Math. Res. Lett. **13** (2006), *no.* 3, 343–357

TOPOLOGICAL INVARIANCE OF GENERIC NON-UNIFORMLY EXPANDING MULTIMODAL MAPS

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Abstract. We introduce a *Topological Slow Recurrence* condition on the orbits of critical points of multimodal maps and show that this condition is equivalent to the simultaneous occurrence of metric *Slow Recurrence* and *Collet-Eckmann* conditions. In particular, if f is non-uniformly expanding and the critical points are generic with respect to the absolutely continuous invariant measures, then any map g topologically conjugate to f is also non-uniformly expanding.

1. Statement of results

A main theme of this paper is the relation between the *topological* structure of a dynamical system and its *measure-theoretic* and *ergodic* properties. Throughout the paper we shall be concerned with the class $\mathcal{S}$ of C^3 interval maps $f : I \to I$ with negative Schwarzian derivative and a finite set $\mathcal{C}$ of non-flat critical points with possibly different orders, we give the precise definitions in section 1.5 on page 346 below.

1.1. The Main Theorem. We say that f satisfies condition CE if all its critical points satisfy the *Collet-Eckmann* [ColEck83] condition

$$\text{(CE)} \qquad \liminf_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \log |Df_{f^i(c)}| > \lambda > 0$$

We say that f satisfies condition (SR) if all its critical points satisfy the *slow recurrence* condition

$$\text{(SR)} \qquad \lim_{\delta\to 0^+} \liminf_{n\to+\infty} \frac{1}{n} \sum_{\substack{1\le i\le n \\ f^i(c)\in\mathcal{C}_\delta}} \log d(f^i(c)) = 0.$$

Here $d(x) = d(x,\mathcal{C}) = \min\{|x-c| : c \in \mathcal{C}\}$ and $\mathcal{C}_\delta = \{x : d(x,\mathcal{C}) \le \delta\}$ is a metric neighbourhood of the critical set. Our main result says that the simultaneous occurrence of both conditions CE and SR is topologically invariant. We recall that f, g are *topologically conjugate* ($f \sim g$) if there exists a homeomorphism h such that $h \circ f = g \circ h$.

Main Theorem. *Suppose that f satisfies both CE and SR. Then every map g topologically conjugate to f also satisfies both CE and SR.*

Received by the editors April 28, 2004.
2000 *Mathematics Subject Classification.* 37C15, 37C05.
The second named author was supported by China NSF 19901035.

In the unimodal case, it was shown in [NowSan98, PrzRoh98, NowPrz98] that CE itself is topologically invariant. This is false, however, in the multimodal case [PrzRivSmi03]. Indeed, recent counterexamples show that several properties of unimodal Collet-Eckmann maps do not extend to the multimodal setting; perhaps these counterexamples may be overcome by assuming the simultaneous occurrence of CE and SR, as in the present paper.

1.2. Topological Slow Recurrence. Our strategy is to define a new condition which we call Topological Slow Recurrence (TSR) condition, see (11) on page 349, which depends only on the combinatorics of the critical orbits. In particular TSR is invariant under topological conjugacy. We will then prove the following double implication:

Main Technical Theorem. *For any $f \in \mathcal{S}$ we have*

$$CE + SR \iff TSR. \tag{1}$$

Our approach is therefore quite distinct from the proof in [NowPrz98] of the invariance of the CE condition in the unimodal case. The present strategy is more direct and in the spirit of [San95] in which, in the unimodal setting, a topological condition is formulated in terms of the kneading sequence of the critical point and this condition is shown to imply the Collet-Eckmann condition. The relation between Sands' topological condition and our Slow Recurrence condition (SR) was studied in [Wan01] in the unimodal case, and a generalization of the topological condition given in the bimodal case in [Wan96]. Here we push this whole approach significantly further by giving a more general definition of this condition which in particular applies to the multimodal setting.

1.3. Non-uniformly expanding maps. Classical results in one-dimensional dynamics imply that any map f satisfying CE is *(non-uniformly) expanding* in the sense that there exists a finite number $\mu_1, .., \mu_q$, $q \geq 1$, of ergodic absolutely continuous invariant probability measures with positive Lyapunov exponents:

$$0 < \int \log |Df| d\mu < \infty \tag{2}$$

for each $\mu = \mu_i$, $i = 1, .., q$. The converse however is not true: there are many examples of non-uniformly expanding maps which do not satisfy CE. A natural question is whether the non-uniform expansivity itself may be topological invariant. A counterexample of Bruin [Bru98a]shows that the answer is negative in general even in the unimodal case. The first examples of (topological classes of) maps with an absolutely continuous invariant measure which is preserved by topological conjugacy were constructed by Bruin [Bru94, Bru98b]. The topological invariance of CE in the unimodal case and of CE+SR in the multimodal case gives some additional conditions which ensure that the non-uniform expansivity property is not destroyed under a topological change of coordinates.

We remark that CE and SR are "abundant" in the sense that they occur with positive probability for quite general families of multimodal maps [Tsu93]. Moreover, they are also both "generic" conditions within the class of non-uniformly expanding maps in the following sense. By Birkhoff's Ergodic Theorem almost every point satisfies

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \log |Df_{f^i(x)}| = \int \log |Df| d\mu. \tag{3}$$

for one of the absolutely continuous invariant measures $\mu = \mu_i$. We say that the critical points are *generic* if they satisfy (3) and it is easy to see that in this case they satisfy both CE and SR. Indeed, CE is immediate, and SR follows by observing that $\log |Df_{f^i(x)}|$ and $\log d(f^i(x))$ are uniformly comparable by the non-flatness of the critical points (4) and therefore Birkhoff's Ergodic Theorem and (3) imply

$$\frac{1}{n} \sum_{\substack{f^i(x)\in\mathcal{C}_\delta \\ 1\le i\le n}} \log |Df_{f^i(c)}| \to \int_{\mathcal{C}_\delta} \log |Df| d\mu$$

and thus integrability condition (2) implies that $\int_{\mathcal{C}_\delta} \log |Df| d\mu \to 0$ as $\delta \to 0$. In particular we have the following consequence of our results.

Corollary. *Suppose that f is non-uniformly expanding, $f \sim g$, and that all critical points of f are generic. Then g is non-uniformly expanding.*

We remark that this result does not imply that the conjugacy h is absolutely continuous and thus preserves the absolute continuity of the measure. In general this will not be true, although both f and g will have absolutely continuous invariant probability measures, these are not mapped to each other by the conjugacy.

1.4. Overview of the paper. In section 2 we give all the details of the combinatorial structure and define condition TSR. We give a brief summary of the key ideas here. The critical set $\mathcal{C}$ defines a partition $\mathcal{P}$ of I which allows us to define a symbolic itinerary for all points which are not preimages of $\mathcal{C}$ and to define cylinder sets made up of points which share the same itinerary up to some finite time. The whole structure of cylinder sets is topological. There is then a natural *topological* separation time: any point x near $\mathcal{C}$ shadows the orbit of the critical point for a time defined by the number of iterates for which x and c have the same symbolic itinerary. This shadowing time tends to ∞ as x tends to c. In particular the symbolic sequence associated to any given point contains information about its pattern of recurrence to the critical set: if it contains long finite blocks which coincide with long initial blocks of the symbolic sequences of one of the critical point then it must have come quite close to the critical set. This simple idea is used in a crucial way in the definition of the condition TSR.

In the remaining three sections we proceed to prove the three steps required to show (1). We first show that CE + SR implies TSR, then that TSR implies SR and finally that TSR implies CE. A particular effort has been made here to make the presentation as self-contained as possible and as much as possible independent of specialized technical notation and constructions familiar mainly to established specialists in one-dimensional dynamics.

1.5. S-multimodal maps with non-flat critical points. Before ending this section we give the formal definition of the class $\mathcal{S}$ of maps under consideration and fix some fairly standard notation and definitions. We shall consider *S-multimodal maps with non-flat critical points* by which we mean maps f satisfying the following conditions. Let $f : I \to I$ be a C^3 map, where $I \subset \mathbb{R}$ is a closed interval. We shall assume without loss of generality that $I = [0, 1]$. We call $c \in I$ a critical point of f if $Df(c) = 0$. Denote by $\mathcal{C}$ the set of all critical points. f has finite number (say q) of *non-flat critical points* $0 < c_1 < c_2 < \cdots < c_q < 1$ if there exist $L_i > 1$, $l_i > 1$ and a neighbourhood $V(c_i)$ of c_i in which

$$|x - c_i|^{l_i - 1}/L_i \le |Df(x)| \le L_i|x - c_i|^{l_i - 1} \tag{4}$$

for any $x \in V(c_i)$ and $1 \le i \le q$. f has *negative Schwarzian derivative*, i.e. $S(f)(x) = (D^3f(x)/Df(x)) - \frac{3}{2}(D^2f(x)/Df(x))^2 < 0$ for $x \in (0, 1)\backslash\mathcal{C}$.

As part of the argument we shall need to obtain some derivative estimates about the iterates of f along the forward orbit of the critical values. If these orbits map to a periodic point then the estimates are easily obtained. Otherwise we need to consider a two-sided neighbourhood of each critical value and the forward images of this neighbourhood. We assume therefore that the forward orbits of all the critical points are disjoint from the boundary of I or, if not, that the boundaries of I are fixed points. This is not restrictive since, if it is not the case, we can extend f to a map $\hat{f}$ defined on neighbourhood $\hat{I}$ of I such that the forward orbits of all the critical points are disjoint from the boundary of $\hat{I}$ and the dynamics on I is unaffected.

For simplicity we shall suppose moreover that all critical points are turning points, i.e. points at which the map fails to be a local homeomorphism. Otherwise the same conclusions hold as long as we assume that the topological conjugacy maps critical points to critical points (this is not automatic in the presence of inflexion-type critical points).

2. Combinatorial structure

In this section we describe in detail the combinatorial structure associated to any S-multimodal map with non-flat critical points and having no neutral or attracting periodic orbits. We emphasize that everything we describe here relies on no other assumptions. Some of the definitions are closely related to the so-called *cutting/co-cutting times* used in [Hof80, HofKel90b, Bru95a, San95, NowSan98] for unimodal maps; we give a generalization to the multimodal setting and, we believe, a simplification of the way those concepts are introduced and applied.

We fix for the rest of the paper a constant

$$\delta_0 > 0 \tag{5}$$

sufficiently small so that the critical neighbourhood $\mathcal{C}_{\delta_0}$ has precisely q disjoint connected components; in particular any $x \in \mathcal{C}_\delta$ for $\delta < \delta_0$ is associated unambiguously to one particular critical point. For any x we shall use the notation $x^i = f^i(x)$, especially, $c^i = f^i(c)$ for a critical point c.

2.1. Symbolic dynamics. The critical points define a natural partition

$$\mathcal{I} = \{I_0, \ldots, I_q\}$$

of I into $q+1$ subintervals. This allows us to associate a symbolic *itinerary* to any point not in the preimage of a critical point by letting

$$\underline{a}(x) = a_0 a_1 a_2 \dots \text{ where } a_i = k \text{ if } f^i(x) \in I_k \text{ for all } i \geq 0.$$

For convenience we associate to each critical point two sequences which differ only in the first term, so that we think of the partition elements as closed intervals and think of each critical point as belonging to both of the adjacent intervals. We define the symbolic *separation time* between two points as

$$s(x,y) = \min\{k \geq 0 : a_k(x) \neq a_k(y)\}$$

as long as the itineraries $\underline{a}(x)$ and $\underline{a}(y)$ are both defined. If $y = c$ is a critical point and x is in one of the adjacent partition elements then we naturally consider the symbolic itinerary for c for which $s(x,y) = s(x,c) \geq 1$. In particular we define the time it takes for a point x to "separate" from the critical set $\mathcal{C}$ by

$$s(x) = s(x, \mathcal{C}) = \max\{s(x,c) : c \in \mathcal{C}\}.$$

Notice that $s(x) \to \infty$ as $d(x) \to 0$. We define *topological neighbourhoods* $\mathcal{C}_n$ of the critical set $\mathcal{C}$ by

$$\mathcal{C}_n = \{x : s(x) \geq n\}.$$

We fix a constant N_0 sufficiently large so that

$$\mathcal{C}_{N_0} \subset \mathcal{C}_{\delta_0}, \tag{6}$$

recall (5), and such that every point x with $s(x) \geq N_0$ is associated to some particular critical point c and $d(x) = d(x,c) \leq \delta_0$ and $s(x) = s(x,c)$.

For any given finite sequence $a_0 \dots a_n$ with $a_i \in \{0, \dots, q\}$ we define the *cylinder set*

$$\hat{I}^{(n)}_{a_0..a_n} = \{x : f^i(x) \in I_{a_i}, 0 \leq i \leq n\}$$

Notice that such a cylinder set may be empty. For a given point x with a symbolic itinerary $\underline{a}(x)$ we define the n'th order cylinder set of x as

$$\hat{I}^{(n)}(x) = \hat{I}^{(n)}_{a_0 \dots a_n}.$$

This is always a two sided neighbourhood of x. We shall often write

$$\hat{I}^{(n)}_-(x) \quad \text{and} \quad \hat{I}^{(n)}_+(x)$$

to denote the part of $\hat{I}^{(n)}(x)$ to the left and right respectively of the point x. Clearly $\hat{I}^{(n+1)}(x) \subseteq \hat{I}^{(n)}(x)$ for any n. Moreover, by the non-existence of wandering intervals [Guc79, MelStr93] the preimages of the critical set are dense and therefore we have

$$|\hat{I}^{(n)}(x)| \to 0 \quad \text{as} \quad n \to \infty \quad \forall\, x \in I.$$

In particular the cylinder sets $\hat{I}^{(n)}(x)$ define a nested sequence of neighbourhoods of the point x.

2.2. Partitions. We now fix some arbitrary point x not contained in any preimage of the critical set. We fix some notation related to the partition $\mathcal{P}(x)$ of a neighbourhood of x defined naturally by the cylinder sets. More precisely, consider a generic cylinder set $\hat{I}^{(n-1)} = \hat{I}^{(n-1)}(x)$. By definition the images $f^j(\hat{I}^{(n-1)})$ all belong to the same element of $\mathcal{I}$ for all $j \le n-1$. Then there are two possibilities:

(1) $f^n(\hat{I}^{(n-1)})$ does not contain any critical points in its interior. In this case all points continue to share the same combinatorics and we have $\hat{I}^{(n)} = \hat{I}^{(n-1)}$, and thus in particular
$$\hat{I}^{(n-1)} \setminus \hat{I}^{(n)} = \emptyset.$$

(2) $f^n(\hat{I}^{(n-1)})$ contains one or more critical points in its interior. In this case the cylinder set $\hat{I}^{(n)} \subset \hat{I}^{(n-1)}$ is given precisely by those points of $\hat{I}^{(n-1)}$ which fall into the same element of $\mathcal{I}$ as $f^n(x)$ and
$$\hat{I}^{(n-1)} \setminus \hat{I}^{(n)} \neq \emptyset.$$
In this case $\hat{I}^{(n-1)} \setminus \hat{I}^{(n)}$ may have either one or two connected components.

We index the times at which case 2 above occurs by two sequences
$$\mathcal{N}_x^- = \{n_i^-\}_{i=1}^\infty \quad \text{and} \quad \mathcal{N}_x^+ = \{n_i^+\}_{i=1}^\infty$$
where
$$n_i^- \text{ denotes a time for which } \hat{I}_-^{(n_i^- -1)} \setminus \hat{I}_-^{(n_i^-)} \neq \emptyset$$
and
$$n_i^+ \text{ denotes a time for which } \hat{I}_+^{(n_i^+ -1)} \setminus \hat{I}_+^{(n_i^+)} \neq \emptyset.$$
For a general point x, the two sequences $\mathcal{N}_x^-$ and $\mathcal{N}_x^+$ are independent even though there may well be one or more pairs i, j such that $n_i^- = n_j^+$. For the special case of the partition $\mathcal{P}(c)$ associated to one of the critical points c, we actually have $\mathcal{N} = \mathcal{N}_c^- = \mathcal{N}_c^+$. For each such time $n_i^\pm$ we define the intervals
$$I^{(n_i^\pm)} = \hat{I}_\pm^{(n_i^\pm -1)} \setminus \hat{I}_\pm^{(n_i^\pm)}$$
which define precisely the elements of the partition
$$\mathcal{P}(x) = \left\{ I^{(n_i^\pm)} : n_i^\pm \in \mathcal{N}_x^\pm \right\}$$
of a neighbourhood of x.

2.3. Shadowing times. We call the elements of the sequences $\mathcal{N}_x^-$ and $\mathcal{N}_x^+$ respectively the *left and right shadowing times* associated to x because every point in $I^{(n_i^\pm)}$ "shadows" the point x for $n_i^\pm$ iterations:
$$s(y,x) = n_i^\pm \iff y \in I^{(n_i^\pm)}(x) \tag{7}$$
The partition $\mathcal{P}(x)$ (and in particular $\mathcal{P}(c)$ for some critical point c) can be thought of as topological versions of the partitions defined by the *binding periods* of Benedicks and Carleson [BenCar85]. Here we lose some good properties of binding periods such as uniformly bounded distortion but gain in other ways, as shall become clear below.

First we state three relatively straightforward but not immediately obvious properties of this partition which will play an important role. First of all notice that

$$f^{n_{i+1}^{\pm}} \text{ is monotone on } \hat{I}_{\pm}^{(n_i^{\pm})}. \tag{8}$$

Indeed, by construction the boundary points of each partition element $I^{(n_i^{\pm})}$ are preimages of some critical point, and one of the boundary points of each cylinder set $\hat{I}_{\pm}^{(n_i^{\pm})}$ is a preimage of some critical point of order $n_i^{\pm}$. Moreover, there are no points in the interior of $\hat{I}_{\pm}^{(n_i^{\pm})}$ which map to a critical point before time $n_{i+1}^{\pm}$. Thus (8) follows.

Secondly, for any y close to x we have

$$|\hat{I}_{\pm}^{s(y,x)-1}| \geq d(y,x) \geq |\hat{I}_{\pm}^{s(y,x)}|. \tag{9}$$

Indeed, by construction we have $\hat{I}_{\pm}^{(m)} = \hat{I}_{\pm}^{(n_{i-1}^{\pm})}$ for $n_{i-1}^{\pm} \leq m < n_i^{\pm}$, and so in particular $\hat{I}_{\pm}^{(n_{i-1}^{\pm})} = \hat{I}_{\pm}^{(n_i^{\pm}-1)}$. Therefore, for any $y \in I^{(n_i^{\pm})}$ the relation (7) gives

$$y \in I^{(n_i^{\pm})} = \hat{I}_{\pm}^{(n_{i-1}^{\pm})} \setminus \hat{I}_{\pm}^{(n_i^{\pm})} = \hat{I}_{\pm}^{(n_i^{\pm}-1)} \setminus \hat{I}_{\pm}^{(n_i^{\pm})} = \hat{I}_{\pm}^{s(y,x)-1} \setminus \hat{I}_{\pm}^{s(y,x)}$$

which implies (9).

Thirdly, for every $i \geq 1$ such that $n_{i+1}^{\pm} - n_i^{\pm} \geq N_0$ we have

$$s(x^{n_i^{\pm}}) = n_{i+1}^{\pm} - n_i^{\pm}. \tag{10}$$

To see this consider the two intervals $[y,x] := \hat{I}_{\pm}^{(n_{i+1}^{\pm})} \subset \hat{I}_{\pm}^{(n_i^{\pm})} =: [z,x]$. By construction we have $f^{n_i^{\pm}}(z) = \tilde{c}$ for some critical point $\tilde{c} \in \mathcal{C}$ and thus $f^{n_i^{\pm}}(\hat{I}_{\pm}^{(n_i^{\pm})}) = f^{n_i^{\pm}}[z,x] = [\tilde{c}, x^{n_i^{\pm}}]$ and $f^{n_i^{\pm}}(y) \in (\tilde{c}, x^{n_i^{\pm}})$, i.e. the interval $[\tilde{c}, x^{n_i^{\pm}}]$ contains the point $f^{n_i^{\pm}}(y)$ in its interior. By (8), $f^{n_{i+1}^{\pm}}$ is monotone on $\hat{I}_{\pm}^{(n_i^{\pm})}$ which implies in particular that $f^j(\hat{I}_{\pm}^{(n_i^{\pm})})$ cannot contain any critical point in its interior before time $n_{i+1}^{\pm}$, i.e. $f^j(\hat{I}_{\pm}^{(n_i^{\pm})}) \cap \mathcal{C} = \emptyset$ for all $j < n_{i+1}^{\pm}$. At this time we have $f^{n_{i+1}^{\pm}}(y) = f^{n_{i+1}^{\pm}-n_i^{\pm}}(f^{n_i^{\pm}}(y)) = \hat{c}$ for some (other) critical point $\hat{c} \in \mathcal{C}$. Therefore $n_{i+1}^{\pm} - n_i^{\pm}$ is exactly the first time at which the iterates of $\tilde{c}$ and of $x^{n_i^{\pm}}$ fall on different sides of some critical point, and therefore is exactly the separation time: $s(x^{n_i^{\pm}}, \tilde{c}) = n_{i+1}^{\pm} - n_i^{\pm}$. If $n_{i+1}^{\pm} - n_i^{\pm} \geq N_0$ this implies that $s(x^{n_i^{\pm}}) = s(x^{n_i^{\pm}}, \tilde{c})$ as in (10).

2.4. Topological Slow Recurrence. We are now ready to formulate the *Topological Slow Recurrence* condition for an arbitrary point x:

$$\lim_{m\to\infty} \limsup_{n\to+\infty} \frac{1}{n} \sum_{\substack{1\leq j\leq n \\ x^j \in \mathcal{C}_m}} s(x^j) = 0. \tag{11}$$

Definition 1. We say that the map f satisfies condition TSR if all the critical points satisfy (11)

Notice that this depends only on the orbit of the critical points with respect to the partitions $\mathcal{P}(c)$ for $c \in \mathcal{C}$. It is therefore invariant under topological conjugacy.

3. CE + SR implies TSR

Lemma 1. *Let f satisfy condition CE. Then there exists a constant $\bar\kappa > 0$ such that*

$$s(c^j) \le \bar\kappa \log d(c^j)^{-1} \quad \forall c \in \mathcal{C},\ \forall\ j \ge 1. \tag{12}$$

We shall use the following

Sublemma 1.1. *[NowPrz98] There exists constants $C, \bar\xi > 0$ such that for all intervals J and integers $s \ge 1$ such that $f^s|_J$ is monotone we have $|J| \le Ce^{-s\bar\xi}$.*

Proof of Lemma 1. We only need to prove the result for $s(c^j) \ge N_0$ as the choice of the constant $\bar\kappa$ can take into account smaller values. Then there is a well-defined critical point $\tilde c$ "closest" to c^j i.e. $s(c^j) = s(c^j, \tilde c)$, and $d(c^j) = d(c^j, \tilde c) = |c^j - \tilde c|$. The map $f^{s(c^j)}$ is monotone on the interval $[c^j, \tilde c]$ and therefore Sublemma 1.1 implies $d(c^j) = |c^j - \tilde c| \le Ce^{-s\bar\xi}$. □

Corollary 1.1. *Let f satisfy conditions CE and SR. Then it satisfies condition TSR.*

Proof. By condition SR, for every $\varepsilon > 0$ there exist $n_\varepsilon, \delta_\varepsilon > 0$ such that

$$\frac{1}{n} \sum_{\substack{1 \le j \le n \\ f^j(c) \in \mathcal{C}_{\delta_\varepsilon}}} \log |d(f^j(c)|^{-1} < \varepsilon \quad \forall\ c \in \mathcal{C},\quad \forall\ n \ge n_\varepsilon.$$

Therefore, choosing T_ε sufficiently large so that $\mathcal{C}_{T_\varepsilon} \subseteq \mathcal{C}_{\delta_\varepsilon}$ and using (12) we have

$$\frac{1}{n} \sum_{\substack{1 \le j \le n \\ f^j(c) \in \mathcal{C}_{T_\varepsilon}}} s(c^j) \le \frac{\bar\kappa}{n} \sum_{\substack{1 \le j \le n \\ f^j(c) \in \mathcal{C}_{\delta}}} \log |d(c^j)|^{-1} < \bar\kappa\varepsilon.$$

Since ε is arbitrary, this implies TSR. □

4. TSR implies SR

Lemma 2. *Let f satisfy condition TSR. Then there exists a constant $\underline{\kappa} > 0$ such that*

$$s(c^j) \ge \underline{\kappa} \log d(c^j)^{-1} \quad \forall c \in \mathcal{C},\ \forall\ j \ge 1. \tag{13}$$

Arguing as in the proof of Corollary 1.1 we then get

Corollary 2.1. *Let f satisfy condition TSR. Then it satisfies condition SR.*

We first give two preliminary results which will be used in the proof.

Sublemma 2.1. *There exists constants $C, \underline{\xi} > 0$ such that for any interval J and any positive integer n such that J is a one-side neighbourhood of a critical point c, $f^n(J)$ contains a one-sided neighbourhood of the same critical point c and $f^{n+1}(J) \supset f(J)$, we have $|J| \ge Ce^{-n\underline{\xi}}$ (we do not assume that $f^{n+1}|_J$ is monotone).*

Proof. By the non-flatness of the critical points, there exists a constant $\theta > 0$ depending only on f such that in either case we have $\theta|J| \le |f^n(J)|$. Moreover, letting $D = \max_{x\in I} |Df(x)|$, the mean value theorem implies $|f^n(J)| \le D^{n-1}|f(J)|$, and, using the non-flatness of the critical points again, we have $|f(J)| \le L|J|^\ell$ for some constants ℓ, L depending only on f. Combining these estimates give $\theta|J| \le LD^{n-1}|J|^\ell$ and therefore $|J|^{\ell-1} \ge \theta L^{-1} D^{-n+1}$ or $|J| \ge (\theta L^{-1} D)^{1/\ell-1} D^{-n/(\ell-1)}$. □

Sublemma 2.2. *Let f satisfy condition TSR. Then for any $\varepsilon > 0$ there exists $N_\varepsilon > 0$ such that*

$$n_{i+1}(c) - n_i(c) \leq \varepsilon n_i(c) \quad \forall\, c \in \mathcal{C}, \quad \forall\, n_i \geq N_\varepsilon. \tag{14}$$

Proof. From TSR we have $s(c^j)/j \to 0$ as $j \to \infty$. By (10) this immediately implies the statement. □

Proof of Lemma 2. We first define in an iterative process which may stop according to one of two possible stopping rules. We then show that in both cases the conclusions of the Lemma are satisfied. We fix some $\varepsilon \in (0, 1)$ and let N_ε be the corresponding integer from Sublemma 2.2.

Step 1. Let $s_0 = s(c^j)$.

- *If $s_0 < \max\{N_0, N_\varepsilon\}$ go to step 3.*

Otherwise we argue as follows. Since $s_0 \geq N_0$, there is a well defined critical point $c_{(0)}$ such that $d(c^j) = d(c^j, c_{(0)})$ and $s(c^j) = s(c^j, c_{(0)})$. By (7) there exists some $n_{i_0} \in \mathcal{N}_{c_{(0)}}$ with $n_{i_0} = s_0$ and

$$\hat{I}^{n_{i_0}} \subset (c^j, c_{(0)}) \subset \hat{I}^{n_{i_0}-1}.$$

Here we omit the subscripts $\pm$ not to overload the notation, and let $\hat{I}^{n_{i_0}} = \hat{I}^{n_{i_0}}_+$ or $\hat{I}^{n_{i_0}}_-$ according to the relative positions of c^j and $c_{(0)}$. By construction

$$f^{s_0}(\hat{I}^{n_{i_0}}) = (c^{s_0}_{(0)}, c_{(1)})$$

for some critical point $c_{(1)}$.

- *If $c_{(1)} = c_{(0)}$ go to step 2.*

Otherwise we repeat the algorithm with $c^{s_0}_{(0)}$ playing exactly the role of c^j above. More precisely, we consider the separation time $s_1 = s(c^{s_0}_{(0)})$. Since $s_0 > N_\varepsilon$, (10) and (14) give

$$s_1 = n_{i_0+1} - n_{i_0} \leq \varepsilon n_{i_0} = \varepsilon s_0.$$

- *If $s_1 < \max\{N_0, N_\varepsilon\}$ go to step 3.*

Otherwise there exists some $n_{i_1} \in \mathcal{N}_{c_{(1)}}$ such that $n_{i_1} = s_1$ and

$$\hat{I}^{n_{i_1}} \subset (c^{s_0}_{(0)}, c_{(1)}) \subset \hat{I}^{n_{i_1}-1}.$$

Again, by construction

$$f^{s_1}(\hat{I}^{n_{i_1}}) = (c^{s_1}_{(1)}, c_{(2)})$$

for some critical point $c_{(2)}$.

- *If $c_{(2)}$ equals either $c_{(0)}$ or $c_{(1)}$ go to step 2.*

Otherwise we repeat the process again with $c^{s(1)}_{(1)}$ playing the role of $c^{s(0)}_{(0)}$ and continue in this way until we go to either step 2 or step 3. In the general case we have $s_j = s(c^{s_{j-1}}_{(j-1)})$ and

$$s_j = n_{i_{j-1}+1} - n_{i_{j-1}} \leq \varepsilon n_{i_{j-1}} \leq \varepsilon^j s_0. \tag{15}$$

- *If $s_j < \max\{N_0, N_\varepsilon\}$ go to step 3.*

Otherwise there exists some $n_{i_j} \in \mathcal{N}_{c_{(j)}}$ with $n_{i_j} = s_j$ such that

$$\hat{I}^{n_{i_j}} \subset (c^{s_{j-1}}_{(j-1)}, c_{(j)}) \subset \hat{I}^{n_{i_j}-1} \text{ and } f^{s_j}(\hat{I}^{n_{i_j}}) = (c^{s_j}_{(j)}, c_{(j+1)})$$

for some critical point $c_{(j+1)}$.

$\bullet$*If* $c_{(j+1)}$ *equals any one of* $c_{(0)}, c_{(1)}, \ldots, c_{(j)}$ *go to step 2.*

The process has to stop in a maximum of $q+1$ steps as by that time the above condition is necessarily satisfied.

Step 2. We suppose here that the procedure described above gives constants $0 \leq k < m \leq q+1$ such that $c_{(m)} = c_{(k)}$ and therefore we have one-sided neighbourhoods $(c^{s_{k-1}}_{(k-1)}, c_{(k)})$ and $(c^{s_{m-1}}_{(m-1)}, c_{(m)}) = (c^{s_{m-1}}_{(m-1)}, c_{(k)})$ of the same critical point $c_{(k)}$. Moreover, by construction we have that $f^{s_k+\ldots+s_{m-1}}(c^{s_{k-1}}_{(k-1)}, c_{(k)}) \supset (c^{s_{m-1}}_{(m-1)}, c_{(m)}) = (c^{s_{m-1}}_{(m-1)}, c_{(k)})$ and therefore we are in a position to apply the estimates of Sublemma 2.1 to the interval $J = (c^{s_{k-1}}_{(k-1)}, c_{(k)})$ if we can show that

$$f(c^{s_{m-1}}_{(m-1)}, c_{(k)}) \supset f(c^{s_{k-1}}_{(k-1)}, c_{(k)}). \tag{16}$$

To see that this is the case we recall that s_j satisfies (15) for all $0 \leq j < m$ (for otherwise we would have gone straight to Step 3) and thus in particular the separation time $s(c^{s_{m-1}}_{(m-1)}, c_{(k)}) = s(c^{s_{m-1}}_{(m-1)}, c_{(m)}) = s_m$ is strictly less than $s(c^{s_{k-1}}_{(k-1)}, c_{(k)}) = s_k$. This implies that the separation time $s(f(c^{s_{m-1}}_{(m-1)}), f(c_{(k)}))$ is strictly less than $s(f(c^{s_{k-1}}_{(k-1)}), f(c_{(k)}))$ where both $(f(c^{s_{m-1}}_{(m-1)}), f(c_{(k)}))$ and $(f(c^{s_{k-1}}_{(k-1)}), f(c_{(k)}))$ are one-sided neighbourhood (on the same side) of the critical value $f(c_{(k)})$. Clearly this implies that $|f(c^{s_{k-1}}_{(k-1)}) - f(c_{(k)})| < |f(c^{s_{m-1}}_{(m-1)}) - f(c_{(k)})|$ which is precisely (16). Therefore, by Sublemma 2.1 we have

$$|(c^{s_{k-1}}_{(k-1)}, c_{(k)})| \geq Ce^{-\underline{\xi}(s_k+\ldots+s_{m-1})}. \tag{17}$$

By construction we also have $f^{s_0+\ldots+s_{k-1}}(c^j, c_{(0)}) \supset (c^{s_{k-1}}_{(k-1)}, c_{(k)})$ and therefore by the mean value theorem we have

$$|(c^j, c_{(0)})| \geq D^{-(s_0+\ldots+s_{k-1})} |(c^{s_{k-1}}_{(k-1)}, c_{(k)})|. \tag{18}$$

Combining (17) and (18) we get that there exist constants $C, \eta > 0$ such that $d(c^j) = |(c^j, c_{(0)})| \geq Ce^{-\eta(s_0+\ldots+s_{m-1})}$. Finally, from (15) we have that there exists a uniform constant $\nu = 1 + \sum \varepsilon^i$ such that $s_0 + \ldots + s_{m-1} \leq \nu s_0 = \nu s(c^j)$, which then gives $d(c^j) = |(c^j, c_{(0)})| \geq Ce^{-\eta\nu s(c^j)}$. This completes the proof in this case.

Step 3. We now consider the situation in which the procedure described in Step 1 leads to the existence of some $0 \leq j \leq q+1$ such that $s_j < \max\{N_0, N_\varepsilon\}$. Since $s_j = s(c^{s_{j-1}}_{(j-1)}, c_{(j)})$ we have that $|c^{s_{j-1}}_{(j-1)} - c_{(j)}| \geq \delta$ where $\delta > 0$ is some constant depending only on f, N_0 and N_ε. Therefore a similar argument to that used in the final part of Step 2 gives the result in this case also. $\square$

5. Topological Slow Recurrence implies CE

We now show that for S-multimodal maps the Topological Slow Recurrence condition implies the Collet-Eckmann condition. We shall be concentrating here on the cylinder sets associated to the *critical values* rather than those associated to the critical points used in the arguments given in the previous section. Consider a critical value c^1 with shadowing times $n_i^\pm$ and corresponding intervals $\hat{I}_\pm^{(n_i^\pm)}$ and $I^{(n_i^\pm)}$ as in section 2.

Definition 2. For any (large) $T > 0$ we say that the gap $n_i^- - n_{i-1}^-$, respectively $n_j^+ - n_{j-1}^+$, between two left, respectively right, shadowing times is *short* if $n_i^- - n_{i-1}^- < T$, respectively $n_j^+ - n_{j-1}^+ < T$. We say that a left short gap and a right short gap are *simultaneous* if they are contained in an interval $\mathcal{T} = [t_1, t_2]$ of times of length $t_2 - t_1 < T$.

In section 5.1 we show that there exist values of T for which there are many intervals of simultaneous short shadowing times. In section 5.2 we show that each such interval implies a uniform estimate on the exponential shrinking of the length of intervals associated to some particular cylinder sets at the corresponding critical value. Finally, in section 5.3 we combine these two results to obtain the Collet-Eckmann property.

5.1. Positive density of simultaneous short shadowing times.

Lemma 3. *For every small $\varepsilon > 0$ there exists $T = T_\varepsilon > 0$, $n_\varepsilon > 0$ and $\eta = \eta_\varepsilon = (1 - 2\varepsilon)/2T_\varepsilon > 0$ such that the number of disjoint simultaneous short (with respect to T) shadowing time intervals associated to any critical values, and occurring before time n, is $\geq \eta n$ for every $n \geq n_\varepsilon$.*

Proof. Suppose that a point x satisfies condition (11). Then for all $\varepsilon > 0$ sufficiently small, there exist $n_\varepsilon, T_\varepsilon$ such that

$$\sum_{\substack{1 \leq n_i^\pm \leq n \\ n_{i+1}^\pm - n_i^\pm > T}} n_{i+1}^\pm - n_i^\pm = \sum_{\substack{1 \leq n_i^\pm \leq n \\ s(x^{n_i^\pm}) > T}} s(x^{n_i^\pm}) \leq \sum_{\substack{1 \leq j \leq n \\ s(x^j) > T}} s(x^j) < \varepsilon n \tag{19}$$

for all $T > T_\varepsilon$ and $n > n_\varepsilon$. Indeed, the equality follows by (10), the first inequality follows simply because the summation on the left is over a subset of times of the summation on the right, and the final inequality follows directly from (11). In particular we have

$$\sum_{\substack{1 \leq n_i^\pm \leq n \\ n_{i+1}^\pm - n_i^\pm \leq T}} n_{i+1}^\pm - n_i^\pm \geq (1 - \varepsilon)n.$$

The calculation holds for both left and right shadowing times independently and therefore there exist at least $(1 - 2\varepsilon)n$ iterates which belong to a small gap for both the left and the right sequence of shadowing times simultaneously. Each such iterate is therefore contained in an interval $\mathcal{T}$ of simultaneous short shadowing times and thus it is possible to define at least $(1 - 2\varepsilon)n/2T_\varepsilon$ disjoint such intervals.

Finally, notice that if some point x satisfies condition (11) then any other point on the orbit of x also satisfies (11). In particular these conclusions hold for all critical values as required.

□

5.2. Exponential shrinking at simultaneous short shadowing times.

Lemma 4. *For any integer $T \geq 1$, there exists a constant $0 < \gamma = \gamma(T) < 1$ such that for any critical value c^1 and any associated pair $[n_{i-1}^-, n_i^-]$ and $[n_{j-1}^+, n_j^+]$ of* *simultaneous left and right short gaps we have*

$$\frac{|\hat{I}_-^{(n_i^-)}|}{|\hat{I}_-^{(n_{i-1}^-)}|} < \gamma \quad \text{and} \quad \frac{|\hat{I}_+^{(n_j^+)}|}{|\hat{I}_+^{(n_{j-1}^+)}|} < \gamma$$

Proof. Let $\mathcal{T} = [k, \tilde{k}]$ be the interval containing $[n_{i-1}^-, n_i^-]$ and $[n_{j-1}^+, n_j^+]$ and satisfying $\tilde{k} - k < T$. We claim first of all that there exists a $\delta = \delta(T) > 0$ such that

$$|f^k(\hat{I}_-^{(n_{i-1}^-)} \setminus \hat{I}_-^{(n_i^-)})| \geq \delta, \quad |f^k(\hat{I}_+^{(n_{j-1}^+)} \setminus \hat{I}_+^{(n_j^+)})| \geq \delta.$$

To see this, let $\mathcal{C}^{(T)} = \{f^i(\mathcal{C})\}_{i=-T}^T$ denote the set of all images and preimages of the critical set between time $-T$ and $+T$ and define $\delta > 0$ to be the minimum distance between any two points in this set. Now let $[x, y] = \hat{I}_-^{(n_{i-1}^-)} \setminus \hat{I}_-^{(n_i^-)}$ and $[z, w] = \hat{I}_+^{(n_{j-1}^+)} \setminus \hat{I}_+^{(n_j^+)}$. Then, by construction we have

$$\{f^{n_{i-1}^-}(x), f^{n_i^-}(y), f^{n_{j-1}^+}(w), f^{n_j^+}(z)\} \subset \mathcal{C}$$

and therefore, since $n_{i-1}^-, n_i^-, n_{j-1}^+, n_j^+$ are all within T iterates of k we have

$$\{f^k(x), f^k(y), f^k(z), f^k(w)\} \subset \mathcal{C}^{(T)}.$$

These are the endpoints of the intervals $f^k(\hat{I}_-^{(n_{i-1}^-)} \setminus \hat{I}_-^{(n_i^-)})$ and $f^k(\hat{I}_+^{(n_{j-1}^+)} \setminus \hat{I}_+^{(n_j^+)})$ and thus the claim follows.

We now choose two points $x' \in (x, y)$ and $w' \in (z, w)$ such that $|f^k[x, x']| = |f^k[x', y]| \geq \delta/2$ and $|f^k[z, w']| = |f^k[w', w]| \geq \delta/2$.

By (8) we observe that the map f^k is monotone on $[x, w]$ and therefore the two extreme intervals $[x, x']$ and $[w', w]$ form the *Koebe space* which guarantees uniformly bounded distortion in $[x', w']$ (see [MelStr93]). In particular the proportion between the lengths of $[x', y]$ and $[y, c^1]$ is uniformly comparable to the proportion between the length of $f^k[x', y]$ and $f^k[y, c^1]$ and therefore we have

$$\begin{aligned}\frac{|\hat{I}_-^{(n_{i-1}^-)}|}{|\hat{I}_-^{(n_i^-)}|} &= \frac{|[x, c^1]|}{|[y, c^1]|} \geq \frac{|[x', c^1]|}{|[y, c^1]|} \\ &\geq 1 + \frac{|[x', y]|}{|[y, c^1]|} \geq 1 + \Gamma \frac{|f^k[x', y]|}{|f^k[y, c^1]|} \geq 1 + \frac{\Gamma\delta}{2|I|}\end{aligned}$$

for a distortion constant Γ which depends only on δ. This proves the first inequality in the statement of the Lemma with $\gamma = (1 + \frac{\Gamma\delta}{2|I|})^{-1}$. Exactly the same argument gives the second inequality. □

5.3. Collet-Eckmann. We are now ready to show that all critical values admit exponentially growing derivative along their orbits. More specifically we shall prove the following

Lemma 5. *Suppose that every critical value satisfies (TSR). Then there exist constants $K, \lambda > 0$ such that for every critical value and every $n \geq 1$ we have*

$$r_n := \min\left\{\frac{|f^n(\hat{I}_-^{(n)})|}{|\hat{I}_-^{(n)}|}, \frac{|f^n(\hat{I}_+^{(n)})|}{|\hat{I}_+^{(n)}|}\right\} \geq Ke^{\lambda n}.$$

The exponential growth of the derivative then follows from the so-called *minimum principle* for maps with negative Schwarzian derivative:

Minimum Principle. *[MelStr93] Let f be an S-multimodal map and $[a,b]\subset I$ be a subinterval. For $i\geq 1$, if $f^i|_{[a,b]}$ is a diffeomorphism, then*

$$|Df^i(x)|\geq \min\left\{\frac{|f^i(b)-f^i(x)|}{|b-x|},\ \frac{|f^i(x)-f^i(a)|}{|x-a|}\right\}$$

for any $x\in(a,b)$.

We therefore get the following

Corollary 5.1. *Suppose that every critical value satisfies (TSR). Then there exist constants $K,\lambda>0$ such that for every critical value c^1 and every $n\geq 1$ we have*

$$|Df^n(c^1)|\geq Ke^{\lambda n}.$$

We begin with a sublemma which follow easily from the estimates obtained above.

Sublemma 5.1. *For all $n\geq 1$ we have*

$$|\hat{I}_\pm^{(n)}|\leq e^{-\eta\gamma n}.$$

Proof. Follows from Lemmas 3 and 4. □

Proof of Lemma 5. We start by considering a (left or right) shadowing time $n_i^\pm$. Let $\hat{I}_\pm^{(n_i^\pm)}=[x,c^1]$ and recall that then $f^{n_i^\pm}(\hat{I}_\pm^{(n_i^\pm)})=(\hat{c},c^{n_i^\pm+1})$ for some critical point $\hat{c}$. By lemma 2 we have

$$d(c^j)\geq e^{-s(c^j)/\underline{\kappa}}.$$

Therefore

$$|f^{n_i^\pm}(\hat{I}_\pm^{(n_i^\pm)})|=|c^{n_i^\pm+1}-\hat{c}|\geq e^{-s(c^{n_i^\pm+1})/\underline{\kappa}} \tag{20}$$

for every critical value and every shadowing time $n_i^\pm\geq 1$.

We now let $n_i^\pm$ be the smallest (left or right) shadowing time larger than n. Then $n_{i-1}^\pm\leq n<n_i^\pm$ and so $\hat{I}_\pm^{(n_i^\pm)}\subset\hat{I}_\pm^{(n)}=\hat{I}_\pm^{(n_{i-1}^\pm)}$. By (8) we know that $f^{n_i^\pm}|_{\hat{I}_\pm^{(n_{i-1}^\pm)}}$ is monotone, so we have by (20),

$$\begin{aligned}|f^n(\hat{I}_\pm^{(n)})|&\geq D^{-(n_i^\pm-n)}|f^{n_i^\pm}(\hat{I}_\pm^{(n)})|=D^{-(n_i^\pm-n)}|f^{n_i^\pm}(\hat{I}_\pm^{(n_{i-1}^\pm)})|\\&\geq D^{-(n_i^\pm-n)}|f^{n_i^\pm}(\hat{I}_\pm^{(n_i^\pm)})|.\end{aligned} \tag{21}$$

Then, by (20) and sublemma 5.1, we obtain

$$\begin{aligned}|f^n(\hat{I}_\pm^{(n)})|\geq D^{-(n_i^\pm-n)}e^{-s(c^{n_i^\pm+1})/\underline{\kappa}}&\geq D^{-(n_i^\pm-n)}e^{-s(c^{n_i^\pm+1})/\underline{\kappa}}e^{\eta\gamma n}|\hat{I}_\pm^{(n)}|\\&=D^{-(n_i^\pm-n)}e^{-s(c^{n_i^\pm+1})/\underline{\kappa}+\eta\gamma n}|\hat{I}_\pm^{(n)}|.\end{aligned} \tag{22}$$

It is therefore sufficient to show that we can choose constants $K,\lambda>0$ so that

$$r_n\geq\ D^{-(n_i^\pm-n)}e^{-s(c^{n_i^\pm+1})/\underline{\kappa}+\eta\gamma n}\geq Ke^{\lambda n}.$$

Since $\lim_{n\to+\infty} s(c^n)/n = 0$, we can choose N sufficiently large such that $s(c^{n_i^\pm+1})/\underline{\kappa} < \eta\gamma(n_i^\pm+1)/2$ when $n_i^\pm \geq N$. So

$$r_n \geq D^{-(n_i^\pm - n)} e^{\eta\gamma n - \eta\gamma(n_i^\pm+1)/2}.$$

We start by taking $\tilde{\lambda} = \eta\gamma/2$, fixing some arbitrary $\tilde{\lambda} > \lambda > 0$ and writing $\tilde{\lambda} = \lambda + (\tilde{\lambda} - \lambda)$ and $n_i^\pm = (n_i^\pm - n) + n$. Then we have $e^{\tilde{\lambda} n_i^\pm} = e^{\lambda n} e^{(\tilde{\lambda}-\lambda)n} e^{-\tilde{\lambda}(n-n_i^\pm)}$ and therefore

$$r_n \geq D^{-(n_i^\pm - n)} e^{\eta\gamma n - \eta\gamma(n_i^\pm+1)/2} = D^{-(n_i^\pm - n)} e^{-\tilde{\lambda}(n_i^\pm - n)} e^{-\tilde{\lambda}} e^{(\tilde{\lambda}-\lambda)n} e^{\lambda n}.$$

It remains to show that

$$(De^{\tilde{\lambda}})^{-(n_i^\pm - n)} e^{-\tilde{\lambda}} e^{(\tilde{\lambda}-\lambda)n} \geq K \tag{23}$$

This follows by the crucial facts that $n_i^\pm - n_{i-1}^\pm > n - n_{i-1}^\pm$ and

$$\lim_{n_{i-1}^\pm \to +\infty} \frac{s(c^{n_{i-1}^\pm})}{n_{i-1}^\pm} = \lim_{n_{i-1}^\pm \to +\infty} \frac{n_i^\pm - n_{i-1}^\pm}{n_{i-1}^\pm} = 0 \tag{24}$$

by TSR and (10). Indeed, We use (24) which implies that the left side of (23) is bigger than 1 for all $n_i^\pm \geq \tilde{N}$ for some sufficiently large $\tilde{N}$ depending only on the map and the constants λ and $\tilde{\lambda}$ and thus, again using (24), for all $n \geq N$ for some sufficiently large N also depending only on the same quantities as $\tilde{N}$. To take care of smaller values of n it is sufficient to choose K sufficiently small depending only on the value of r_n for $n \leq N$. This completes the proof.

□

Acknowledgements

Lanyu Wang would like to thank his PhD supervisor Professor Zhifen Zhang for her advise and encouragement, and Professors Zhujun Jing and Lan Wen for their help and encouragement for many years.

References

[BenCar85] M. Benedicks and L. Carleson, *On iterations of* $1 - ax^2$ *on* $(-1, 1)$, Ann. of Math. **122** (1985), 1–25.

[Bru94] H. Bruin, *Topological conditions for the existence of invariant measures for unimodal maps*, Ergodic Theory Dynam. Systems **14** (1994), 433–451.

[Bru95a] ———, *Combinatorics of the kneading map*, Thirty Years after Sharkovskiĭ's Theorem: New Perspectives (Murcia, 1994), World Sci. Ser. Nonlinear Sci. Ser. B Spec. Theme Issues Proc., vol. 8, World Sci. Publishing, River Edge, NJ, 1995, pp. 77–87.

[Bru98a] ———, *The existence of absolutely continuous invariant measures is not a topological invariant for unimodal maps*, Ergodic Theory Dynam. Systems **18** (1998), 555–565.

[Bru98b] ———, *Topological conditions for the existence of absorbing Cantor sets*, Trans. Amer. Math. Soc. **350** (1998), 2229–2263.

[ColEck83] P. Collet and J.-P. Eckmann, *Positive Lyapunov exponents and absolute continuity for maps of the interval*, Ergodic Theory Dynam. Systems **3** (1983), 13–46.

[Guc79] J. Guckenheimer, *Sensitive dependence to initial conditions for one-dimensional maps*, Comm. Math. Phys. **70** (1979), 133–160.

[Hof80] F. Hofbauer, *The topological entropy of the transformation* $x \mapsto ax(1-x)$, Monatsh. Math. **90** (1980), 117–141.

[HofKel90b] F. Hofbauer and Gerhard Keller, *Quadratic maps without asymptotic measure*, Comm. Math. Phys. **127** (1990), 319–337.

[MelStr93] W. de Melo and Sebastian van Strien, *One-dimensional dynamics*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 25, Springer-Verlag, Berlin, 1993.

[NowPrz98] T. Nowicki and F. Przytycki, *Topological invariance of the Collet-Eckmann property for S-unimodal maps*, Fund. Math. **155** (1998), 33–43.

[NowSan98] T. Nowicki and D. Sands, *Non-uniform hyperbolicity and universal bounds for S-unimodal maps*, Invent. Math. **132** (1998), 633–680.

[PrzRivSmi03] F. Przytycki, J. Rivera-Letelier, and S. Smirnov, *Equivalence and topological invariance of conditions for non-uniform hyperbolicity in the iteration of rational maps*, Invent. Math. **151** (2003), 29–63.

[PrzRoh98] F. Przytycki and S. Rohde, *Porosity of Collet-Eckmann Julia sets*, Fund. Math. **155** (1998), 189–199.

[San95] D. Sands, *Topological conditions for positive Lyapunov exponents* (1995), `http://topo.math.u-psud.fr/~sands/Papers/thesis.ps` (Preprint).

[Tsu93] M. Tsujii, *Positive Lyapunov exponents in families of one-dimensional dynamical systems*, Invent. Math. **111** (1993), 113–137.

[Wan01] L. Wang, *Topological and metrical conditions for Collet-Eckmann unimodal maps*, Acta Math. Appl. Sinica (English Ser.) **17** (2001), 350–360.

[Wan96] ———, *Dynamics of multimodal maps*, Phd Thesis (1996). (Chinese)

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