Math. Res. Lett. **13** (2006), *no.* 3, 463–465

ERRATUM AND ADDENDUM FOR "RATIONAL HOMOLOGY 5-SPHERES WITH POSITIVE RICCI CURVATURE"

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In our paper [BG02] we claimed the following (cf. Theorem A.)

Theorem 1. *For every integer $k > 1$, there exists a simply connected rational homology 5-sphere M_k^5 such that $H_2(M_k^5, \mathbb{Z})$ has order k^2, $w_2(M_k^5) = 0$, and M_k^5 admits a Sasakian metric with positive Ricci curvature.*

This statement is false without any extra assumption on the integer k. This is due to an error in the proof which occurred since the orbifold Fano index was not computed correctly. Instead we had used the ordinary Fano index of the algebraic variety $\mathcal{Z}_f$ of Lemma 2.6. In other terminology the orbifold $\mathcal{Z}_f$ is Fano if and only if the induced Sasakian structure on the link L_f is positive, and the correct statement for positivity is:

Lemma 2. *Let L_f be the link of an isolated hypersurface singularity of a weighted homogeneous polynomial f of degree d and weight vector $\mathbf{w}$. Suppose further that $|\mathbf{w}| - d > 0$. Then L_f admits a Sasakian metric with positive Ricci curvature.*

Given this it is easy to see that Theorem 1 will hold with the added hypothesis that $\gcd(k, 3) = 1$. To see this we notice that the link in question is the link L_f of the polynomial

$$f = z_0^k + f_3(z_1, z_2, z_3)$$

of Proposition 2.1 of [BG02] with weights $\mathbf{w} = (d_3, k\mathbf{w}_3)$ where k is an integer > 1 where f_3 is a weighted homogeneous polynomial of degree d_3 with weights $\mathbf{w}_3 = (w_1, w_2, w_3)$ as above. The degree of f is $d = kd_3$ and we take $\gcd(k, d_3) = 1$. Now the idea of Proposition 2.8 is that by taking d_3 to be primes of the form $4l - 1$ we can satisfy $\gcd(k, d_3) = 1$ for all k. However, the orbifold is not Fano for all values of k and l as we now show. By Lemma 2 the condition that the orbifold be Fano is $|\mathbf{w}| - d > 0$. We have

$$|\mathbf{w}| - d = k(1 - l) + 4l - 1 = (k - 4)(1 - l) + 3$$

which is > 0 for all k if $l = 1$. But then $d_3 = 3$, so we need to impose the condition $\gcd(k, 3) = 1$. Notice also that both positivity and the gcd condition hold for $k = 3$ or 6 by taking $l = 2$. Thus, Theorem 1 holds for $k = 3$ or 6 and for all k relatively prime to 3. Summarizing the example given in our paper proves only the weaker result

Theorem 3. *For every integer $k > 1$ relatively prime to 3, there exists a simply connected rational homology 5-sphere M_k^5 such that $H_2(M_k^5, \mathbb{Z})$ has order k^2, $w_2(M_k^5) = 0$, and M_k^5 admits a Sasakian metric with positive Ricci curvature. Furthermore, M_3 and M_6 both admit a Sasakian metric with positive Ricci curvature.*

Received by the editors December 9, 2005.

Accordingly, Corollary B of [BG02] needs to be corrected as follows

Corollary 4. *For every positive integer* r *and every list of distinct primes* $p_1, \cdots, p_r$, *different than* 3 *the manifolds*

$$M^5 = M^5_{p_1} \# \cdots \# M^5_{p_r}$$

admits Sasakian metrics with positive Ricci curvature.

Since the appearance of [BG02] much progress has been made on the positivity question for rational homology spheres. First in [BG03, BG05] the authors constructed three infinite series (with an additional handful of "sporadic" examples) of rational homology 5-spheres that admit Sasakian-Einstein metrics. One of these series is precisely the series constructed above with $d_3 = 3$. With these new examples, we slightly improved Theorem 3 showing that any 1-connected M_p with $H_2(M_p, \mathbb{Z}) = \mathbb{Z}_p \oplus \mathbb{Z}_p$, vanishing second Stiefel-Whitney class, and $\gcd(p, 6) = 1$ admits a positive Ricci curvature metric. The sporadic examples mentioned in [BG05] give, in addition, positive Ricci curvature metrics on $2M_2, 3M_3, 4M_3, 2M_5, 4M_2, 6M_2$ and $7M_2$.

For a while we were hoping that we could "fix" the proof of Theorem 1 with some additional examples which would have the order of the torsion group divisible by 6, thus filling in the gap. On the other hand, the examples in [BG03, BG05] have suggested that positivity puts some restrictions on the structure of $H_2(M^5, \mathbb{Z})$. Indeed, it has been demonstrated that these restrictions are quite severe. The problem has now been completely solved by Kollár who showed precisely which simply-connected rational homology 5-spheres with $w_2(M^5) = 0$ admit a positive Sasakian structure [Kol04]. Again, let M_p be characterized via Smale's theorem [Sma62] by $\pi_1(M) = 0$, $H_2(M_p, \mathbb{Z}) = \mathbb{Z}_p \oplus \mathbb{Z}_p$, and $w_2(M_p) = 0$. Then Kollár shows

Theorem 5. *Let* M *be a rational homology 5-sphere with* $\pi_1(M) = 0$ *and* $w_2(M) = 0$. *Then* M *admits a positive Sasakian structure if and only if* M *is one of the following*

(1) M_p *with* $\gcd(p, 30) = 1$,
(2) nM_2 *for any* $n > 1$,
(3) $2M_3, 3M_3, 4M_3,\ 2M_4,\ 2M_5$.

In particular, all rational homology spheres listed above admit metrics of positive Ricci curvature.

Kollár's Theorem shows that Theorem 1 is actually false as stated. It further shows serious limitations of our method for proving the existence of positive Ricci curvature metrics on 1-connected compact spin rational homology 5-spheres. Conjecturally, all such spheres should admit metrics of positive Ricci curvature, but positive Sasakian geometry alone is not sufficient to prove this. In his most recent paper on the subject, Kollár [Kol05] shows that infinitely many rational homology spheres do not admit even a differentiable fixed point free circle action. Hence, while there are infinitely many simply connected rational homology 5-spheres that admit Sasakian metrics of positive Ricci curvature, there are also infinitely many simply connected rational homology 5-spheres that have **no Sasakian structure of any kind**.

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