0(S_{2,i}, K_{S_{2,i}} + \ulcorner \frac{a-m_2-m_3}{a} E' \urcorner|_{S_{2,i}}) \longrightarrow 0.\end{aligned}$$

The above exact sequence gives

$$h^0(S_{2,i}, K_{S_{2,i}} + \ulcorner \frac{a-m_2-m_3}{a} E' \urcorner|_{S_{2,i}}) \geq p_g(S_{2,i}) = 2$$

and

$$\begin{aligned}(2.2) \qquad & h^0(X', K_{X'} + \ulcorner \pi^*(K_X) - \sum_{j=1}^{m_3} S_{3,j} - \frac{m_2+m_3}{a} E' \urcorner) \\ & \geq h^0(K_{X'} + \ulcorner \pi^*(K_X) - \sum_{i=1}^{m_2} S_{2,i} - \sum_{j=1}^{m_3} S_{3,j} - \frac{m_2+m_3}{a} E' \urcorner) + 2m_2.\end{aligned}$$

We shall go on studying the group

$$H^0(X', K_{X'} + \ulcorner \pi^*(K_X) - \sum_{i=1}^{m_2} S_{2,i} - \sum_{j=1}^{m_3} S_{3,j} - \frac{m_2+m_3}{a} E' \urcorner).$$

Apparently, it is slightly bigger than $H^0(X', K_{X'} + S_0)$.

We set $\delta := 2 - h^2(\mathcal{O}_X)$. By looking at the exact sequence:

$$0 \longrightarrow \mathcal{O}_{X'}(K_{X'}) \longrightarrow \mathcal{O}_{X'}(K_{X'} + S_0) \longrightarrow \mathcal{O}_{S_0}(K_{S_0}) \longrightarrow 0,$$

one has

$$(2.3) \qquad h^0(K_{X'} + S_0) \geq p_g(X) + \delta.$$

Combining the above inequalities (2.1)$\sim$(2.3), we have

$$(2.4) \qquad P_2(X) = h^0(K_{X'} + \pi^*(K_X)) \geq 3m_3 + 2m_2 + p_g(X) + \delta.$$

Applying Reid's plurigenus formula (see the last section of [10] and Lemma 8.3 of [8]):

$$P_2(X) = \frac{1}{2}K_X^3 - 3\chi(\mathcal{O}_X) = \frac{1}{2}K_X^3 - 3(1 - b + h^2(\mathcal{O}_X) - p_g(X)),$$

we get the Noether type inequality:

$$(2.5) \qquad K_X^3 \geq 6m_3 + 4m_2 - 4p_g(X) + 4h^2(\mathcal{O}_X) - 6b + 10.$$

2.4. A problem on surfaces. As we have seen, the general fiber S has the invariants $(K_{S_0}^2, p_g(S_0)) = (1, 2)$. We have a divisor $E'_{|S} \sim \sigma^*(K_{S_0})$ which has normal crossings. So there is a divisor $D_0 \in |K_{S_0}|$ with $E'_{|S} = \sigma^*(D_0)$. We expect to find a real number $\alpha_3 \in (0, 1)$ such that $h^0(S, K_S + \ulcorner \alpha E'_{|S} \urcorner) \geq 3$ for all $\alpha > \alpha_3$. Furthermore we hope α_3 to be as small as possible.

3. The rounding up problem for (1,2) surfaces

Assume that Y is the canonical model of a surface of general type with $p_g(Y) = 2, K_Y^2 = 1$, that $\tau : S_0 \to Y$ is its minimal model, and finally that $f : S \to S_0$ is a sequence of point blow ups.

We set up the following notation and assumptions:

- $\Gamma \subset Y$ is a canonical divisor
- D is the full transform $\tau^*(\Gamma)$
- we assume that $f^*(D)$ is a normal crossing divisor
- for $t \in (0,1)$ we consider the round up divisor $\Delta_t := \ulcorner tf^*(D) \urcorner$

Remark 3.1. Observing that since $H^1(\mathcal{O}_Y) = H^1(\mathcal{O}_S) = H^1(K_S) = 0$ (cf. [1]), one has $h^0(K_S + \Delta_t) = p_g(S) + h^0(\omega_{\Delta_t})$ where $\omega_{\Delta_t} := \mathcal{O}_{\Delta_t}(K_S + \Delta_t)$.

Theorem 3.2. *Assume that $p_g(Y) = 2, K_Y^2 = 1$, that $f^*(D)$ is a normal crossings divisor, and that $3/10 < t$. Then $h^0(K_S + \Delta_t) = 2 + h^0(\omega_{\Delta_t}) \geq 3$.*

Proof. 1) Since $K_Y^2 = 1$, and K_Y is ample, Γ is irreducible. (Note that $|K_Y|$ has one smooth and simple base point and the general member of $|K_Y|$ is a smooth curve of genus 2 (cf. page 225 in [1]). It is well known and easy to show that Y is a hypersurface of degree 10 in the weighted projective space $\mathbb{P}(1,1,2,5)$, so Y is a finite double cover of $\mathbb{P}(1,1,2)$ and the involution σ on Y induced by the hyperelliptic involution of those genus 2 curves has exactly one isolated fixed point – the base point of $|K_Y|$. We shall also denote by the same symbol σ its lift to a biregular involution on S_0, observing that again there is exactly one isolated fixed point – the base point of $|K_{S_0}|$.

The quotient $Q_2 = Y/\langle\sigma\rangle = \mathbb{P}(1,1,2)$ is isomorphic to a quadric cone in $\mathbb{P}^3$ and Γ is isomorphic to a double cover of $\mathbb{P}^1$ branched in a point P_∞ and in a disjoint sub-scheme of length 5 (cf. [1], page 231, a construction due to Horikawa).

2) Observe that if $D \geq D'$, and $\Delta'_t := \ulcorner tf^*(D') \urcorner$, then $h^0(K_S + \Delta_t) \geq h^0(K_S + \Delta'_t)$.

3) Set $K := K_{S_0}$. Write $D = \tilde\Gamma + \tilde Z$, where $\tilde\Gamma$ is the strict transform of Γ. Thus $\tilde\Gamma \cdot K = 1, \tilde Z \cdot K = 0$. Since Γ is a Cartier divisor, it follows that the support of $\tilde Z$ is a union of the support of certain fundamental cycles Z_i corresponding to the rational double points $P_i \in X$ such that $P_i \in \Gamma$, and moreover $\tilde Z = \sum_i \tilde Z_i$, where $\tilde Z_i \geq Z_i$.

4) If we take an effective decomposition $D = D' + W$, where $D' \cdot K = 1$, then $(D')^2 = D' \cdot (K - W) = 1 - D' \cdot W \leq -1$, since a canonical curve is 2-connected ([1], VII (6.2)).

5) If $Z' \cdot K = 0$, and Z' is (effective and) reduced, then $(Z')^2 = -2k$, where k is the number of connected components of Z'. In fact, it suffices to prove the formula for Z' connected, but Z' is contained in a fundamental cycle, and corresponds therefore to a rational subtree of the Dynkin diagram. Thus, if n is the number of edges of the subtree, then $(Z')^2 = -2(n+1) + 2n = -2$.

We pass now to the strategy of proof:

[S1] if the arithmetic genus $p(\tilde\Gamma) \geq 1$ then we pick $D' = \tilde\Gamma$ (see point 2)).

Observe now that $p(\tilde\Gamma) \geq 1$ is equivalent, since $\tilde\Gamma \cdot K = 1$, to $\tilde\Gamma^2 \geq -1$, or to $D = \tilde\Gamma$, in view of 4). If the first strategy is not allowed, this means that $\tilde\Gamma^2 = -3$, and $\tilde\Gamma \cong \mathbb{P}^1$.

If $\tilde\Gamma \cong \mathbb{P}^1$ we consider the reduced divisor $\tilde\Gamma + Z'_i$, where $Z'_i = (Z_i)_{red}$ is the reduced curve corresponding to one of the divisors $\tilde Z_i$ appearing in 3). By 5) and 4) it follows that the odd number $(\tilde\Gamma + Z_i')^2 = -5 + 2(\tilde\Gamma \cdot Z_i')$ equals -3 or -1, accordingly $(\tilde\Gamma \cdot Z_i') = 1$ or 2.

[S2] If $(\tilde\Gamma \cdot Z_i') = 2$, there are four cases:

[S2.1] $\tilde\Gamma + Z_i'$ is a normal crossing divisor (of arithmetic genus 1), and we pick $D' = \tilde\Gamma + Z_i'$.

[S2.2] $\tilde\Gamma$ is tangent to a smooth (-2)-curve $A \subset Z_i'$, and then we take $D' = \tilde\Gamma + A$.

[S2.3] A fundamental cycle $Z_1 < \tilde Z$ is of type A_4 and $\tilde\Gamma$ passes through the central point transversally. Take $D' = D$ (see the claim below).

[S2.4] A fundamental cycle $Z_1 < \tilde Z$ is of type A_2 and $\tilde\Gamma$ passes through the central point transversally. Take $D' = \tilde\Gamma + Z_2$ (see the claim below).

Claim 3.3. *(1) Cases [S2.1] – [S2.4] are the only possible cases if [S2] holds.*

(2) In Case [S2.3], one has $K_{S_0} \sim D = \tilde\Gamma + \tilde Z$ with $\tilde Z = A_1 + 2A + 2A' + A_4$, so that $Z_1 = A_1 + A + A' + A_4$ is a fundamental cycle of type A_4.

(3) In Case [S2.4], there is another fundamental cycle $Z_2 < \tilde Z$ of type A_m which together with $\tilde\Gamma$ forms a rational loop (of arithmetic genus 1).

Proof. (of the claim) If $\tilde\Gamma + Z_i'$ is not a normal crossings divisor, then, the intersection number being 2, either [S2.2] holds or $\tilde\Gamma$ meets Z_i' at a singular point P where two components A, A' meet, and all intersections are transversal. We observed that on S_0 we have a canonical biregular involution σ, induced from the hyperelliptic involution on the (genus two) canonical curves.

P is then a fixed point for the involution σ, which has only the point lying over P_∞ as isolated fixed point. Since P lies in a fundamental cycle, P is a different point than the above isolated fixed point. So there is a σ-fixed curve C (on S_0) through P. If both A, A' are σ-stable, then the action σ_* on the tangent space at P will have three eigenvectors (along $A, A', \tilde\Gamma$) and hence it equals $(-1)id$, contradicting the fact that P is not an isolated σ-fixed point.

Thus σ must interchange A and A'.

Let $\tilde Z_1$ contain A, A'. Then σ acts on the graph of $\tilde Z_1$ fixing $P = A \cap A'$. So $\tilde Z_1$ is of Dynkin type A_{2n} $(n \geq 1)$ and P is the central point of $\tilde Z_1$. Therefore, A, A' are the inverse images in the double cover $S_0 \to Q_2$ of the last exceptional curve of the blow up of a singular point P' of the branch curve B on Q_2. Indeed, $P' \in B$ is a cusp of type $(2, 2n+1)$ with fibre F the only tangent at $P' \in B$. By point 1) follows that $5 \geq (F.B)_{P'} = 2n + 1$. Thus $n = 1, 2$. This proves the first assertion.

The second assertion follows from point 1).

Concerning assertion (3), by point 1) and observing that $\tilde\Gamma \cong \mathbb{P}^1$, our F has one further intersection point P_2 with B, with $(F.B)_{P_2} = 2$, and with P_2 a singular point for B of type A_n. Then assertion (3) follows. □

6) If strategies [S1] and [S2] are both not allowed, this means that $\tilde\Gamma \cong \mathbb{P}^1$, and that $(\tilde\Gamma \cdot Z_i') = 1$ for each i.

7) Consider the intersection number $\tilde\Gamma \cdot \tilde Z_i$ which equals $K \cdot \tilde Z_i - (\tilde Z_i)^2 = -(\tilde Z_i)^2$ as is therefore a strictly positive even number. By 4), since $(\tilde\Gamma + \tilde Z_i)^2 = (\tilde\Gamma)^2 - (\tilde Z_i)^2 \leq -1$ this number equals 2, or $\tilde Z_i = \tilde Z$, in which case we get 4 (indeed, note that $1 = \tilde\Gamma \cdot (\tilde\Gamma + \tilde Z)$).

8) Assume still that strategies [S1] and [S2] are both not allowed, thus $1 = K^2 = \tilde{\Gamma}^2 + \sum_i (2(\tilde{\Gamma} \cdot \tilde{Z}_i) + \tilde{Z}_i^2)$, which, by 7), equals $-3 + \sum_i 2$ if there is more than one fundamental cycle. Therefore we conclude that $\tilde{\Gamma}$ intersects precisely one or two fundamental cycles, and in the former case $\tilde{\Gamma} \cdot \tilde{Z}_i = 4$.

9) Let us consider first the case where there are two fundamental cycles intersecting $\tilde{\Gamma}$, and observe the following inequalities: $2 = \tilde{\Gamma} \cdot \tilde{Z}_i \geq \tilde{\Gamma} \cdot Z_i \geq \tilde{\Gamma} \cdot Z'_i = 1$, and write $\tilde{Z}_i = Z_i + W_i$. We have $(\tilde{\Gamma} + \tilde{Z}_i) \cdot Z_i = 0 = \tilde{\Gamma} \cdot Z_i + W_i \cdot Z_i + Z_i^2$. By the well known properties of a fundamental cycle, we have $Z_i^2 = -2$, and $W_i \cdot Z_i \leq 0$, therefore $\tilde{\Gamma} \cdot Z_i \geq 2$, and we conclude by the previous inequality that $\tilde{\Gamma} \cdot Z_i = 2$.

10) By 6), 8) and 9) it follows that if we have two fundamental cycles which are intersected by $\tilde{\Gamma}$, both are not reduced. By the standard classification of fundamental cycles, this means that the corresponding rational double points are not of type A_n, or, equivalently, that on the fibre $F \cong \mathbb{P}^1$ of which Γ is the inverse image, we have two triple points. This however contradicts 1), and shows that one of the cases [S1] or [S2] occurs.

11) Let us consider then the former case in 8), where $\tilde{\Gamma} \cdot \tilde{Z}_1 = 4$, and there is only one fundamental cycle which is intersected by $\tilde{\Gamma}$, so we have $\tilde{Z}_1 = \tilde{Z}$ and we may write accordingly Z for the fundamental cycle, and $Z' = Z_{red}$. Since $\tilde{Z}^2 = -4$, $Z^2 = -2$, we can write as in 9) $\tilde{Z} = Z + W$, and $-4 = \tilde{Z}^2 = Z^2 + W^2 + 2W \cdot Z$, and we get a sum of non positive terms, where the first two are even and strictly negative. Hence follows that $-2 = W^2$, $W \cdot Z = 0, \tilde{\Gamma} \cdot Z = \tilde{\Gamma} \cdot W = 2$ (note that $0 = Z \cdot K_{S_0} = Z \cdot (\tilde{\Gamma} + Z + W)$).

Thus again the fundamental cycle corresponds to a triple point of the branch curve, and $\tilde{\Gamma}$ intersects Z' in a smooth point, belonging to a (-2)-curve A which appears with multiplicity 2 in both Z and W. Write $W = \sum r_i A_i$ with $r_i > 0$, then $A_i \cdot Z = 0$ for all i. So the equation $A_i \cdot (\tilde{\Gamma} + Z + W) = 0$ implies $A_i \cdot W = -A_i \cdot \tilde{\Gamma} \leq 0$. Also we have seen that W is a sum of only those A_i's which are orthogonal to Z. Moreover, since the point $P = A \cap \tilde{\Gamma}$ is invariant under the involution σ, we see that A is pointwise σ-fixed. Indeed, both A and $\tilde{\Gamma}$ are σ-stable and their tangents are eigenvectors of the action σ_* on the tangent space at P, but $\tilde{\Gamma}$ is not pointwise fixed and if also A were not we would have an isolated fixed point, a contradiction.

Thus after we divide by the involution we obtain a (-4)-curve E, image of A, such that $\tilde{\Gamma}$ is the inverse image of a transversal curve $\tilde{F}$ meeting E precisely in the point p image of P.

12) Let us analyse this last case in terms of the double covering $\Gamma \to F$, where $F \cong \mathbb{P}^1$. Since, on S_0, $\tilde{\Gamma}$ is smooth of genus 0, it follows that this covering is branched on the point P_∞ and on another point $P \in F \cap B$, where the branch locus B of the double covering meets F with intersection multiplicity $(B \cdot F)_P = 5$ (observe also that B does not contain F as a component, else $K_{S_0} \geq 2\tilde{\Gamma}$, absurd.)

Because Y has only Rational Double Points as singularities,the branch curve B of the double covering has only simple singularities (see [1]). Since the fundamental cycle is not reduced, then B has a triple point at P. After blowing up P we get a (-1)-curve E_1 and the full transform of F is then $F' + E_1$, and the new branch locus is $B' + E_1$, where B' is the proper transform of B. We know that the curve E occurring in the normal crossing resolution of the branch locus has multiplicity 2 in the full transform of F (since A has multiplicity 4 in $D = \tilde{\Gamma} + \tilde{Z}$), so E cannot be the proper transform of E_1, and the new branch locus has a point of multiplicity 3 at the point $P' = F' \cap E_1$, and we must blow up P', obtaining a (-1)-curve E_2 which is part of the new branch locus, together with the strict transform B'' of B' and the strict transform E_1' of E_1.

Since we had $B' \cdot F' = 2$, and B' has multiplicity 2 at P', it follows that now $F'' \cap B'' = \emptyset$. Moreover also $F'' \cap E_1' = \emptyset$, therefore E is the strict transform of E_2. Since E has self intersection equal to -4 we need three further blow ups of points (possibly infinitely near) on E_2.

13) Since the proper transform B' is singular we can exclude that we have a rational double point of type E_6 or of type E_7. The other two cases are separated accordingly as follows (see [1], II (8.1) and III (7.1) for the one-to-one correspondence between the type of curve singularity of the branch locus $B \subset Q_2$ of the double cover $Y \to Q_2$ and the type of surface singularity at the corresponding point on the canonical model Y).

[S3.1] $(B' \cdot E_1)_{P'} = 2$ implies that we have the D_n case, since B has then two distinct tangents at P.

[S3.2] $(B' \cdot E_1)_{P'} = 3$ implies that we have the E_8 case, since B has then only one tangent at P.

14) In both cases, we observe that $\tilde{Z}$ is the pull-back of $E_1' + 2E_2$, i.e., the pull back of the maximal ideal of P plus the pull back of the maximal ideal of P'. Moreover, in case [S3.2], there is only one (-2)-curve A which occurs in Z with multiplicity two such that $A \cdot Z = 0$. In case [S3.1] we see instead that A is the curve corresponding to the vertex at distance three from the asymmetrical end (observe that our assumptions imply $n \geq 6$).

15) We proceed by observing that it suffices to verify the statement for one blow up of S_0 where we have normal crossings for $C := f^*(D)$.

Lemma 3.4. *Let $C \subset S$ be a normal crossing divisor, and let $g : S' \to S$ the blow up of a point P. Then, if we set $C' = g^*(C)$, and $\Theta_t := \ulcorner tC \urcorner$, $\Theta_t' := \ulcorner tC' \urcorner$, then $h^0(K_S + \Theta_t) = h^0(K_{S'} + \Theta_t')$, for all $t \in (0,1)$. More generally, let $C = \sum_i n_i C_i$ be the decomposition of C as a sum of irreducible analytic branches at P, and let $m_i := \mathrm{mult}_P(C_i)$, then the above equality holds if there are two smooth local branches, or just one branch (i.e., $n_1 = 1, n_j = 0\ \forall j \geq 2$) of multiplicity $m = 2$, provided $t \in (0,1)$.*

Proof. Let E be the exceptional divisor of g, let $C = \sum_i n_i C_i$ be the decomposition of C as a sum of irreducible divisors: then

$$C' = g^*(C) = \sum_i n_i C_i' + \sum_i n_i m_i E,$$

where C_i' is the proper transform of C_i, and $m_i := \mathrm{mult}_P(C_i)$.

Taking the round up, we obtain

$$\begin{aligned} \Theta_t' = \ulcorner tC' \urcorner &= \sum_i \ulcorner t n_i \urcorner C_i' + \ulcorner t \sum_i m_i n_i \urcorner E \\ &= g^*(\Theta_t) + (\ulcorner t \sum_i m_i n_i \urcorner - \sum_i \ulcorner t n_i \urcorner m_i) E. \end{aligned}$$

Since $K_{S'} = g^*(K_S)+E$, $K_{S'}+\Theta'_t = g^*(K_S+\Theta_t)+[1+\ulcorner t\sum_i m_in_i\urcorner - \sum_i \ulcorner tn_i \urcorner m_i]E$ and it suffices that the integer in the square brackets is non negative in order to conclude the desired equality. Notice that the calculation is entirely local, so that we can replace the global decomposition by the local decomposition in analytic branches.

The normal crossings case is a special case of the one where all the multiplicities satisfy $m_i = 1$: in this case we want the inequality

$$(**) \qquad 1+\ulcorner \sum_i tn_i \urcorner \geq \sum_i \ulcorner tn_i \urcorner$$

to hold. This is obvious if there are exactly two terms, since for any two real numbers a, b holds $1+\ulcorner a+b\urcorner \geq \ulcorner a\urcorner + \ulcorner b\urcorner$.

For only one branch, we want $1+\ulcorner tm\urcorner \geq \ulcorner t\urcorner m$, and this is true for $m=2$, since $t<1$.

□

Case [S1]: we take $C = \tilde{\Gamma}$: it is irreducible of arithmetic genus equal to $p \in \{1,2\}$, therefore, for each $t \in (0,1)$ C is equal to the round up of tC, and $h^0(\omega_C) = p$. If C has normal crossings, we are done by the previous lemma (choose $D' = C$ in 2)).

Assume the contrary and assume first $p=1$: then C has an ordinary cusp, thus the hypothesis of the lemma above applies. After a blow up we get two smooth tangent branches ($n_1 = 1, n_2 = 2$), and the lemma still applies. We then get three smooth transversal branches where $n_1 = 1, n_2 = 2, n_3 = 3$: the inequality (**) holds, provided $1/6 < t \leq 1/3$ (since it is equivalent then to $\ulcorner 6t\urcorner \geq 2$) and we are done, since after this blow up we get global normal crossings for the full transform.

Assume now that C does not have normal crossings, and that $p=2$: we have just verified that an ordinary cusp gives no problem (as well as a node). We use now the fact that C has only double points as singularities, so we have to verify that a tacnode $y^2 = x^4$ and a higher cusp $y^2 = x^5$ give no problem (higher singularities are excluded by point 1)).

For a tacnode we get two smooth branches, so the lemma applies, and after the first blow up we get three smooth transversal branches , with $n_1 = 1, n_2 = 1, n_3 = 2$, thus (**) applies again if $1/4 < t \leq 1/3$ (since (**) is equivalent then to $\ulcorner 4t\urcorner \geq 2$) and after this blow up we get normal crossings.

In the case of the higher cusp, we get one branch of multiplicity 2, so the lemma applies; after the first blow up we get a reduced ordinary cusp transversal to a smooth branch, occurring with multiplicity 2. In this case we have to verify that $1+\ulcorner 4t\urcorner \geq \ulcorner 2t\urcorner + 2\ulcorner t\urcorner$, but this clearly holds for $1/4 < t \leq 1/3$.

After a further blow up, we get two smooth branches, tangent, and with $n_1 = 1, n_2 = 4$, so the lemma applies. A further blow up, the last before we get normal crossings, yields a point where three smooth branches meet transversally, and $n_1 = 1, n_2 = 4, n_3 = 5$: we have to verify whether (**) holds, i.e., $1+\ulcorner 10t\urcorner \geq \ulcorner t\urcorner + \ulcorner 4t\urcorner + \ulcorner 5t\urcorner$. But this holds clearly for $3/10 < t \leq 1/3$ (else , for $1/5 < t \leq 3/10$ there is a loss by 1, which however would not trouble us since we started with $p=2$, and we only want the arithmetic genus above to be at least 1). We now treat the remaining cases one by one, using 2).

Case [S2.1]: D' already has normal crossings, and is reduced, thus there is nothing to prove.

Case [S2.2]: here D' consists of two smooth tangent divisors $\cong \mathbb{P}^1$, so its arithmetic genus is $p = 1$. This is exactly the case of the tacnode, which we already treated, thus this case is also settled.

Case [S2.3]: D ($\sim K_{S_0}$) now has arithmetic genus 2, and does not have normal crossings exactly at the point where $A, A', \tilde{\Gamma} \cong \mathbb{P}^1$ meet transversally. The local multiplicities are $2, 2, 1$, thus for $1/5 < t \leq 1/3$ we obtain $1 + \ulcorner 5t \urcorner \geq \ulcorner t \urcorner + \ulcorner 2t \urcorner + \ulcorner 2t \urcorner$. Thus we are done as in the above lemma.

Case [S2.4]: D' already has normal crossings and has arithmetic genus 1. So there is nothing to prove.

Case [S3.1]: an explicit calculation, probably well known, (cf. [1], page 65, lines 2-3) shows that the full transform of the maximal ideal of P is the fundamental cycle Z of D_n, while the full transform of the maximal ideal of P', which is then W, is the fundamental cycle of the D_{n-2} configuration obtained by deleting the asymmetric end and its neighbour.

In this case let us choose as $D' = \tilde{\Gamma} + 2W < D = \tilde{\Gamma} + \tilde{Z}$: since all the multiplicities of the components of D' are then either 1, 2 or 4, it follows that for $1/4 < t \leq 1/3$ the round up $\Delta'_t := \ulcorner tD' \urcorner$ equals $\tilde{\Gamma} + W$. Since $W^2 = -2$, $\tilde{\Gamma} \cdot W = 2$, the self intersection $(\Delta'_t)^2 = -1$, thus Δ'_t has arithmetic genus 1 (topologically it is of elliptic type D^*_{n-2} or I^*_{n-6} in Kodaira's notation) and this case is settled by virtue of 2).

Case [S3.2]: Also in this case an explicit calculation, probably well known, (cf. [1], page 65, lines 2-3) shows that the full transform of the maximal ideal of P is the fundamental cycle Z of E_8, while the full transform of the maximal ideal of P', which is then W, is the fundamental cycle of the E_7 configuration obtained by deleting the furthest end.

In this case we write the multiplicities for the components of $\tilde{Z}$ starting from left to right (i.e., from middle length end (i.e., A) to longest end), and then we give the multiplicity for the shortest end: we get the sequence $4, 7, 10, 8, 6, 4, 2$ and then 5. We may choose for convenience D' as $\tilde{\Gamma} + \tilde{Z}$ minus twice the longest end and minus its neighbor, i.e., we change the sequence to $4, 7, 10, 8, 6, 3, 0, 5$. If we now choose $3/10 < t \leq 1/3$, one can easily calculate that the round up $\Delta'_t := \ulcorner tD' \urcorner$ equals $\tilde{\Gamma} + W$ (topologically, it is of elliptic type E^*_7 or III^* in Kodaira's notation), and we are done as in the previous case. □

4. The Noether inequality

Theorem 4.1. *Let X be a minimal Gorenstein 3-fold of general type with canonical singularities. Assume either $p_g(X) \leq 2$ or that $|K_X|$ is composed with a pencil of surfaces of type (1,2). Then*

$$K_X^3 \geq \frac{7}{5} p_g(X) - 2.$$

Proof. As we have seen in 1.5, we may take X to be factorial with only terminal singularities. Because $K_X^3 \geq 2$, the inequality is automatically true for $p_g(X) \leq 2$.

We may suppose, from now on, that $p_g(X) \geq 3$. Denote by $f : X' \longrightarrow B$ the fibration induced from $\Phi_{|K_X|}$. Let S be a general fiber of f. Lemma 4.5 of [3] says $0 \leq b = g(B) \leq 1$. Theorem 3.2 says that we may take $\alpha_3 = \frac{3}{10}$ for a general fiber S of f; see the first part of 2.3.

Case 1. $b = 1$. We may write $a = p_g(X) = 10m + c$ where $m \geq 0$ and $0 \leq c \leq 9$ (obviously, for $m = 0$ we have $3 \leq c \leq 9$).

When $m > 0$, we take $m_3 := 7m + \Box_c$, where $\Box_c := -1, 0, 1, 2, 2, 3, 4, 4, 5, 6$ respectively when $0 \leq c \leq 9$. Then one sees that

$$1 - \frac{m_3}{a} > \alpha_3 = \frac{3}{10}.$$

Take $m_2 = a - 1 - m_3 = 3m + \nabla_c$, where $\nabla_c := c - 1 - \Box_c$. Then the inequality (2.5) gives

$$\begin{aligned} K_X^3 &\geq 6(7m + \Box_c) + 4(3m + \nabla_c) - 4p_g(X) + 4 \\ &= 54m - 4p_g(X) + 6\Box_c + 4\nabla_c + 4 \\ &= \frac{7}{5}p_g(X) - \frac{7}{5}c + 2\Box_c \\ &\geq \frac{7}{5}p_g(X) - 2. \end{aligned}$$

When $m = 0$, we have $3 \leq a = c \leq 9$. Take m_3 =2, 2, 3, 4, 4, 5, 6 respectively when $3 \leq c \leq 9$. We may easily check that $1 - \frac{m_3}{a} > \frac{3}{10}$. Take $m_2 := a - 1 - m_3$ =0, 1, 1, 1, 2, 2, 2 respectively when $3 \leq c \leq 9$. By inequality (2.5), we may verify case by case that $K_X^3 > \frac{7}{5}p_g(X) - 2$.

Case 2. $b = 0$. We may write $a = p_g(X) - 1 = 10m + c$ where $m \geq 0$ and $0 \leq c \leq 9$ (for $m = 0$ we have $2 \leq c \leq 9$).

Again when $m > 0$, we take $m_3 := 7m + \Box_c$ where $\Box_c = -1, 0, 1, 2, 2, 3, 4, 4, 5, 6$ respectively when $0 \leq c \leq 9$. Then the calculation is similar to Case 1. Take $m_2 = a - 1 - m_3 = 3m + \nabla_c$ where $\nabla_c = c - 1 - \Box_c$. Then the inequality (2.5) gives

$$\begin{aligned} K_X^3 &\geq 6(7m + \Box_c) + 4(3m + \nabla_c) - 4p_g(X) + 4h^2(\mathcal{O}_X) + 10 \\ &\geq 54m - 4p_g(X) + 6\Box_c + 4\nabla_c + 10 \\ &= \frac{7}{5}p_g(X) - \frac{7}{5}c + 2\Box_c + \frac{3}{5} \\ &\geq \frac{7}{5}p_g(X) - \frac{7}{5}. \end{aligned}$$

When $m = 0$ and $2 \leq a = c \leq 9$, one can in a similar way verify that $K_X^3 > \frac{7}{5}p_g(X) - \frac{7}{5}$. $\Box$

4.2. Proof of the main results.

Proof. Now both Theorem 1.2 and Corollary 1.4 follow directly from 2.2 and Theorem 4.1. $\Box$

Remark 4.3. A quite natural problem left to us is the possibility of generalizing Theorem 1.2 to the case where X is Gorenstein minimal. Unfortunately the method of Theorem 4.3 of [3] only works when X is smooth. One needs a new method to treat the difficult case where X is canonically fibred by curves of genus 2. However we would like to put forward the following:

Conjecture 4.4. The Noether inequality

$$K^3 \geq \frac{2}{3}(2p_g - 5)$$

holds for any projective minimal Gorenstein 3-fold of general type.

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