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LOCALLY CONFORMAL PARALLEL G_2 AND Spin(7) MANIFOLDS

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Abstract. We characterize compact locally conformal parallel G_2 (respectively, Spin(7)) manifolds as fiber bundles over S^1 with compact nearly Kähler (respectively, compact nearly parallel G_2) fiber. A more specific characterization is provided when the local parallel structures are flat.

1. Introduction

Locally conformal parallel structures have been studied for a long time with respect to different natural geometries. The Kähler condition is the oldest example of parallel structure and locally conformal Kähler geometry is by now a quite developed subject with several interactions to other geometries [DO98, Orn04]. Other choices include locally conformal hyperkähler and quaternion Kähler metrics [OP97]. In particular, the useful technique of the quotient construction, coming from symplectic geometry, has been described both for locally conformal Kähler and locally conformal hyperkähler metrics [GOP05, GOPP05]. By looking at further groups whose holonomy has significance, the choices of G_2 and Spin(7) appear as deserving of attention.

In fact locally conformal parallel G_2 and Spin(7) structures are both natural classes in the frameworks of the the symmetries considered in [FG82, Fer86]. More recently, conformal parallel G_2 structures have been studied in details on solvmanifolds [CF04]. A locally conformal parallel structure is encoded in the Lee form θ, the 1-form representing the irreducible component of the covariant derivative of the fundamental form in the standard representation. When θ can be assumed to be parallel for one metric in the conformal class the geometry of the manifold is described, in the Kähler and hyperkähler settings, by a fibration over a circle [OV03, Ver04, GOPP05]. In fact, we obtain here the corresponding characterization in the G_2 and Spin(7) cases.

Theorem A A compact Riemannian 7-manifold M admits a locally conformal parallel G_2 structure if and only if there exists a fibre bundle $M \to S^1$ with abstract fibre N/Γ, where N is a compact simply connected nearly Kähler 6-manifold and Γ is a finite subgroup of complex isometries of N acting freely. Moreover, the cone $\mathcal{C}(N/\Gamma)$ covers M with cyclic infinite covering transformations group.

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Theorem B A compact Riemannian 8-manifold M admits a locally conformal parallel Spin(7) structure if and only if there exists a fibre bundle $M \to S^1$ with abstract fibre N/Γ, where N is a compact simply connected nearly parallel G_2 manifold and Γ is a finite subgroup of isometries of N acting freely. Moreover, the cone $\mathcal{C}(N/\Gamma)$ covers M with cyclic infinite covering transformations group.

When the local parallel structures are flat then this statement simplifies. In the G_2 case, one gets a trivial circle bundle whose fibre is the standard nearly Kähler 6-sphere S^6 (**Theorem 3.13**). In the Spin(7) case, the fibre is a spherical space form S^7/Γ, where Γ is a finite subgroup of Spin(7) (**Theorem 3.14**). In particular, the problem of describing all locally conformal parallel flat Spin(7) structures is thus reduced to the possibility of recognizing, inside the list of finite subgroup of SO(8), the ones which are contained in Spin(7).

An independent proof of Theorem A has recently been given by M. Verbitsky in [Ver05]. Moreover, the statements given here concerning the universal coverings have been recently formulated for manifolds of any dimension in the setting of geometric structures of vectorial type by I. Agricola and T. Friedrich in [AF05].

2. Preliminaries

Let $e_1, \dots, e_7$ be an oriented orthonormal basis of $\mathbb{R}^7$. The subgroup of GL(7) fixing the 3-form

$$\omega = e_{127} - e_{236} + e_{347} + e_{567} - e_{146} - e_{245} + e_{135} \tag{1}$$

is the compact Lie group $G_2 \subset SO(7)$, and ω corresponds to a real spinor so that G_2 can be identified with the isotropy group of a non-trivial real spinor.

A Riemannian manifold (M^7, g) is a G_2 manifold if its structure group reduces to G_2, or equivalently if there exists a 3-form ω on M locally given by (1). A G_2 manifold is said to be parallel if ω is parallel (namely, if $\mathrm{Hol}(g) \subset G_2$) and *locally conformal parallel* if the 1-form θ defined by

$$\theta = -\frac{1}{3} *(*d\omega \wedge \omega)$$

satisfies

$$d\omega = \frac{3}{4}\theta \wedge \omega, \qquad d{*\omega} = \theta \wedge *\omega. \tag{2}$$

The latter terminology is justified by the fact that if (2) holds, then θ is closed [Cab96] and the functions locally defined by $\theta = df$ gives rise to local parallel G_2 structures $e^{3f}\omega$. As usual, the choice of constants is a matter of habits and convenience.

Finally, a G_2 structure with co-closed Lee form is called a *Gauduchon* G_2 structure.

Next, let $e_0, \dots, e_7$ be an oriented orthonormal basis of $\mathbb{R}^8 = \mathbb{R} \oplus \mathbb{R}^7$. The subgroup of GL(8) fixing the 4-form

$$\phi = e_0 \wedge \omega + *_7\omega \tag{3}$$

is the compact Lie group $\mathrm{Spin}(7) \subset \mathrm{SO}(8)$. Like in the G_2 case, ϕ corresponds to a real spinor and Spin(7) can be identified with the isotropy group of a non-trivial real spinor.

A Spin(7) structure on (M^8, g) is a reduction of its structure group to Spin(7), or equivalently a nowhere vanishing 4-form ϕ on M, locally written as (3). A Spin(7) manifold (M, g, ϕ) is said to be parallel if $\mathrm{Hol}(g) \subset \mathrm{Spin}(7)$, or equivalently if ϕ is parallel. The Lee form of a Spin(7) manifold is now

$$\Theta = -\frac{1}{7} *(*d\phi \wedge \phi)$$

and if $d\phi = \Theta \wedge \phi$ then $d\Theta = 0$ [Cab95] and ϕ is locally conformal parallel. Again, a Spin(7) structure with co-closed Lee form will be called a Gauduchon Spin(7) structure.

3. Structure of compact manifolds

We will need the following fact concerning the homotheties of a cone over a compact Riemannian manifold.

Theorem 3.1. [GOPP05] *Let N be a compact Riemannian manifold, and denote by $\mathcal{C}(N)$ its Riemannian cone. Let Γ be a discrete subgroup of homotheties of $\mathcal{C}(N)$ acting freely and properly discontinously. Then Γ is a finite central extension of $\mathbb{Z}$, and the finite part is a finite subgroup of* $\mathrm{Isom}(N)$.

Recall that an almost Hermitian (N^6, J, g) with Kähler form F is *nearly Kähler* if dF decomposes as a $(3,0) + (0,3)$-form. Similarly a G_2 manifold (N^7, ω, g) is nearly parallel if $d\omega = \lambda{*\omega}$ with constant $\lambda > 0$. In both cases g is Einstein with positive scalar curvature. It is easy to see that N is nearly Kähler or nearly parallel G_2 if and only if its Riemannian cone $\mathcal{C}(N)$ is a parallel G_2 or Spin(7) manifold, respectively [Bär93].

3.1. Proof of Theorem A. Denote by $[\omega]$ the conformal class of the fundamental 3-form, and observe that we can more appropriately define locally conformal parallel G_2 manifolds as pairs $(M, [\omega])$ such that any representative of the conformal class is locally conformal parallel. Of course, one has an associated conformal class $[g]$ of Riemannian metrics. Note also that any G_2 manifold M can be viewed through its Lee form as a Weyl manifold. Thus, whenever M is compact, a Gauduchon G_2 structure can be obtained [Gau95, FI03].

Examples 3.2. *Consider a nearly Kähler manifold N^6, with fundamental form F. Then $N \times S^1$ admits a locally conformal parallel* G_2 *structure defined by* [CS02]

$$\omega = dt \wedge F + dF. \tag{4}$$

Thus the four known compact nearly Kähler 6-manifolds, namely S^6, $S^3 \times S^3$, $\mathbb{CP}^3$ and the flag manifold $F = \mathrm{U}(3)/(\mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{U}(1))$ give rise to products with S^1 that admit a locally conformal parallel G_2 *structure* [Hit01, Sal01]. *Further examples can be obtained by choosing N to be the twistor orbifold of the* *weighted projective planes $\mathbb{CP}^2_{p_0,p_1,p_2}$, the case $(p_0,p_1,p_2)=(1,1,1)$ corresponding to the flag manifold F. These orbifolds N admit a nearly Kähler structure and therefore their products with S^1 are locally conformal parallel G_2 manifolds.*

Proposition 3.3. *Let $(M,[\omega])$ be a compact locally conformal parallel G_2 manifold. If ω is a Gauduchon G_2 form, θ its Lee form and g the metric with Levi-Civita connection ∇^g, then $\nabla^g\theta=0$. Moreover:*

(1) *The Riemannian universal cover of M is isometric, up to homotheties, to the product $N\times\mathbb{R}$, where N is a compact 6-dimensional Einstein manifold with positive scalar curvature.*
(2) *The scalar curvature of g is the positive constant $s=\frac{15}{8}|\theta|^2$.*
(3) *The Lee flow preserves the Gauduchon G_2 structure, that is $\mathcal{L}_{\theta^\sharp}\omega=0$, where $\theta^\sharp$ denotes the Lee vector field.*
(4) *If ∇ is the unique G_2 connection with totally skew-symmetric torsion T, then $\nabla T=0$.*

Proof: The Weyl structure of M defines a Weyl connection D such that $Dg=\theta\otimes g$. Since θ is closed and the Ricci tensor vanishes, Theorem 3 of [Gau95] can be applied. This gives $\nabla^g\theta=0$ as well as property i), where the compactness of N is due to Myers Theorem. Statement ii) is a direct consequence of the formula (4.19) in [FI03]. To prove iii), observe that $\mathcal{L}_{\theta^\sharp}\omega=\nabla^g_{\theta^\sharp}\omega$. Using the fact that locally $\theta=-4df$ and that the Weyl connection D coincides locally with the Levi-Civita connection of the parallel G_2 structure $(\omega'=e^{3f}\omega, g'=e^{2f}g)$ we get

$$D_{\theta^\sharp}X=\nabla^g_{\theta^\sharp}X-\frac{1}{4}|\theta|^2X,\qquad D\omega=\frac{3}{4}\theta\otimes\omega. \tag{5}$$

On the other hand, using the first equation in (5) we calculate $D_{\theta^\sharp}\omega=\nabla^g_{\theta^\sharp}\omega+\frac{3}{4}|\theta|^2\omega$. Compare with the second equation in (5) to conclude $0=\nabla^g_{\theta^\sharp}\omega=\mathcal{L}_{\theta^\sharp}\omega$. Finally iv) follows from the fact that the unique G_2 connection having as torsion a 3-form is given by [FI03]

$$g(\nabla_XY,Z)=g(\nabla^g_XY,Z)+\frac{1}{2}T(X,Y,Z),$$

where $T=\frac{1}{4}*(\theta\wedge\omega)=-\frac{1}{4}i_\theta(*\omega)$. The last two equalities lead to $\nabla\theta=0$ and hence to $\nabla T=0$. $\square$

Corollary 3.4. *A 7-dimensional compact Riemannian manifold M carries a locally conformal parallel G_2 structure if and only if M admits a covering which is a Riemannian cone over a compact nearly Kähler 6-manifold and such that the covering transformations are homotheties preserving the corresponding parallel G_2 structure.*

Proof: According to Proposition 3.3, the universal covering $\tilde{M}$ is isometric to $N\times\mathbb{R}$ with the product metric $\tilde{g}=g_N+dt^2$ and $\tilde{\theta}=dt$. The local conformal transformations making the structures parallel give rise to a global G_2 structure on $\tilde{M}$. Therefore

$$\tilde{g}'=e^{2t}\tilde{g}=e^{2t}\left(g_N+dt^2\right),\qquad \tilde{\omega}'=e^{3t}\tilde{\omega},$$

defines a parallel G_2 structure on $\tilde{M}$. Observe that $\tilde{g}'$ is the cone metric and use [Bär93] to get the conclusion. $\square$

Remark 3.5. *The universal covering used in the proof of Corollary 3.4 can be replaced by any Riemannian covering of M such that the Lee form is exact, say $\tilde{\theta} = df$. Then $f\colon \tilde{M} \to \mathbb{R}$ is a Riemannian submersion, because $|\tilde{\theta}|$ is constant. Moreover, f is globally trivial, for the Lee flow acts by isometries and $\tilde{M}$ is complete.*

The previous Remark together with Theorem 3.1 gives that among all the cones covering a compact M, one can always choose the one whose covering transformations group is cyclic infinite.

Proposition 3.6. *Let $(M, [\omega])$ be a compact locally conformal parallel* G_2 *manifold. Then there exists a covering $\tilde{M} \stackrel{\mathbb{Z}}{\to} M$, where $\tilde{M}$ is a globally conformal parallel* G_2 *manifold.*

Proof: As a consequence of the property $\nabla^g \theta = 0$, we can always assume the Lee form to be of constant length and harmonic. Moreover, the Lee field preserves ω, so that it is Killing and the Lee flow acts as G_2 isometries. Next, Corollary 3.4 gives a globally conformal parallel covering $\mathcal{C}(N) \stackrel{\Gamma}{\to} M$, where N is a compact nearly Kähler 6-manifold. Denote by $\rho\colon \Gamma \to \mathbb{R}^+$ the map given by the dilation factors of elements of Γ. The isometries in Γ are then $\ker \rho$ and by Theorem 3.1 we obtain

$$\Gamma / \ker \rho \simeq \mathrm{Im}(\rho) \simeq \mathbb{Z}.$$

Define $\tilde{M} \stackrel{\mathbb{Z}}{\to} M$ as $\mathcal{C}(N/\ker \rho) \stackrel{\Gamma/\ker\rho}{\to} M$. $\square$

If $\mathcal{C}(N) \stackrel{\mathbb{Z}}{\to} M$ is the covering given by Proposition 3.6 and π the projection of the cone onto its radius, it is easy to check that π is equivariant with respect to the action of the covering maps on $\mathcal{C}(N)$ and the action of $n \in \mathbb{Z}$ on $t \in \mathbb{R}$ given by $n + t$. From all of this, Theorem A follows.

3.2. Proof of Theorem B. Again, denote by $[\phi]$ the conformal class of the fundamental 4-form, and note that locally conformal parallel Spin(7) manifolds can be viewed as pairs $(M, [\phi])$ such that any representative of the conformal class is locally conformal parallel. Of course, one has an associated conformal class $[g]$ of Riemannian metrics.

Remark 3.7. *Like in the* G_2 *case, any compact* Spin(7) *manifold admits a Gauduchon* Spin(7) *structure* [Iva04]. *In fact, if (N^7, ω) is any nearly parallel* G_2 *manifold, the product $N^7 \times S^1$ carries a natural locally conformal parallel* Spin(7) *structure defined by*

$$\phi = dt \wedge \omega + *_{N^7}\omega$$

where $*_{N^7}$ is the Hodge star operator on N^7.

Examples 3.8. *Through the above formula one obtains several classes of examples of locally conformal parallel* Spin(7) *manifolds. In fact, according to* [FKMS97] *nearly parallel* G_2 *manifolds split in 4 classes, in correspondance with the number of Killing spinors. These 4 classes are (I) proper nearly parallel* G_2 *manifolds, (II) Sasakian-Einstein manifolds, (III) 3-Sasakian manifolds and (IV) the class containing only* S^7. *In particular, the class (I) contains all squashed 3-Sasakian metrics, including* S^7_{sq} *with its Einstein metric. Moreover, we have in this class the homogeneous real Stiefel manifold* SO(5)/SO(3) *and the Aloff-Wallach spaces* [CMS96]. *The class (II) includes* S^7 *with any of its differentiable structures and any Einstein metric in the families constructed in* [BGK05]. *The class (III) is known to contain examples with arbitrarily large second Betti number* [BGMR98]. *Note that the topological* S^7 *can be endowed with nearly parallel* G_2 *structures in any of the above mentioned 4 classes.*

These 4 classes correspond to the sequence of inclusions $\mathrm{Spin}(7) \supset \mathrm{SU}(4) \supset \mathrm{Sp}(2) \supset \{id\}$. *From this point of view, the products* $N \times S^1$ *carry a proper locally conformal parallel* Spin(7) *structure, a locally conformal Kähler Ricci-flat metric, a locally conformal hyperkähler metric and a locally conformal flat metric according to whether* N *belongs to the class (I), (II), (III) or (IV).*

One can easily check that if I, J, K *denote the hypercomplex structure on* $N \times S^1$ *(class (III)), then the* Spin(7) *form* ϕ *is related to the Kähler forms* $\omega_I, \omega_J, \omega_K$ *by* $2\phi = \omega_I^2 + \omega_J^2 - \omega_K^2$. *Thus the locally conformal hyperkähler property* $d\omega_I = \theta \wedge \omega_I, \dots$ *implies* $\theta = \Theta$. *This holds of course also for* N *in class (II), where the conformality factors* f_α *on* $N \times S^1$ *can be choosen in the same way for both the local Kähler metrics and the local parallel* Spin(7) *forms.*

Remark 3.9. *The existence of a Killing spinor on a nearly parallel* G_2 *manifold turns out to be equivalent to the existence of a parallel spinor with respect to the unique* G_2 *connection with totally skew-symmetric torsion* [FI02]. *The different classes of structures correspond to the fact that the holonomy group of the torsion connection is contained in* Sp(1), SU(3) *and* G_2, *respectively.*

An easy consequence of the considerations in [Iva04] is that the unique Spin(7) connection with totally skew-symmetric torsion on a locally conformal parallel Spin(7) manifold is determined by the formula

$$g(\nabla_X Y, Z) = g(\nabla^g_X Y, Z) + \frac{1}{2} T(X, Y, Z)$$

where

$$T = -\frac{1}{6} *(\Theta \wedge \phi) = -\frac{1}{6} i_{\Theta^\sharp}(*\phi).$$

Now, in the exactly same way as in the G_2 case, we have the following.

Proposition 3.10. *Let* $(M^8, [\phi])$ *be a compact locally conformal parallel* Spin(7) *manifold, where* ϕ *is a Gauduchon* Spin(7) *structure. Let* Θ *be its Lee form,* g *be the metric and* ∇^g *its Levi-Civita connection. Then* $\nabla^g \Theta = 0$. *Moreover:*

(1) *The Riemannian universal cover is isometric, up to homothety, to the product* $N^7 \times \mathbb{R}$*, where* N *is a compact 7-dimensional Einstein manifold with positive scalar curvature.*
(2) *The scalar curvature of* g *is the positive constant* $s = \frac{21}{36}|\Theta|^2$.
(3) *The Lee flow preserves the Gauduchon* Spin(7) *structure, that is* $\mathcal{L}_{\Theta^\sharp}\phi = 0$*, where* $\Theta^\sharp$ *denotes the Lee vector field.*
(4) *If* ∇ *denotes the unique* Spin(7) *connection with totally skew-symmetric torsion* T*, then* $\nabla T = 0$.

Moreover, similarly to the G_2 case, we get:

Corollary 3.11. *A 8-dimensional compact Riemannian manifold* M *carries a locally conformal parallel* Spin(7) *structure if and only if* M *admits a covering which is a Riemannian cone over a compact nearly parallel* G_2 *manifold and such that the covering transformations are homotheties preserving the corresponding parallel* Spin(7) *structure.*

Proposition 3.12. *Let* $(M, [\phi])$ *be a compact locally conformal parallel* Spin(7) *manifold. Then there exists a covering* $\tilde{M} \xrightarrow{\mathbb{Z}} M$*, where* $\tilde{M}$ *is a globally conformal parallel* Spin(7) *manifold.*

Then Theorem B follows.

3.3. Locally conformal flat structures and further remarks. The following statements specialize Theorems A and B whenever the local metrics are flat.

Theorem 3.13. *Let* $(M, [\omega])$ *be a compact locally conformal parallel flat* G_2 *manifold. Then* $(M, [\omega]) = (S^6 \times S^1, [\omega_0 = \eta \wedge F_0 + dF_0])$*, where* F_0 *is the standard nearly Kähler structure on* S^6.

Proof: The flatness of the local parallel G_2 structures is equivalent to the flatness of the Weyl connection of M, and from Corollary at page 11 in [Gau95] the associated compact simply connected nearly Kähler manifold is isometric to the round sphere S^6. Thus Theorem A implies that M is a fibre bundle over S^1 with fibre S^6/Γ, where Γ is a finite subgroup of SO(7). Since M is orientable, Γ must be trivial, and the connectedness of SO(7) implies that M is isometric to the trivial bundle $S^6 \times S^1$. The G_2 structure is given by $[\omega] = [\eta \wedge F + dF]$, where F is a nearly Kähler structure on the round sphere S^6. But the standard nearly Kähler structure on S^6 inherited from the imaginary octonions is the only nearly Kähler structure on S^6 compatible with the round metric [Fri05], and this ends the proof. $\square$

Theorem 3.14. *Let* $(M, [\phi])$ *be a compact locally conformal parallel flat* Spin(7) *manifold. Then* $(M, [\phi]) = (N \times S^1, [\phi_0 = \eta \wedge \omega_0 + *_N\omega_0])$*, where* $N = S^7/\Gamma$ *is a spherical space form and* ω_0 *is the standard nearly parallel* G_2 *structure on* S^7.

Proof: As for the G_2 case, the flatness of the local parallel Spin(7) structures gives that the associated compact simply connected nearly parallel G_2 manifold is isometric to the round sphere S^7. Thus Theorem B implies that M is a fibre bundle over S^1 with fibre S^7/Γ, where Γ is a finite subgroup of SO(8), and the connectedness of SO(8) implies that M is isometric to the trivial bundle $(S^7/\Gamma) \times S^1$. The Spin(7) structure is given by $[\phi] = [\eta \wedge \omega + *_7\omega]$, where ω is a Γ-invariant nearly parallel G_2 structure on the round sphere S^7 and $*_7$ is the associated Hodge star operator. But the standard nearly parallel G_2 structure on S^7 inherited from the standard Spin(7) structure on $\mathbb{R}^8$ is the only nearly parallel G_2 structure on S^7 compatible with the round metric [Fri05], and this completes the proof. □

Remark 3.15. *Note that* $\mathbb{R}^8 - \{0\}$ *admits a* Spin(7) *structure that is not in the conformal class generated by the standard one. This can be seen by looking at the proper nearly parallel* G_2 *structure on the squashed sphere* S^7_{sq} *and hence at the locally conformal parallel* Spin(7) *structure on* $S^7_{sq} \times S^1$*, that is not conformally flat.*

We restrict our attention to compact locally conformal parallel flat Spin(7) manifolds covered by $\mathbb{R}^8 - \{0\}$. The Spin(7) invariant 4-form ϕ on $\mathbb{R}^8$ is given, in terms of octonion multiplication, by (see [HL82])

$$\phi(x, y, z, v) = \langle x \cdot (\bar{y} \cdot z) - z \cdot (\bar{y} \cdot x), v\rangle.$$

Then the 4-form on $\mathbb{R}^8 - \{0\}$

$$\tilde{\phi}(x, y, z, v) = \frac{\phi(x, y, z, v)}{(|x|^2 + |y|^2 + |z|^2 + |v|^2)^2}$$

descends to a locally conformal parallel flat Spin(7) structure on $S^7 \times S^1$. For any finite $\Gamma \subset$ Spin(7) that acts freely on S^7 we get an example of locally conformal parallel Spin(7) structure on $(S^7/\Gamma) \times S^1$ as well as a new nearly parallel G_2 structure (respectively Sasaki-Einstein) on the factor S^7/Γ.

A class of finite subgroups of Spin(7) can be described by recalling that Spin(7) is generated by the right multiplication on the octonions $\mathbb{O}$ by elements of $S^6 \subset \mathrm{Im}(\mathbb{O})$. Let $1 \le m \le 7$ and let $V_m(\mathbb{R}^7)$ be the Stiefel manifold of orthonormal m-frames in $\mathbb{R}^7$. Any $\sigma \in V_m(\mathbb{R}^7)$ gives rise to mutually orthogonal $\sigma_1, \dots, \sigma_m \in S^6$ generating a finite subgroup $G_{\sigma(m)} \subset$ Spin(7) of order 2^{m+1}. This is due to the identity $(o\bar{y})x = -(o\bar{x})y$ that holds for any $o \in \mathbb{O}$ and any orthogonal octonions x, y. For example, the choice of $\sigma(4) = \{i, j, k, l\}$ gives rise to the group $G_{\sigma(4)}$:

$$\begin{gathered} o \to \pm o, \quad o \to \pm oi, \quad o \to \pm oj, \quad o \to \pm ok, \\ o \to \pm ol, \quad o \to \pm(oi)j, \quad o \to \pm(oi)k, \quad o \to \pm(oi)l, \\ o \to \pm(oj)k, \quad o \to \pm(oj)l, \quad o \to \pm(ok)l, o \to \pm((oi)j)k, \\ o \to \pm((oi)k)l, \quad o \to \pm((oi)j)l, \quad o \to \pm((oj)k)l, \quad o \to \pm(((oi)j)k)l. \end{gathered}$$

Other examples include the finite subgroup of $\mathrm{Sp}(2) \subset \mathrm{Spin}(7)$ obtained by looking at $\mathbb{O} = \mathbb{H} \oplus \mathbb{H}$. A simple example is given by the group:

$$o = (q, q') \to (\pm(q, q'), \pm(qi, q'i), \pm(qj, q'j), \pm(qk, q'k)).$$

Remark 3.16. *Note that most of these finite groups do not act freely on S^7 and it would be interesting to recognize which of the quotients are actually smooth and can appear as spherical space forms, thus fitting in J. Wolf classification* [Wol84, Wol01]*. For any such space form S^7/Γ the product manifold $(S^7/\Gamma) \times S^1$ will carry a locally conformal parallel flat* Spin(7) *structure. Moreover, all compact locally conformal parallel flat* Spin(7) *manifold could be obtained in this way selecting from the J. Wolf classification* [Wol84, Wol01] *of finite subgroups of $SO(8)$ those which are finite subgroups of* Spin(7)*.*

Another way of producing examples is to look at the locally conformal parallel Spin(7) and non locally conformal Kähler Ricci flat $S^7_{\mathrm{sq}} \times S^1$. Since the isometry group of the squashed sphere S^7_{sq} is $\mathrm{Sp}(2) \cdot \mathrm{Sp}(1)$, any finite $\Gamma \subset \mathrm{Sp}(2) \cdot \mathrm{Sp}(1)$ acting freely gives rise to such a structure on $(S^7_{\mathrm{sq}}/\Gamma) \times S^1$. These are pure examples in the sense that they are not locally conformal Kähler Ricci flat. More generally, any 3-Sasakian structure in dimension 7 gives rise to a 'squashed' nearly parallel G_2 structure [FKMS97], that admits exactly one Killing spinor and similar arguments apply.

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References

[AF05] I. Agricola and T. Friedrich, *Geometric structures of vectorial type*, preprint, arXiv:math.DG/0509147.

[Bär93] C. Bär, *Real Killing spinors and holonomy,* Comm. Math. Phys. **154** (1993), no. 3, 509–521.

[BGK05] C. P. Boyer, K. Galicki, and J. Kollár, *Einstein metrics on spheres*, Ann. of Math. (2) **162** (2005) 1–24.

[BGMR98] C. P. Boyer, K. Galicki, B. M. Mann, and E. G. Rees. *Compact 3-Sasakian 7-manifolds with arbitrary second Betti number*, Invent. Math. **131** (1998), no. 2, 321–344.

[Cab95] F. M. Cabrera, *On Riemannian manifolds with Spin(7)-structure*, Publ. Math. Debrecen **46** (1995) 271–283.

[Cab96] ———, *On Riemannian manifolds with G_2-structure*, Bollettino U.M.I., sezione A, 99–112, 1996.

[CF04] S. Chiossi and A. Fino, *Conformally parallel G_2 structures on a class of solvmanifolds*, preprint, arXiv:math.DG/0409137.

[CMS96] F. M. Cabrera, M. D. Monar, and A. F. Swann, *Classification of G_2-Structures,* J. Lond. Math. Soc., II. Ser. **53** (1996), no. 2, 407–416.

[CS02] S. Chiossi and S. Salamon, *The intrinsic torsion of $SU(3)$ and G_2 structures.* Differential geometry, Valencia, 2001, 115–133, World Sci. Publishing, River Edge, NJ, 2002.

[DO98] S. Dragomir and L. Ornea, Locally conformal Kähler geometry, Progress in Math Vol. 155. Birkhäuser, 1998.

[Fer86] M. Fernández, *A Classification of Riemannian Manifolds with Structure Group Spin*(7). Ann. di Mat. Pura e Appl. **148** (1986) 101–122.

[FG82] M. Fernández and A. Gray, *Riemannian manifolds with structure group G_2,* Ann. Mat. Pura Appl., IV. Ser. **132** (1982) 19–45.

[FI02] T. Friedrich and S. Ivanov, *Parallel spinors and connections with skew-symmetric torsion in string theory*, Asian J. Math. **6** (2002), no. 2, 303–335.

[FI03] ———, *Killing spinor equations in dimension 7 and geometry of integrable G_2-manifolds.* J. Geom. Phys. **48** (2003), no. 1, 1–11.

[FKMS97] T. Friedrich, I. Kath, A. Moroianu, and U. Semmelmann, *On nearly parallel G_2-structures*, J. Geom. Phys. **23** (1997), no. 3–4, 259–286.

[Fri05] T. Friedrich. *Nearly Kähler and nearly parallel* G_2*-structures on spheres*, preprint, arXiv:math.DG/0509146.

[Gau95] P. Gauduchon, *Structures de Weyl-Einstein, espaces de twisteurs et variétés de type $S^1 \times S^3$*. J. Reine Angew. Math. **469** (1995) 1–50.

[GOP05] R. Gini, L. Ornea, and M. Parton, *Locally conformal Kähler reduction*, J. Reine Angew. Math. **581** (2005) 1–21.

[GOPP05] R. Gini, L. Ornea, M. Parton, and P. Piccinni, *Reduction of Vaisman structures in complex and quaternionic geometry*, preprint, arXiv:math.DG/0502572.

[Hit01] N. Hitchin, *Stable forms and special metrics*, Global differential geometry: the mathematical legacy of Alfred Gray (Bilbao, 2000), 70–89, Contemp. Math., 288, Amer. Math. Soc., Providence, RI, 2001.

[HL82] R. Harvey and H. B. Lawson, *Calibrated geometries*, Acta Math. **148** (1982) 47–157.

[Iva04] S. Ivanov, *Connection with torsion, parallel spinors and geometry of Spin(7) manifolds*, Math. Res. Lett. **11** (2004), no. 2–3, 171–186.

[OP97] L. Ornea and P. Piccinni, Locally conformal Kähler structures in quaternionic geometry, Trans. Am. Math. Soc. **349** (1997), no. 2, 641–655.

[Orn04] L. Ornea, *Locally conformally Kähler manifolds. A selection of results*, preprint, arXiv:math.DG/0411503.

[OV03] L. Ornea and M. Verbitsky, Structure theorem for compact Vaisman manifolds. Math. Res. Lett. **10** (2003), no. 5-6, 799–805.

[Sal01] S. Salamon, *Almost parallel structures*, Global differential geometry: the mathematical legacy of Alfred Gray (Bilbao, 2000), 162–181, Contemp. Math., 288, Amer. Math. Soc., Providence, RI, 2001.

[Ver04] M. Verbitsky, *Theorems on the vanishing of cohomology for locally conformally hyper-Kähler manifolds (Russian)*, Tr. Mat. Inst. Steklova, Algebr. Geom. Metody, Svyazi i Prilozh, **246** (2004) 64–91

[Ver05] ———, *An intrinsic volume functional on almost complex 6-manifolds and nearly Kähler geometry*, preprint, arXiv:math.DG/0507179.

[Wol84] J. A. Wolf, Spaces of constant curvature. 5th ed. Boston, Mass.: Publish Perish, Inc. XV, 1984.

[Wol01] ———, *Isospectrality for spherical space forms*, Result. Math. **40** (2001), no. 1-4, 321–338.

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