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TITS ALTERNATIVE IN HYPEKÄHLER MANIFOLDS

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Dedicated to Professor Yukihiko Namikawa on the occasion of his sixtieth birthday

Abstract. We show an analogous result of the famous Tits alternative for a group G of birational automorphisms of a projective hyperkäher manifold: Either G contains a non-commutative free group or G is an almost abelian group of finite rank. As an application, we show that the automorphism group of the so-called singular K3 surface contains a non-commutative free group.

1. Introduction–background and main results

Our main results are Theorems (1.1)-(1.3) and (2.6). Though our actual proof is entirely algebraic, these results are much motivated by recent important works about automorphisms of K3 surfaces from the view of complex dynamics, notably, a work of McMullen [Mc] and works of Cantat [Ca 1, 2].

1. Throughout this note, we work over the complex number field $\mathbf{C}$. By a *hyperkähler manifold* we mean a compact simply-connected complex Kähler manifold M having everywhere non-degenerate holomorphic 2-form σ_M such that $H^2(M, \Omega_M) = \mathbf{C}\sigma_M$. Such manifolds are even dimensional and form one of the three fundamental building blocks of compact Kähler manifolds of vanishing first Chern class. For example, a K3 surface S is a hyperkähler manifold of dimension 2, and the Hilbert scheme $\mathrm{Hilb}^n(S)$, of 0-dimensional closed subschemes of length n of S, is a hyperkähler manifold of dimension $2n$. (About basics of hyperkäher manifolds, we refer the readers to the excellent account [GHJ, Part III] by Huybrechts. All of what we need in this note is also reviewed in [Og1], which we will quote later.) We denote the bimeromorphic (resp. biholomorphic) automorphism group of M by $\mathrm{Bir}\, M$ (resp. $\mathrm{Aut}\,(M)$). Let G be a group. We call G an *almost abelian group of finite rank*, say r, if G has a normal subgroup A of finite index and A has a finite subgroup F fitting in with the exact sequence

$$1 \longrightarrow F \longrightarrow A \longrightarrow \mathbf{Z}^r \longrightarrow 0\ .$$

Roughly speaking, an almost abelian group of rank r is nothing but a group isomorphic to $\mathbf{Z}^r$ up to finite kernel and finite cokernel. (See eg. [Og1, Section

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9] for elementary properties which we will use later.) The extreme counter part of an almost abelian group is a group which contains a non-commutative free group, or in other words, a group having a subgroup isomorphic to the free product $\mathbf{Z} * \mathbf{Z}$.

2. The aim of this note is to study groups of birational automorphisms of a projective hyperkähler manifold. Note that any bimeromorphic automorphism is birational when a manifold is projective. Our main results are as follows:

Theorem 1.1. *Let M be a projective hyperkähler manifold and let G be a subgroup of* $\mathrm{Bir}\,(M)$*. Then G satisfies either:*

(1) G is an almost abelian group of finite rank[1] *, or*
(2) G contains a non-commutative free group.

In particular, so are $\mathrm{Bir}\,(M)$ *and* $\mathrm{Aut}\,(M)$*.*

Theorem 1.2. (Crescent Theorem)[2] *Let M be a hyperkähler manifold having at least two complex torus fibrations, say $\varphi_i : M \longrightarrow B_i$ ($i = 1, 2$). Then M is projective. Assume furthermore that both φ_i are Jacobian fibrations of positive Mordell-Weil rank. Then* $\mathrm{Bir}\,(M)$ *contains a non-commutative free group.*

Here by a complex torus fibration, we mean a surjective morphism $f : M \longrightarrow B$ over a normal projective variety B, whose general fiber is a positive dimensional complex torus. (See [Ma] for fibered hyperkähler manifolds.) We call f a Jacobian fibration if M is projective and f admits a rational section, say O. The rational sections of a Jacobian fibration f naturally form an abelian subgroup of $\mathrm{Bir}\,(M)$. We call this group the *Mordell-Weil group* of f and denote it by $\mathrm{MW}\,(f)$. By Theorem (1.1), $\mathrm{MW}\,(f)$ is finitely generated. So, it is of positive rank iff it contains an element of infinite order. See also [Ca 1, 2], [CF], [Og2, 3] for relevant results.

Theorem 1.3. *Let S be a singular K3 surface and put $M := \mathrm{Hilb}^n\,(S)$. Then, both* $\mathrm{Aut}\,(S)$ *and* $\mathrm{Aut}\,(M)$*, hence* $\mathrm{Bir}\,(M)$*, contain a non-commutative free group.*

Here a *singular* K3 surface means a K3 surface having maximum Picard number 20. Theorem (1.3) is a generalization of a famous result of Shioda and Inose [SI], that *every singular K3 surface has an automorphism of infinite order*, in a non-commutative direction. Note that there is a projective K3 surface S s.t. $\rho(S) = \rho$ and $|\mathrm{Aut}\,(S)| < \infty$ for each integer ρ with $1 \le \rho \le 19$ ([Kn, Ni]). So a statement similar to Theorem (1.3) is not true for K3 surfaces of smaller Picard number $\rho \le 19$. Theorem (1.3) also shows that the second alternative in Theorem (1.1) really occurs in each dimension.

[1] In [Og3], it will be shown that rank $\le \max\,(1, \rho(M) - 2)$.

[2] Theorem (1.2) is inspired by a discussion with Y. Kawamata about remarkable difference of the appearance of crescent in Japan and Korea; it looks vertical in Japan while horizontal in Korea. This reminded me of the situation in Theorem (1.2).

3. It is also interesting to compare these results with the following existing result and interesting open Questions:

Theorem 1.4. [Og1] *Let M be a non-projective kyperkähler manifold. Then both* $\mathrm{Bir}\,(M)$ *and* $\mathrm{Aut}\,(M)$ *are almost abelian groups of finite rank. In particular, they are finitely generated.*

Question 1.5. Let M be a projective kyperkähler manifold. Are $\mathrm{Bir}\,(M)$ and $\mathrm{Aut}\,(M)$ finitely generated?

Question 1.6. (J.M. Hwang) Let M be a projective kyperkähler manifold. Is $\mathrm{Aut}\,(M)$ is of finite index in $\mathrm{Bir}\,(M)$?

In dimension 2, the first question is affirmative by Sterk [St] and the second one is trivial. The second question is asked by J.M. Hwang after my talk on a relevant subject at KIAS on March 2005.

4. Our proof of Theorems (1.1)-(1.3) is based on two famous, deep results in linear algebraic groups; Lie-Kolchin Theorem and Tits Theorem ([Hm], [Ti]; see also [Ha] and Section 2 for the statements) and the notion of Salem polynomial (see eg. [Mc] and also Section 2). Unfortunately, our proof does not tell us much about algebro-geometrical reason why non-abelian free groups should be in the birational automorphism groups in Theorem (1.2). It would be interesting to find a more "geometrically visible" proof of Theorem (1.2), especially when M is a K3 surfrace. For this, an observation of Cantat [Ca2] might give us some hint.

2. Algebraic preparations

The goal of this section is Theorem (2.6), the technical heart in the proof of our main results. In **1**, we recall Lie-Kolchin Theorem and Tits Theorem (Tits alternative). Both are very important for our proof. We recall basic notions about lattices in **2** and a few facts about Salem polynimials in **3**. We show Theorem (2.6) in **4**.

1. For simplicity, we shall work only over $\mathbf{C}$. Let $V \neq \{0\}$ be a finite dimensional vector space over $\mathbf{C}$. We regard the general linear group $\mathrm{GL}\,(V)$ as an algebraic group defined over $\mathbf{C}$, with Zariski topology. We identify $\mathrm{GL}\,(V)$ with the group $\mathrm{GL}\,(V)(\mathbf{C})$ of $\mathbf{C}$-valued points in a usual way. A subgroup of $\mathrm{GL}\,(V) = \mathrm{GL}\,(V)(\mathbf{C})$ simply means a subgroup as an abstract group. If G is a subgroup of $\mathrm{GL}\,(V)$, then its Zariski closure $\overline{G}$, as well as the identity component of $\overline{G}$, is an algebraic subgroup of $\mathrm{GL}\,(V)$. First, we notice the following well-known:

Lemma 2.1. *Let G be a solvable subgroup of* $\mathrm{GL}\,(V)$*. Then:*

(1) Any subgroup of G and any quotient group of G are solvable.

(2) $\overline{G}$ *is also solvable.*

Proof. We only show (2). It suffices to check that $\overline{[G,G]} = [\overline{G}, \overline{G}]$. Note that $[\overline{G}, \overline{G}]$ is closed in $\overline{G}$ (See for instance [Hm, 17.2]). Thus $\overline{[G,G]} \subset [\overline{G}, \overline{G}]$.

Let us show the other inclusion. Take $g \in G$. Let us define the map α_g by

$$\alpha_g : \overline{G} \longrightarrow \overline{G} \; ; \; f \mapsto f^{-1}g^{-1}fg \; .$$

Clearly, α_g is continuous and satisfies $\alpha_g(G) \subset [G,G]$. Thus $\alpha_g(\overline{G}) \subset \overline{[G,G]}$. Hence $[\overline{G}, G] \subset \overline{[G,G]}$. Let $f \in \overline{G}$. Let us define the map β_f by

$$\beta_f : \overline{G} \longrightarrow \overline{G} \; ; \; g \mapsto f^{-1}g^{-1}fg \; .$$

Clearly, β_f is continuous and satisfies $\beta_f(G) \subset [\overline{G}, G]$. Since $[\overline{G}, G] \subset \overline{[G,G]}$, we have $\beta_f(G) \subset \overline{[G,G]}$ as well. Thus $\beta_f(\overline{G}) \subset \overline{[G,G]}$ and hence $[\overline{G}, \overline{G}] \subset \overline{[G,G]}$. □

Lie-Kolchin Theorem and Tits alternative are the following:

Theorem 2.2. (Lie-Kolchin Theorem, see eg. [Hm, Chap. VII, 17.6]) *Let* G *be a connected solvable subgroup of* $\mathrm{GL}\,(V)$*. Then* G *has a common eigenvector in* V*.*

Theorem 2.3. (Tits alternative [Ti]) *Let* G *be a subgroup of* $\mathrm{GL}\,(V)$*. Then* G *is either virtually solvable or contains a non-commutative free group.*

Here a group G is called *virtually solvable* if G contains a solvable subgroup of finite index. We also notice that any non-commutative free group contains $\mathbf{Z} * \mathbf{Z}$, i.e. the free group of rank 2.

2. By a *lattice* $L = (L, (*, **))$, we mean a pair consisting of a free abelian group $L \simeq \mathbf{Z}^r$ and its (possibly degenerate) integral-valued symmetric bilinear form $(*, **) : L \times L \longrightarrow \mathbf{Z}$. By L_K, we denote the scalar extension $L \otimes_{\mathbf{Z}} K$ of L by a field K. The signature of L is the pair of the numbers of positive-, zero- and negative-eigenvalues of a symmetric matrix associated to $(*, **)$. We call L *hyperbolic* (resp. *parabolic*, *elliptic*) if the signature is $(1, 0, r-1)$ (resp. $(0, 1, r-1)$, $(0, 0, r)$).

We call an element $v \in L \setminus \{0\}$ (resp. a subgroup V) *primitive* if $L/\langle v \rangle$ (resp. L/V) is torsion-free. We denote by $\mathrm{O}\,(L)$ the group of isometries of a lattice L. Note that $|\mathrm{O}\,(L)| < \infty$ if L is elliptic.

From now on until the end of this section, we choose and fix a hyperbolic lattice L *of rank* r*.*

Then the set $\{v \in L_{\mathbf{R}} | (v^2) > 0\}$ consists of two connected components (w.r.t. Euclidean topology of $L_{\mathbf{R}}$). We choose and fix one of them and denote it by $\mathcal{P}(L)$. We call $\mathcal{P}(L)$ the *positive cone* of L. In general, there is no canonical choice of the positive cone. (However, when L is the Néron-Severi group of a projective hyperkähler manifold, with Beauville-Bogomolov-Fujiki's bilinear form, we always choose the positive cone so that it contains ample classes.)

Let $\overline{\mathcal{P}}(L)$ (resp. $\partial\mathcal{P}(L)$) be the closure (resp. the boundary) of the positive cone in $L_{\mathbf{R}}$ (w.r.t. Euclidean topology). By the Schwartz inequality, we have $(x, y) \geq 0$ for $x, y \in \overline{\mathcal{P}}(L)\backslash\{0\}$ and the equality holds iff $\mathbf{R}_{>0}x = \mathbf{R}_{>0}y \in \partial\mathcal{P}(L)$.

Let M be a primitive sublattice of L. Then M is either hyperbolic, parabolic, or elliptic, and $M_L^{\perp}$ is elliptic, parabolic, hyperbolic respectively. Here and here after we denote by $M_L^{\perp}$, the primitive subgroup $\{\, v \in L \,|\, (v, w) = 0 \ \forall w \in M \,\}$. Note that $M \cap M_L^{\perp} = \{0\}$ and $M \oplus M_L^{\perp}$ is of finite index in L when M is hyperbolic or elliptic, while $M \cap M_L^{\perp} = \mathbf{Z}e$ with $(e^2) = 0$ and $M + M_L^{\perp}$ is corank 1 in L when M is parabolic.

3. In [Mc] and [Og1], the notion of Salem polynomial plays a very crucial role. In our proof, it also plays an important role. We recall the definition:

Definition 2.4. An irreducible monic polynomial $\Phi(t) \in \mathbf{Z}[t]$ of degree n is called a Salem polynomial if the complex roots of $\Phi(t) = 0$ consists of two real roots a and $1/a$ s.t. $a > 1$ and $n - 2$ roots on the unit circle S^1.

In our argement, we need the following result. This is a formal generalization of [Mc, Corollary 3.3]:

Proposition 2.5. *Let L be a hyperbolic lattice of rank r and g be an element of $\mathrm{O}(L)$, preserving the positive cone $\mathcal{P}(L)$. Then the irreducible factors of the characteristic polynomial $\Phi_g(t) := \det(tI - g)$ of g includes at most one Salem polynomial; and the remaining factors are cyclotomic. In particular, if c is an eigenvalue of g, then so is $1/c$.*

Proof. Since $\Phi_g(t) \in \mathbf{Z}[t]$ is monic, all the eigenvalues of g are algebraic integers. Since L is hyperbolic and $g \in \mathrm{O}(L)$, it follows that g has at most one eigenvalue (counted with multiplicity) *outside* S^1. Therefore by $|\det g| = 1$, g satisfies either that all eigenvalues are on S^1, or that g has exactly one eigenvalue outside S^1, say a, exactly one eigenvalue inside S^1, say b, and $r - 2$ eigenvalues on S^1.

In the first case, the eigenvalues are all roots of unity by Kronecker's Theorem. So, the irreducible factors of $\Phi_g(t)$ are cyclotomic.

Let us consider the second case. Both a and b are real by the uniqueness. Then, one can choose a real eigenvector $v \in L_{\mathbf{R}}$ with eigenvalue a. Then, by $(v^2) = (g(v)^2) = a^2(v^2)$, we have $(v^2) = 0$. Thus, $v \in \partial\mathcal{P}(L)$ (after replacing v by $-v$ if necssary). The same holds for an eigenvector with eigenvalue b. Thus $a > 1$ and $ab = 1$ by $g(\mathcal{P}(L)) = \mathcal{P}(L)$ and $|\det(g)| = 1$. Thus $\det g = 1$ as well. Let $f(t) \in \mathbf{Z}[t]$ be the minimal monic polynomial of a. Then $f(t)|\Phi_g(t)$. Since $\Phi_g(0) = \pm 1$, we have also $f(b) = 0$. Thus $f(t)$ is a Salem polynomial. The zeros of the monic polynomial $\Phi_g(t)/f(t) \in \mathbf{Z}[t]$ are now on S^1. Thus, again by Kronecker's Theorem, $\Phi_g(t)/f(t)$ includes only cyclotomic polynomials as its irreducible factors. The last statement is now clear. Indeed, if $c \in S^1$, then $1/c = \bar{c}$ is a zero of $\Phi_g(t)$. If $c \notin S^1$, then $c = a$ or b, and $1/c = b$ or a is also a zero of $\Phi_g(t)$. □

4. Let us now formulate our key result:

Theorem 2.6. *Let L be a hyperbolic lattice of rank r and let G be a subgroup of $\mathrm{O}(L)$. Assume that G is virtually solvable. Then G is almost abelian of finite rank.*

Proof. We shall show Theorem (2.6) by induction on r. The result is clear if $r = 1$. So, we may assume that $r \geq 2$. We shall prove dividing into five steps.

Step 1. By the assumption, G has a solvable subgroup N s.t. $[G : N] < \infty$. Note that G is almost abelian of finite rank iff so is N (cf. [Og1, Section 9]). Then by replacing G by N, we may assume that G is solvable. *We will do so from now on.*

Step 2. Put $V := L_{\mathbf{C}}$. We have a natural embedding:

$$G \subset \mathrm{O}\,(L) \subset \mathrm{GL}\,(V)\ .$$

Let $\overline{G}$ be the Zariski closure of G in $\mathrm{GL}\,(V)$ and let S be the identity component of $\overline{G}$, i.e. the irreducible component containing the identity 1. Since G is solvable, so is $\overline{G}$ by Lemma (2.1)(2). Thus, by Lemma (2.1)(1), S is a connected solvable subgroup of $\mathrm{GL}\,(V)$. Since $\overline{G}$ is an algebraic subset of a noetherian space $\mathrm{GL}\,(V)$, it has only finitely many irreducible components. Thus, $[\overline{G} : S] < \infty$. For the same reason as in Step 1, we may assume that G is a subgroup of a connected solvable group by replacing G by its finite index subgroup $G \cap S$, and that the positive cone $\mathcal{P}(L)$ is G-stable by further replacing G by its index two subgroup (if necessary). *We will do so from now on.*

Step 3. Recall that $G \subset \mathrm{GL}(r, \mathbf{Z})$. Assume for a moment that all elements of G is of finite order. Let $g \in G$. Then g is diagonalizable in $\mathrm{GL}(r, \mathbf{C})$ and each eigenvalue of g, say, α, is a root of unity, say, a primitive n-th root of unity. Since the characteristic polynomial $\Phi_g(x)$ of g is in $\mathbf{Z}[x]$ and of degree r, it follows that $\varphi(n) \leq r$. Here $\varphi(n)$ is the Euler function. Thus $n \leq N(r)$ for some integer $N(r)$ depending only on r. Hence $\mathrm{ord}\,(g) \leq N(r)!$, that is, the orders of elements of G are universally bounded by $N(r)!$. Thus G is a finite subgroup (of a general complex linear group) by Burnside's Theorem [Bu1, 2]. In particular, G is then an almost abelian group of rank 0 and we are done. So, we may now assume that there is $g_0 \in G$ s.t. $\mathrm{ord}\, g_0 = \infty$. *We will do so from now on.*

Step 4. Since G is in a connected solvable subgroup, say S, of $\mathrm{GL}\,(V)$, by applying Lie-Kolchin Theorem (2.2) for S, we find a common eigenvector of

G, say $v \in V = L_{\mathbf{C}}$. Set $g(v) = \alpha(g)v$ for $g \in G$. Then α defines a group homomorphism

$$\alpha : G \longrightarrow \mathbf{C}^{\times} \ ; \ g \mapsto \alpha(g) \ .$$

Let M be the minimal primitive sublattice of L s.t. $v \in M_{\mathbf{C}}$. This M is G-stable. Indeed, since $g \in G$ is defined over $\mathbf{Z}$ and $g(v) = \alpha(g)v$, one has $M \cap g(M) \subset M$, $(M \cap g(M))_{\mathbf{C}} = M_{\mathbf{C}} \cap g(M)_{\mathbf{C}}$ and $v \in M_{\mathbf{C}} \cap g(M)_{\mathbf{C}}$. Thus $v \in (M \cap g(M))_{\mathbf{C}}$, and therefore $g(M) = M$ by the minimality of M.

Step 5. Note that M is either elliptic, parabolic, or hyperbolic. We completes the proof by dividing into these three cases.

The case where M is elliptic. In this case $M_L^{\perp}$ is hyperbolic. Let $K := \mathrm{Ker}\,(r_M : G \longrightarrow \mathrm{O}\,(M))$. Then K is of inite index of G and $K \subset \mathrm{O}\,(M_L^{\perp})$. Since $\mathrm{rank}\, M_L^{\perp} < \mathrm{rank}\, L$, K is almost abelian of finite rank by the induction hypothesis. Thus so is G.

The case where M is parabolic. In this case there is a unique primitive element $u \in M$ s.t. $(u^2) = 0$ and $u \in \partial\mathcal{P}(L)$. By the uniqueness of u and by $G(\mathcal{P}(L)) = \mathcal{P}(L)$, this u is G-stable. Since $u \in \partial\mathcal{P}(L) \cap L \setminus \{0\}$, it follows from [Og2, Proposition 2.9] that G is almost abelian of finite rank.

The case where M is hyperbolic. In this case $M_L^{\perp}$ is elliptic (possibly 0).

Consider first the case where $M_L^{\perp} \neq \{0\}$, i.e. the case where $M \neq L$. Let $K := \mathrm{Ker}\,(r_M : G \longrightarrow \mathrm{O}\,(M_L^{\perp}))$. Then K is of finite index in G and $K \subset \mathrm{O}\,(M)$. Since $\mathrm{rank}\, M < \mathrm{rank}\, L$ by the case assumption, K is almost abelian of finite rank by the induction hypothesis. Thus so is G.

It remains to consider the case where $M = L$. In this case, the result follows from the next Lemma (2.7)(4).

Lemma 2.7. *Assume that $M = L$. Then:*

(1) The homomorphism $\alpha : G \longrightarrow \mathbf{C}^{\times}$ is injective. In particular, G is an abelian group.

(2) The characteristic polynomial $\Phi_{g_0}(t)$ is a Salem polynomial. Here g_0 is an element of G with $\mathrm{ord}\, g_0 = \infty$ in Step 3.

(3) Let $g \in G$. Then there is $\varphi(t) \in \mathbf{Q}[t]$ s.t. $g = \varphi(g_0)$.

(4) G is an almost abelian group of finite rank.

Proof. Let us show (1). If $\alpha(g) = 1$, then $v \in E(g, 1)$. Here $E(g, 1)(\subset L_{\mathbf{C}})$ is the eigenspace of eigenvalue 1 of g. Since g is defined over $\mathbf{Z}$, there is a primitive sublattice $E \subset L$ s.t. $E_{\mathbf{C}} = E(g, 1)$. Thus, by the minimality of $L = M$, one has $E = L$, i.e. $g = 1$. Thus α is injective.

Let us show (2). If $\alpha(g_0)$ is a root of unity, then there is a positive integer m s.t. $\alpha(g_0^m) = 1$. Then $g_0^m = 1$ by (1), a contradiction to $\mathrm{ord}\, g_0 = \infty$. Thus $\alpha(g_0)$ is not a root of unity. Thus, by Proposition (2.5), $\Phi_{g_0}(t)$ has a Salem polynomial, say $f(t)$, as its irreducible factor. If $\Phi_{g_0}(t) \neq f(t)$, then one can write $\Phi_{g_0}(t) = f(t)h(t)$. Here $h(t)$ is a product of cyclotomic polynomials by Proposition (2.5). However, since $(f(t), h(t)) = 1$ and $f(\alpha(g_0)) = 0$, the decomposition $\Phi_{g_0}(t) = f(t)h(t)$ would lead a non-trivial rational decomposition of $L_{\mathbf{Q}}$, say $L_{\mathbf{Q}} = L_1 \oplus L_2$, s.t. $v \in (L_2)_{\mathbf{C}}$. However this contradicts the minimality of $L = M$. Thus $\Phi_{g_0}(t) = f(t)$ and the results follows.

Let us show (3). Since $\Phi_{g_0}(t)$ is irreducible by (2), the eigenvalues of g_0 are mutaully distinct. Thus the $\mathbf{C}$-linear space $W' := \{h \in \mathrm{End}_{\mathbf{C}}(L_{\mathbf{C}}) | hg_0 = g_0h\}$ is of dimension r. Since g_0 is defined over $\mathbf{Q}$, one has $W' = W_{\mathbf{C}}$, where $W := \{h \in \mathrm{End}_{\mathbf{Q}}(L_{\mathbf{Q}}) | hg_0 = g_0h\}$. Note that g_0^k $(0 \leq k \leq r-1)$ are linearly independent over $\mathbf{Q}$ by the irreducibility of $\Phi_{g_0}(t)$. Thus $\langle g_0^k \rangle_{k=0}^{r-1}$ forms a basis of W over $\mathbf{Q}$. Since $G \subset W$ by (1), the result follows.

Let us show (4). Let $F(\subset \mathbf{C})$ be the minimal splitting field of $\Phi_{g_0}(t)$ over $\mathbf{Q}$. Let O_F be the ring of algebraic integers of F and let U_F be the unit group of O_F. Both $\alpha(g)$ and $1/\alpha(g)$ $(g \in G)$ are zero of the characteristic polynomial $\Phi_g(t) \in \mathbf{Z}[t]$ by Proposition (2.5). Thus both $\alpha(g)$ and $1/\alpha(g)$ are algebraic integers in F. Hence $\alpha(G) \subset U_F$. By the Dirichlet unit Theorem, U_F is a finitely generated abelian group. Thus so is its subgroup $\alpha(G)$. Since α is injective, this implies the result. □

This completes the proof of Theorem (2.6). □

3. Proof of Theorems (1.1)-(1.3)

In this section, we shall prove Theorems (1.1)-(1.3). Let M be a projective hyperkähler manifold. Then the Néron-Severi group $NS(M)$ is a hyperbolic lattice w.r.t. the Beauville-Bogomolov-Fujiki's form. Let G be a subgroup of $\mathrm{Bir}\,(M)$. Then, there is a natural group homomorphism $r_{NS} : G \longrightarrow \mathrm{O}\,(NS(M))$. The kernel of r_{NS} is a finite group ([Og1, Corollary 2.7]). So, G contains a non-commutative free group iff so does $r_{NS}(G)$, and G is almost abelian of finite rank iff so is $r_{NS}(G)$ ([Og1, Section 9]).

Proof of Theorem (1.1). Assume that $r_{NS}(G)$ does not contain a non-commutative free subgroup. Then, by the Tits alternative (2.3), $r_{NS}(G)$ is virtually solvable. Thus $r_{NS}(G)$ is almost abelian of finite rank by Theorem (2.6). □

Proof of Theorem (1.2). Since B_i are assumed to be projective, one can write $\varphi_i = \Phi_{|\varphi_i^* H_i|}$. Here H_i is a very ample divisor of B_i and $\Phi_{|\varphi_i^* H_i|}$ is the morphism associated with the complete linear system $|\varphi_i^* H_i|$. Put $e_i := [\varphi_i^* H_i] \in NS(M)$. Then $(e_i^2) = 0$ (cf. [Ma]). Since $\rho(B_i) = 1$ by Matsushita [ibid], it follows that $\mathbf{R}_{>0}e_1 \neq \mathbf{R}_{>0}e_2$. (Here we note that all the arguments in [ibid] are valid if we assume that the base space is normal and projective, even if M is not assumed to be projective. Indeed, one can rewrite his argument [ibid] by using a Kähler class on M, instead of an ample class there). Thus $((e_1+e_2)^2) > 0$ and hence $NS(M)$ is hyperbolic. Thus M is projective by the fundamental result of Huybrechts [Hu].

Choose $f_i \in \mathrm{MW}\,(\varphi_i)$ s.t. $\mathrm{ord}\,(f_i) = \infty$. We naturally regard both f_i as elements of $\mathrm{Bir}\,(M)$. Set $G := \langle f_1, f_2\rangle$. Then $r_{NS}(G) = \langle r_{NS}(f_1), r_{NS}(f_2)\rangle$ and $r_{NS}(f_i)(e_i) = e_i$ for each $i = 1, 2$. By Theorem (1.1), it suffices to check that $r_{NS}(G)$ is not almost abelian of finite rank. However, the proof of this fact is the same as [Og2, Theorem 1.6(1)] under a few obvious modifications. □

Remark 3.1. Our proof of Theorem (1.2) is based on Theorem (1.1) which, as we see above, involves a very deep result, the Tits alternative. By using the so-called table-tennis Lemma and some elementary results in hyperbolic geometry (see eg. [Ha, Sections 1, 3, 4]), one can also find a non-commutative free group in $\langle r_{NS}(f_1), r_{NS}(f_2)\rangle$ more directly. This is pointed out to us by S. Cantat.

Proof of Theorem (1.3). By [Og2, Theorem 1.6(1)], every singular K3 surface S admits at least two Jacobina fibrations of positive Mordell-Weil rank. Thus $\mathrm{Aut}\,(S)$ contains a non-commutative free group by Theorem (1.2). By the universality of the Hilbert scheme, $\mathrm{Aut}\,(S)$ naturally acts on $M := \mathrm{Hilb}^n\,(S)$. This action is also faithful. Thus $\mathrm{Aut}\,(M)$, and hence $\mathrm{Bir}\,(M)$, contains a non-commutative free group. □

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