

NEGATIVE SECTIONAL CURVATURE AND THE PRODUCT COMPLEX STRUCTURE

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ABSTRACT. Let $M = M_1 \times M_2$ be a product of complex manifolds. We prove that M cannot admit a complete Kähler metric with sectional curvature $K < c < 0$ and Ricci curvature $Ric > d$, where c and d are arbitrary constants.

In particular, a product domain in \mathbb{C}^n cannot cover a compact Kähler manifold with negative sectional curvature.

On the other hand, we observe that there are complete Kähler metrics with negative sectional curvature on \mathbb{C}^n . Hence the upper sectional curvature bound is necessary.

1. Introduction

The interplay between the curvature and the underlying complex structure of a Kähler manifold is a central theme in complex differential geometry. In this article we prove a general result ruling out the existence of negatively curved Kähler metrics on product complex manifolds. The inspiration for our result is the classical Preissmann theorem stating that the fundamental group of a compact negatively curved Riemannian manifold does not contain $\mathbb{Z} \oplus \mathbb{Z}$ as a subgroup. In fact, if the factors are assumed compact, then our theorem follows from Preissmann's theorem. In the general situation the obstruction to negative curvature arises from the complex structure rather than the topology of M .

Theorem 1.1. *Let $M = M_1 \times M_2$ be a product of complex manifolds M_1 and M_2 with $\dim M_i \geq 1$, $i = 1, 2$. Then M cannot admit a complete Kähler metric with sectional curvature $K < c < 0$ and Ricci curvature $Ric > d$, where c and d are arbitrary constants.*

An important feature of Theorem 1.1 is that no assumptions are made about the factors M_i . Let us compare our result with the work of P. Yang [9] and F. Zheng [11] which again consider the interaction between the product complex structure and negative curvature. On the one hand, both these papers assume only negative or nonpositive *holomorphic bisectional* curvature B . On the other hand, the assumptions on the factors M_i are more stringent. Yang's paper rules out the existence of complete Kähler metrics with $d < B < c < 0$ on *polydiscs* (more generally, bounded symmetric domains of rank > 1) while Zheng classifies all metrics with $B \leq 0$ on products of *compact* manifolds.

Let us consider Theorem 1.1 in the context of a basic question regarding negatively curved Kähler manifolds: Is every simply-connected, complete Kähler manifold M

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with sectional curvatures bounded between two negative constants biholomorphic to a bounded domain in \mathbb{C}^n ? (cf. [1], [8] and [4]). This question is still open, even in the special case of M being the universal cover of a compact Kähler manifold with negative sectional curvature. In this case if one imposes further restrictions on M , there are interesting results due to B. Wong [7], J.-P. Rosay [6] and S. Frankel [2]. In these works M is only assumed to be the universal cover of a compact complex manifold. The Wong-Rosay theorem implies that if such an M is a domain in \mathbb{C}^n with C^2 -smooth boundary, then M has to be biholomorphic to \mathbb{B}^n . According to the work of Frankel, if M is a bounded convex domain, then M has to be biholomorphic to a bounded symmetric domain. As a corollary of our theorem, we have

Corollary 1.2. *A product domain in \mathbb{C}^n cannot cover a compact Kähler manifold with negative sectional curvature. In fact, product domains do not admit complete Kähler metrics with pinched negative sectional curvature.*

Remark: Theorem 1.1 has other intriguing complex-analytic implications. For instance, it follows that the unit ball in \mathbb{C}^n is not biholomorphic to a product of complex manifolds, a fact which is probably known.

Regarding the necessity of the assumptions on the curvature in Theorem 1.1, we show in the last section that the calculations in [5] can be adapted to get a complete Kähler metric on \mathbb{C}^n with sectional curvature $d < K < 0$. Hence the upper bound $K < c < 0$ is necessary in Theorem 1.1. However, the following question is still open: Is Theorem 1.1 still valid if we drop the lower bound on Ric ? Also, it would be interesting to know if the result remains true if one replaces sectional curvature with holomorphic bisectional curvature.

The proof of Theorem 1.1 is “soft”, modulo the use of Yau’s Schwarz Lemma. By this we mean that only “coarse” geometric ideas in the sense of Gromov are used. It can be summarized as follows: Without loss of generality, one can assume that M is simply-connected. The key observation is that by using Yau’s Schwarz Lemma one can show that (M, g) has to be bi-Lipschitz to a product of non-compact Riemannian manifolds $(M_1, g_b) \times (M_2, g_a)$. Finally we prove that such a product cannot be Gromov-hyperbolic. But (M, g) being a simply-connected Riemannian manifold of negative sectional curvature (bounded away from zero) has to be Gromov-hyperbolic.

2. Proof

If (X, d) , (X_1, d_1) and (X_2, d_2) are metric spaces, we write $(X, d) = (X_1, d_1) \times (X_2, d_2)$ to mean that $X = X_1 \times X_2$ and $d(p, q) = d_1(p_1, q_1) + d_2(p_2, q_2)$ for all $p = (p_1, p_2)$, $q = (q_1, q_2)$ in X .

For $0 < L < \infty$, metrics d_1 and d_2 on X are said to be *L-bi-Lipschitz* (or just *bi-Lipschitz*) if

$$L^{-1} d_1(x, y) \leq d_2(x, y) \leq L d_1(x, y)$$

for all $x, y \in X$.

Let $M = M_1 \times M_2$ be a complex manifold with Kähler metric g . For $a \in M_1$, $b \in M_2$, let $i_b : M_1 \rightarrow M_1 \times M_2$ and $i_a : M_2 \rightarrow M_1 \times M_2$ be $i_b(x) = (x, b)$ and

$i_a(y) = (a, y)$. Also, let $g_a := i_a^*(g)$ and $g_b := i_b^*(g)$ denote the induced Kähler metrics on M_2 and M_1 respectively.

Key Lemma 2.1. *Let $M = M_1 \times M_2$ be a product complex manifold. Suppose that g is a complete Kähler metric on M with $Ric > d$ and $H < c < 0$, where H denotes holomorphic sectional curvature.*

Then (M, d) is bi-Lipschitz to $(M_1, d_b) \times (M_2, d_a)$, where d, d_b and d_a are the distance functions associated to g, g_b and g_a respectively. Here $a \in M_1$ and $b \in M_2$ are arbitrary.

Proof. We recall Yau’s Schwarz Lemma which is the main ingredient in the proof :

Theorem 2.2. (Yau’s Schwarz Lemma [10]) *Let (M, g) be a complete Kähler manifold and with Ricci curvature $Ric \geq d$ and let (N, h) a Hermitian manifold with holomorphic sectional curvature $H \leq c < 0$, where c and d are constants. If $f : M \rightarrow N$ is a non-constant holomorphic map, then $d < 0$ and $f^*(h) \leq \frac{d}{c}g$.*

Hence $d_h(f(p), f(q)) \leq \sqrt{\frac{d}{c}} d_g(p, q)$ for any $p, q \in M$.

We note that one only needs an upper bound on the holomorphic sectional curvature H , as opposed to the full sectional curvature, of N to apply this theorem (in Yau’s original paper [10], an upper bound on holomorphic bisectional curvature of N is assumed. For a proof involving only H , see [12].).

Now we restrict to $M = M_1 \times M_2$. Observe that the holomorphic sectional curvatures of g_a and g_b are less than c , since $\{a\} \times M_2$ and $M_1 \times \{b\}$ are complex submanifolds of M .

Let $\pi_i : M \rightarrow M_i$ denote the projections. Let $x, y \in M_1$ and $b, b' \in M_2$. Applying Theorem 2.2 to $\pi_1 : (M, g) \rightarrow (M_1, g_b)$, we conclude that

$$d_b(x, y) \leq L d((x, b'), (y, b')),$$

where $L = \sqrt{\frac{d}{c}}$. Since $g_{b'}$ is induced from g , we have $d((x, b'), (y, b')) \leq d_{b'}(x, y)$ and hence

$$d_b(x, y) \leq L d_{b'}(x, y).$$

Interchanging b and b' , we see that the distance functions d_b and $d_{b'}$ are L-bi-Lipschitz on M_1 for any $b, b' \in M_1$. Similarly d_a and $d_{a'}$ are L-bi-Lipschitz on M_2 for any $a, a' \in M_1$.

Fix $a \in M_1$ and $b \in M_2$. Let $p = (p_1, p_2), q = (q_1, q_2)$ be arbitrary points in M . Again, by Theorem 2.2 applied to $\pi_1 : (M, g) \rightarrow (M_1, g_{q_2})$, we get

$$d_{q_2}(p_1, q_1) \leq L d(p, q).$$

Similarly $d_{p_1}(p_2, q_2) \leq L d(p, q)$. Adding these two inequalities and using the L-bi-Lipschitz equivalence of all metrics on M_1 and M_2 (as proved in the previous paragraph), we get

$$(2.1) \quad d_b(p_1, q_1) + d_a(p_2, q_2) \leq 2L^2 d(p, q).$$

On the other hand, letting $r = (p_1, q_2)$ and applying the triangle inequality we have

$$(2.2) \quad \begin{aligned} d(p, q) &\leq d(p, r) + d(r, q) \\ &\leq d_{p_1}(p_2, q_2) + d_{q_2}(p_1, q_1) \\ &\leq L (d_a(p_2, q_2) + d_b(p_1, q_1)). \end{aligned}$$

In the second inequality we have again used the fact that if $p, q \in M_1 \times \{b\}$ for some $b \in M_2$, then $d_b(p, q) \geq d(p, q)$, with similar inequality holding for $p, q \in \{a\} \times M_2$.

Combining (2.1) and (2.2), we see that (M, d) is bi-Lipschitz to $(M_1, d_a) \times (M_2, d_b)$. \square

Remark: The use of Yau's Schwarz Lemma in this proof was inspired by Yang's article [9].

Let us now recall the concept of *Gromov-hyperbolicity* of metric spaces. For the sake of clarity, we give definitions that are more general than are actually needed. For further details, we refer the reader to [3].

Definition 2.3. Let (X, d) be a metric space. A *geodesic* between $p, q \in X$ is an isometric map $\gamma : [0, d(p, q)] \rightarrow X$ with $\gamma(0) = p$, $\gamma(d(p, q)) = q$. A metric space is *geodesic* if any there is a geodesic between any two points in X .

A *geodesic triangle* Δ in X is a union of images of three geodesics $\gamma_i : [a_i, b_i] \rightarrow X$ with $\gamma_i(b_i) = \gamma_{i+1}(a_{i+1})$, where i is taken mod 3. The image of each γ_i is called a *side* of Δ .

A geodesic metric space X is *Gromov-hyperbolic* or δ -*hyperbolic* if there exists $\delta > 0$ such that for any geodesic triangle Δ in X , the δ -neighbourhood of any two sides in Δ contains the third side.

An important class of Gromov-hyperbolic spaces are the simply-connected, complete Riemannian manifolds with sectional curvature bounded above by a negative constant. The Gromov-hyperbolicity of these is a consequence of Toponogov's comparison theorem.

The following basic lemma captures the coarse geometric nature of Gromov-hyperbolicity. This lemma is actually valid under the weaker hypothesis of *quasi-isometric* equivalence, but we only require the bi-Lipschitz case.

Lemma 2.4. (cf. [3]) *Let d and d' be bi-Lipschitz metrics on X . If (X, d) is Gromov-hyperbolic, then so is (X, d') .*

For the next lemma, recall that a *geodesic ray* in X is an isometric map $\gamma : [0, \infty) \rightarrow X$.

Lemma 2.5. *Let $(X, d) = (X_1, d_1) \times (X_2, d_2)$. If there is a geodesic ray in each (X_i, d_i) for $i=1,2$, then (X, d) is not Gromov-hyperbolic.*

Proof. For $i = 1, 2$, let γ_i be a geodesic ray in (X_i, d_i) . For each $n \in \mathbb{Z}^+$, define the map $\sigma_n : [0, 2n] \rightarrow X$ by

$$\begin{aligned} \sigma_n(t) &= (\gamma_1(t), \gamma_2(n)) && \text{for } 0 \leq t \leq n \\ &= (\gamma_1(n), \gamma_2(2n - t)) && \text{for } n \leq t \leq 2n \end{aligned}$$

Then $\sigma_n : [0, 2n] \rightarrow (X, d)$ is a geodesic and if $S_1 = \gamma_1([0, n]) \times \{\gamma_2(0)\}$, $S_2 = \{\gamma_1(0)\} \times \gamma_2([0, n])$ then $\Delta_n = S_1 \cup S_2 \cup \sigma_n([0, 2n])$ is a geodesic triangle. But the distance between $\sigma_n(n) = (\gamma_1(n), \gamma_2(n))$ and $S_1 \cup S_2$ is at least n . Hence (X, d) cannot be δ -hyperbolic for any δ . □

Now we can complete the proof of Theorem 1.1: By taking the universal cover of M if necessary, we can assume that M is simply-connected. As noted earlier, M will then be Gromov-hyperbolic. On the other hand, since M is diffeomorphic to \mathbb{R}^{2n} by the theorem of Cartan-Hadamard, both M_1 and M_2 are noncompact. Moreover, since (M, g) is complete, (M_1, g_b) and (M_2, g_a) are complete for any $a \in M_1$ and $b \in M_2$. Now one can always find a geodesic ray in any complete noncompact Riemannian manifold. By Lemma 2.5, $(M_1, d_b) \times (M_2, d_a)$ is *not* Gromov-hyperbolic. Hence, neither is (M, d) by Lemma 2.1 and Lemma 2.4. This contradiction completes the proof. □

3. Negatively curved Kähler metrics on \mathbb{C}^n

The example in this section is adapted from [5] and we refer the reader to it for further details. It is clear from Klembeck’s computations that if

$$g = \sum_{i,j=1}^n \frac{\partial f(r^2)}{\partial z_i \partial \bar{z}_j} dz_i \otimes d\bar{z}_j,$$

where $f : \mathbb{R} \rightarrow \mathbb{R}$ and $r^2 = |z_1|^2 + \dots + |z_n|^2$, then the following are sufficient conditions for g to be a complete Kähler metric of strictly negative sectional curvature on \mathbb{C}^n :

- (a) $f'(r^2) + r^2 f''(r^2) > 0$,
- (b) $\int_0^\infty \sqrt{f'(r^2) + r^2 f''(r^2)} dr = \infty$,
- (c) $f''(r^2) > 0$,
- (d) $f'''(r^2) + r^2 f''''(r^2) - r^2 \frac{f''(r^2)^2}{f'(r^2)} > 0$,
- (e) $\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \ln \left(f'(r^2) + r^2 f''(r^2) \right) \right) > 0$

If we let $f(x) = e^x$, then it can be checked that all the above conditions are satisfied. Moreover, the sectional curvatures of g will be bounded below. Indeed, some sectional curvatures will tend to zero exponentially fast while some will equal -2 as $r \rightarrow \infty$.

Hence the upper sectional curvature bound in Theorem 1.1 is necessary.

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