

\mathbb{P} -OBJECTS AND AUTOEQUIVALENCES OF DERIVED CATEGORIES

DANIEL HUYBRECHTS AND RICHARD THOMAS

Besides abelian varieties, there are essentially two types of smooth projective variety with trivial canonical bundle, Calabi–Yau and holomorphic symplectic manifolds. They are distinguished, among other things, by their holonomy groups being $SU(n)$, respectively $Sp(n)$. The difference between these two types are in many respect analogous to the difference between spheres S^n and complex projective spaces \mathbb{P}^n . This analogy is of course difficult to make precise, but it comes up in various contexts:

- For a compact manifold X of real dimension $2n$, respectively $4n$, with holonomy $SU(n)$, respectively $Sp(n)$, there exists a ring isomorphism $H^*(X, \mathcal{O}) \cong H^*(S^n, \mathbb{C})$, respectively $H^*(X, \mathcal{O}) \cong H^*(\mathbb{P}^n, \mathbb{C})$.

- The SYZ conjecture predicts in a large complex structure limit the existence of a lagrangian fibration $X \rightarrow S^n$ for any simply connected Calabi–Yau manifold X , whereas for a holomorphic symplectic manifold this should become a lagrangian fibration $X \rightarrow \mathbb{P}^n$.

- On the symplectic side, Seidel has studied Dehn twists associated to any lagrangian sphere contained in a Calabi–Yau manifold, and an analogously defined symplectomorphism associated to a lagrangian complex projective space (see [10, Section 4b]). In complex dimension 2 (where $S^2 \cong \mathbb{P}^1$), the latter twist is the square of the former; this should be compared to our Proposition 2.9.

- Spherical twists, studied in detail in [12], are supposed to mirror Dehn twists associated with lagrangian spheres. These are autoequivalences of the derived category of coherent sheaves $\mathbf{D}^b(X)$ of a Calabi–Yau manifold associated to ‘spherical’ objects $\mathcal{E} \in \mathbf{D}^b(X)$. By definition, an object is spherical if $\text{Ext}^*(\mathcal{E}, \mathcal{E})$ is isomorphic to $H^*(S^n, \mathbb{C})$.

This note aims at completing the picture by introducing the notion of \mathbb{P} -objects and the induced \mathbb{P} -twists, which are autoequivalences of the derived category $\mathbf{D}^b(X)$ of a smooth projective variety X (typically a holomorphic symplectic manifold).

For the low-dimensional case $SU(2) = Sp(1)$ (reflected by $S^2 = \mathbb{P}^1$) the newly defined \mathbb{P}^1 -twist is just the square of the usual spherical twist (see Proposition 2.9). In higher dimensions, however, \mathbb{P}^n -twists describe genuinely new derived equivalences. In fact, spherical objects ought not exist on manifolds with holonomy $Sp(n \geq 2)$.

Received by the editors July 12, 2005.

In higher dimensions one can sometimes nevertheless establish a relation between \mathbb{P}^n -twists on $\mathrm{Sp}(n)$ -manifolds and spherical twists on $\mathrm{SU}(2n+1)$ -manifolds. Roughly, a \mathbb{P}^n -object that does not deform sideways in a one-dimensional family becomes a spherical object in the ambient $(2n+1)$ -dimensional manifold (see Proposition 1.4). In such a situation Proposition 2.7 shows that the spherical twist becomes the \mathbb{P}^n -twist on the special fibre.

1. \mathbb{P}^n -objects

Throughout X will be a smooth projective variety and $\mathbf{D}^b(X)$ denotes the bounded derived category of coherent sheaves on X . All functors that we use between derived categories are derived functors; we omit \mathbf{L} s and \mathbf{R} s from left and right derived functors.

Definition 1.1. *An object $\mathcal{E} \in \mathbf{D}^b(X)$ is called a \mathbb{P}^n -object if $\mathcal{E} \otimes \omega_X \cong \mathcal{E}$ and $\mathrm{Ext}^*(\mathcal{E}, \mathcal{E})$ is isomorphic as a graded ring to $H^*(\mathbb{P}^n, \mathbb{C})$.*

Remark 1.2. i) Serre duality shows that if $\mathcal{E} \in \mathbf{D}^b(X)$ is a \mathbb{P}^n -object then $\dim(X) = 2n$.

ii) The notion of \mathbb{P}^n -objects is designed for holomorphic symplectic manifolds (i.e. hyperkähler manifolds). Of course, in this case the first condition $\mathcal{E} \otimes \omega_X \cong \mathcal{E}$ is automatic.

Examples 1.3. i) Let X be holomorphic symplectic of dimension $2n$ and $P := \mathbb{P}^n \subset X$. Then $\mathcal{N}_{P/X} \cong \Omega_P$ and hence $\mathrm{Ext}^q(\mathcal{O}_P, \mathcal{O}_P) \cong \Omega_P^q$. Thus the spectral sequence

$$E_2^{p,q} = H^p(X, \mathrm{Ext}^q(\mathcal{O}_P, \mathcal{O}_P)) \Rightarrow \mathrm{Ext}_X^{p+q}(\mathcal{O}_P, \mathcal{O}_P)$$

yields an isomorphism $\mathrm{Ext}_X^*(\mathcal{O}_P, \mathcal{O}_P) \cong H^*(P, \Omega_P^*) = H^*(\mathbb{P}^n, \mathbb{C})$. This is a ring isomorphism; one roundabout way to see this is given in the remark following Example 1.5. Hence, $\mathcal{O}_P \in \mathbf{D}^b(X)$ is a \mathbb{P}^n -object. The same arguments also show that $\mathcal{O}_P(i) \in \mathbf{D}^b(X)$ is a \mathbb{P}^n -object for all i .

ii) Suppose X is an irreducible holomorphic symplectic manifold, i.e. in addition $H^*(X, \mathcal{O}_X) \cong H^*(\mathbb{P}^n, \mathbb{C})$. Then any line bundle L on X is a \mathbb{P}^n -object. Indeed, $\mathrm{Ext}^*(L, L) \cong H^*(X, \mathcal{O}_X)$.

iii) Let X be a K3 surface and $C \cong \mathbb{P}^1 \subset X$. Due to i), $\mathcal{O}_C \in \mathbf{D}^b(X)$ is a \mathbb{P}^1 -object. Note that \mathcal{O}_C is also spherical, which reflects $S^2 \cong \mathbb{P}^1$.

iv) Let $\pi : X \rightarrow \mathbb{P}^n$ be a Lagrangian fibration of an irreducible symplectic manifold. If $\mathcal{E} \in \mathbf{D}^b(\mathbb{P}^n)$ is an object with a graded ring isomorphism $\bigoplus \mathrm{Ext}^p(\mathcal{E}, \mathcal{E} \otimes \Omega_{\mathbb{P}^n}^q) \cong \bigoplus H^p(\mathbb{P}^n, \Omega_{\mathbb{P}^n}^q)$ (in particular, \mathcal{E} is exceptional), then $\pi^* \mathcal{E} \in \mathbf{D}^b(X)$ is a \mathbb{P}^n -object. This uses Matsushita's result: $R^i \pi_* \mathcal{O}_X \cong \Omega_{\mathbb{P}^n}^i$ [8].

In dimension > 2 the notions of a spherical and a \mathbb{P}^n -object are different. In many examples, \mathbb{P}^n -objects should be thought of as hyperplane sections of spherical objects. This is made more precise by the following result.

Suppose $\mathcal{X} \rightarrow C$ is a smooth family over a smooth curve C with distinguished fibre $j : X := \mathcal{X}_0 \hookrightarrow \mathcal{X}$, $0 \in C$. Let us denote the Kodaira-Spencer class of this family by $\kappa(\mathcal{X}) \in H^1(X, \mathcal{T}_X)$ and the Atiyah class of an object $\mathcal{E} \in \mathbf{D}^b(X)$ by

$A(\mathcal{E}) \in \text{Ext}^1(\mathcal{E}, \mathcal{E} \otimes \Omega_X^1)$. The product $A(\mathcal{E}) \cdot \kappa(\mathcal{X}) \in \text{Ext}^2(\mathcal{E}, \mathcal{E})$ is the obstruction to deforming \mathcal{E} sideways to first order to neighbouring fibres in the family [3, 7].

Proposition 1.4. *Suppose $\mathcal{E} \in \mathbf{D}^b(X)$ is a \mathbb{P}^n -object such that $0 \neq A(\mathcal{E}) \cdot \kappa(\mathcal{X}) \in \text{Ext}^2(\mathcal{E}, \mathcal{E})$. Then $j_*(\mathcal{E}) \in \mathbf{D}^b(\mathcal{X})$ is spherical.*

Proof. Firstly, one checks $j_*(\mathcal{E}) \otimes \omega_{\mathcal{X}} \cong j_*(\mathcal{E})$, which follows from $\omega_{\mathcal{X}}|_X \cong \omega_X$ and the assumption on \mathcal{E} .

Next, we use the existence of a distinguished triangle of the form

$$\mathcal{E} \otimes \mathcal{O}_X[1] \longrightarrow j^* j_* \mathcal{E} \longrightarrow \mathcal{E} \longrightarrow \mathcal{E} \otimes \mathcal{O}_X[2]$$

with boundary morphism given by $A(\mathcal{E}) \cdot \kappa(\mathcal{X}) : \mathcal{E} \rightarrow \mathcal{E}[2]$. (For the convenience of the reader we give a short proof of this standard result in the appendix.) This yields the long exact sequence

$$\text{Ext}_X^k(\mathcal{E}, \mathcal{E}) \longrightarrow \text{Ext}_{\mathcal{X}}^k(j_* \mathcal{E}, j_* \mathcal{E}) \longrightarrow \text{Ext}_X^{k-1}(\mathcal{E}, \mathcal{E}) \xrightarrow{\delta} \text{Ext}_X^{k+1}(\mathcal{E}, \mathcal{E}),$$

where we use adjunction $\text{Ext}_{\mathcal{X}}^k(j_* \mathcal{E}, j_* \mathcal{E}) = \text{Ext}_X^k(j^* j_* \mathcal{E}, \mathcal{E})$.

The boundary morphism δ is given by cup-product with $A(\mathcal{E}) \cdot \kappa(\mathcal{X}) : \mathcal{E} \rightarrow \mathcal{E}[2]$ considered as an element in $\text{Ext}^2(\mathcal{E}, \mathcal{E})$. By assumption this may be taken to be the degree 2 generator of $\text{Ext}^*(\mathcal{E}, \mathcal{E})$. Therefore δ is an isomorphism for $1 \leq k \leq 2n - 1$, yielding

$$\text{Ext}^k(j_* \mathcal{E}, j_* \mathcal{E}) = \begin{cases} \mathbb{C} & k = 0, 2n + 1, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, $j_* \mathcal{E}$ is indeed a spherical object in $\mathbf{D}^b(\mathcal{X})$ of the $(2n + 1)$ -dimensional variety \mathcal{X} . \square

Example 1.5. The typical example for the situation considered above is the twistor space of a hyperkähler manifold. The twistor space is, unfortunately, never a projective manifold (not even Kähler), but the notion of a spherical object makes sense also in the analytic category.

For the \mathbb{P}^n -object \mathcal{O}_P given by a projective space $P \subset X$ one can in fact find a projective \mathcal{X} for which $\mathbb{P}^n \cong P \subset X$ does not deform to $P_t \subset \mathcal{X}_t$ even to first order (see [6]).

Note that there are \mathbb{P}^n -objects for which such a family does not exist. E.g. the \mathbb{P}^n -object \mathcal{O}_X always deforms in families. On the other hand, any non-trivial line bundle admits such a family.

Presumably this might be fixed by deforming X in noncommutative directions, i.e. allowing \mathcal{X} to be a noncommutative variety, providing other examples of the same phenomenon, namely that a \mathbb{P}^n -object becomes a spherical object in an ‘ambient’ derived category (the analogue of $\mathbf{D}^b(\mathcal{X})$).

We remark that the proof of Proposition 1.4 shows that if $\text{Ext}^*(\mathcal{E}, \mathcal{E})$ is isomorphic to $H^*(\mathbb{P}^N, \mathbb{C})$ as a vector space but *not* as a ring, then $j_* \mathcal{E}$ is *not* a spherical object. In the case of \mathcal{O}_P given by a projective space $\mathbb{P}^N \cong P \subset X$ with normal bundle Ω_P as in Example 1.3 i), fix an \mathcal{X} for which $P \subset X$ does

not deform to first order. It is then standard that the normal bundle of $P \subset \mathcal{X}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus(n+1)}$, from which it is easy to see it is spherical (see e.g. [5, Ch. 8]). Thus the ring structure on $\text{Ext}^*(\mathcal{O}_P, \mathcal{O}_P)$ is indeed that of $H^*(\mathbb{P}^N, \mathbb{C})$.

2. \mathbb{P}^n -twists

We shall try to imitate the construction of the spherical twist $T_{\mathcal{E}}$ associated to any spherical object (see [12]) and define a \mathbb{P}^n -twist for any \mathbb{P}^n -object. This is done in two steps. We first describe the Fourier–Mukai kernel and then show that the induced Fourier–Mukai transform is an equivalence. The second step is straightforward, whereas the description of the Fourier–Mukai kernel itself is interesting in as much as it uses a double cone construction.

Suppose $\mathcal{E} \in \mathbf{D}^b(X)$ is a \mathbb{P}^n -object. A generator $\bar{h} \in \text{Ext}^2(\mathcal{E}, \mathcal{E})$ will be viewed as a morphism $h : \mathcal{E}[-2] \rightarrow \mathcal{E}$. The ring $\text{Ext}^*(\mathcal{E}, \mathcal{E})$ is then isomorphic to $\mathbb{C}[\bar{h}] / (\bar{h}^{n+1})$. The image of \bar{h} under the natural isomorphism $\text{Ext}^2(\mathcal{E}, \mathcal{E}) \cong \text{Ext}^2(\mathcal{E}^\vee, \mathcal{E}^\vee)$ will be denoted \bar{h}^\vee , which represents a morphism $h^\vee : \mathcal{E}^\vee[-2] \rightarrow \mathcal{E}^\vee$.

Then introduce $H := h^\vee \boxtimes \text{id} - \text{id} \boxtimes h$ on $X \times X$ which is thus a morphism

$$H : (\mathcal{E}^\vee \boxtimes \mathcal{E})[-2] \longrightarrow \mathcal{E}^\vee \boxtimes \mathcal{E}.$$

The cone $\mathcal{H} := C(H)$ of this morphism fits in a distinguished triangle

$$(\mathcal{E}^\vee \boxtimes \mathcal{E})[-2] \xrightarrow{H} \mathcal{E}^\vee \boxtimes \mathcal{E} \longrightarrow \mathcal{H} \longrightarrow (\mathcal{E}^\vee \boxtimes \mathcal{E})[-1].$$

Recall that the kernel of the spherical twist associated to a spherical object is by definition the cone of the trace morphism $\text{tr} : \mathcal{E}^\vee \boxtimes \mathcal{E} \rightarrow \mathcal{O}_\Delta$, which is actually the composition of the restriction to the diagonal $\Delta \subset X \times X$ and the trace map on the diagonal.

In order to define the kernel of the \mathbb{P}^n -twist we have to go one step further.

Lemma 2.1. *The natural trace map $\text{tr} : \mathcal{E}^\vee \boxtimes \mathcal{E} \rightarrow \mathcal{O}_\Delta$ factorizes uniquely over the cone \mathcal{H} , i.e. there exists a unique morphism t that makes the following diagram commutative*

$$\begin{array}{ccc} \mathcal{E}^\vee \boxtimes \mathcal{E} & \longrightarrow & \mathcal{H} \\ & \searrow \text{tr} & \downarrow t \\ & & \mathcal{O}_\Delta. \end{array}$$

Proof. Apply $\text{Hom}(\cdot, \mathcal{O}_\Delta)$ to the distinguished triangle defining \mathcal{H} . Use $\text{Ext}_{X \times X}^i(\mathcal{E}^\vee \boxtimes \mathcal{E}, \mathcal{O}_\Delta) \cong \text{Ext}_X^i(\mathcal{E}, \mathcal{E})$ and the definition of H , to show that the boundary maps $\text{Ext}^i(\mathcal{E}, \mathcal{E}) \rightarrow \text{Ext}^{i+2}(\mathcal{E}, \mathcal{E})$ become $\bar{h} - \bar{h} = 0$. Hence $\text{Hom}(\mathcal{H}, \mathcal{O}_\Delta) \rightarrow \text{Hom}(\mathcal{E}^\vee \boxtimes \mathcal{E}, \mathcal{O}_\Delta)$ is an isomorphism, giving the unique lift t of the trace map. \square

Definition 2.2. To any \mathbb{P}^n -object $\mathcal{E} \in \mathbf{D}^b(X)$ one associates the cone

$$\mathcal{Q}_{\mathcal{E}} := C(\mathcal{H} \xrightarrow{t} \mathcal{O}_{\Delta}) \in \mathbf{D}^b(X \times X).$$

While we used the assumption that \mathcal{E} is a \mathbb{P}^n -object to simplify the above construction, an alternative construction, using locally free and Čech resolutions, shows that in fact t always exists.

Definition 2.3. Let $\mathcal{E} \in \mathbf{D}^b(X)$ be a \mathbb{P}^n -object and let $\mathcal{Q}_{\mathcal{E}} \in \mathbf{D}^b(X \times X)$ be the object associated to it by the above construction. The \mathbb{P}^n -twist $P_{\mathcal{E}}$ induced by a \mathbb{P}^n -object $\mathcal{E} \in \mathbf{D}^b(X)$ is the Fourier–Mukai transform

$$P_{\mathcal{E}} := \Phi_{\mathcal{Q}_{\mathcal{E}}} : \mathbf{D}^b(X) \longrightarrow \mathbf{D}^b(X)$$

with kernel $\mathcal{Q}_{\mathcal{E}}$.

Remark 2.4. Note that the induced actions $P_{\mathcal{E}}^K : K(X) \rightarrow K(X)$ and $P_{\mathcal{E}}^H : H^*(X, \mathbb{Q}) \rightarrow H^*(X, \mathbb{Q})$ are both the identity. This follows from the observation that $[\mathcal{H}] = [\mathcal{E}^{\vee} \boxtimes \mathcal{E}] + [\mathcal{E}^{\vee} \boxtimes \mathcal{E}[-1]] = 0$ in K-theory. Later, in Remark 2.8, we will see that in most cases $P_{\mathcal{E}}$ can even be deformed to the identity on a deformation of X .

It is often useful to have a cone description of the image of an object under the \mathbb{P}^n -twist. It is not difficult to see that for any $\mathcal{F} \in \mathbf{D}^b(X)$ the image $P_{\mathcal{E}}(\mathcal{F})$ is isomorphic to the double cone

$$C \left(C(\mathrm{Ext}^{*-2}(\mathcal{E}, \mathcal{F}) \otimes \mathcal{E} \xrightarrow{\bar{h}^{\vee} \cdot \mathrm{id} - \mathrm{id} \cdot h} \mathrm{Ext}^*(\mathcal{E}, \mathcal{F}) \otimes \mathcal{E}) \longrightarrow \mathcal{F} \right).$$

Let us spell this out in the case of the \mathbb{P}^n -object itself and objects that are orthogonal to it.

Lemma 2.5. Let $\mathcal{E} \in \mathbf{D}^b(X)$ be a \mathbb{P}^n -object. Then

- i) $P_{\mathcal{E}}(\mathcal{E}) \cong \mathcal{E}[-2n]$ and ii) $P_{\mathcal{E}}(\mathcal{F}) \cong \mathcal{F}$ for any $\mathcal{F} \in \mathcal{E}^{\perp} := \{\mathcal{G} : \mathrm{Ext}^*(\mathcal{E}, \mathcal{G}) = 0\}$.

Proof. The second assertion is trivial, for $\Phi_{\mathcal{H}}(\mathcal{F}) \cong 0$ for any $\mathcal{F} \in \mathcal{E}^{\perp}$ and, therefore, $P_{\mathcal{E}}(\mathcal{F}) \cong \Phi_{\mathcal{O}_{\Delta}}(\mathcal{F})$.

For the first one we use $\Phi_{\mathcal{E}^{\vee} \boxtimes \mathcal{E}}(\mathcal{E}) = \bigoplus \bar{h}^i \cdot \mathcal{E}[-2i]$ to compute $\Phi_{\mathcal{H}}(\mathcal{E})$ as the cone of the morphism

$$\begin{array}{ccc} \mathrm{id} \cdot \mathcal{E}[-2] & \xrightarrow{-h} & \mathrm{id} \cdot \mathcal{E} \\ \oplus & \searrow \bar{h} \cdot & \oplus \\ \bar{h} \cdot \mathcal{E}[-4] & \xrightarrow{-h} & \bar{h} \cdot \mathcal{E}[-2] \\ \oplus & \searrow \bar{h} \cdot & \oplus \\ \vdots & & \vdots \\ \oplus & & \oplus \\ \bar{h}^n \cdot \mathcal{E}[-2n-2] & \xrightarrow{-h} & \bar{h}^n \cdot \mathcal{E}[-2n]. \end{array}$$

The evaluation map to \mathcal{E} is the obvious one, taking the r th term in the right hand column to \mathcal{E} by the map h^r , for all $0 \leq r \leq n$.

$C(\mathcal{E} \xrightarrow{\text{id}} \mathcal{E})$ maps into the cone on the evaluation map in the obvious way (with the first factor mapping isomorphically to the top right hand corner of the above diagram). Taking the cone on this shows that $P_{\mathcal{E}}(\mathcal{E})$ is quasi-isomorphic to the cone on the above diagram with the top right hand corner removed. Repeating this procedure with the subcone $C(\text{id} \cdot \mathcal{E}[-2] \xrightarrow{\bar{h}} \bar{h} \cdot \mathcal{E}[-2])$ shows we can further remove the top left hand corner and the next right hand term. Iterating leaves us with just the bottom left hand term $\bar{h}^n \cdot \mathcal{E}[-2n-2]$ which, due to its position, makes $P_{\mathcal{E}}(\mathcal{E}) \cong \mathcal{E}[-2n]$. \square

Proposition 2.6. *For any \mathbb{P}^n -object $\mathcal{E} \in \mathbf{D}^b(X)$ the associated \mathbb{P}^n -twist is an autoequivalence*

$$P_{\mathcal{E}} : \mathbf{D}^b(X) \xrightarrow{\sim} \mathbf{D}^b(X).$$

Proof. We follow Ploog's simplified proof (see [9]) of the analogous result for spherical twists in [12].

Let $\mathcal{E} \in \mathbf{D}^b(X)$ be any object. Then $\Omega := \{\mathcal{E}\} \cup \mathcal{E}^{\perp}$ is a spanning class (cf. [5, Ch.8]). On this spanning class the \mathbb{P}^n -twist acts by shifting $[-2n]$ on \mathcal{E} and as the identity on the rest. This immediately shows that $P_{\mathcal{E}}$ is fully faithful.

In order to show that $P_{\mathcal{E}}$ is an equivalence use the assumption $\mathcal{E} \otimes \omega_X \cong \mathcal{E}$ which ensures that $\mathcal{Q}_{\mathcal{E}} \otimes \pi_1^* \omega_X \cong \mathcal{Q}_{\mathcal{E}} \otimes \pi_2^* \omega_X$, where π_i is the projection of $X \times X$ onto its i th factor. Hence the left and right adjoint of $P_{\mathcal{E}}$ coincide, which suffices to conclude (cf. [1, 5]). \square

Let us compare \mathbb{P}^n -twists and spherical twists. We study the situation of Proposition 1.4. So, let $\mathcal{X} \rightarrow C$ be a smooth family over a smooth curve C with distinguished fibre $j : X := \mathcal{X}_0 \hookrightarrow \mathcal{X}$, $0 \in C$.

Proposition 2.7. *Suppose $\mathcal{E} \in \mathbf{D}^b(X)$ is a \mathbb{P}^n -object with $A(\mathcal{E}) \cdot \kappa(\mathcal{X}) \neq 0$. Then j_* intertwines the \mathbb{P}^n -twist $P_{\mathcal{E}}$ and the spherical twist $T_{j_*\mathcal{E}}$, i.e. one has the following commutative diagram*

$$\begin{array}{ccc} \mathbf{D}^b(X) & \xrightarrow{j_*} & \mathbf{D}^b(\mathcal{X}) \\ P_{\mathcal{E}} \downarrow & & \downarrow T_{j_*\mathcal{E}} \\ \mathbf{D}^b(X) & \xrightarrow{j_*} & \mathbf{D}^b(\mathcal{X}). \end{array}$$

Proof. This is an application of Chen's lemma (see [4] or [5, Ch. 11]). One simply has to show that there exists an object \mathcal{L} on $\mathcal{X} \times_C \mathcal{X}$ with

$$l_*\mathcal{L} \cong \mathcal{P}_{j_*\mathcal{E}} \text{ and } f^*\mathcal{L} \cong \mathcal{Q}_{\mathcal{E}}.$$

Here $\mathcal{P}_{j_*\mathcal{E}} = \mathbb{C}(j_*(\mathcal{E})^\vee \boxtimes j_*\mathcal{E} \rightarrow \mathcal{O}_\Delta)$ is the kernel of the spherical twist $T_{j_*\mathcal{E}}$, and notations for the relevant morphisms are fixed as follows

$$\begin{array}{ccccc} X & \xrightarrow{j} & \mathcal{X} & & \\ \downarrow \iota_0 & & \downarrow k & \searrow \iota & \\ X \times X & \xrightarrow{f} & \mathcal{X} \times_C \mathcal{X} & \xrightarrow{\ell} & \mathcal{X} \times \mathcal{X}, \end{array}$$

where ι_0 , ι , and k are the diagonal embeddings.

We shall define \mathcal{L} as the cone $\mathbb{C}(\xi)$ of a morphism

$$\xi : f_*(\mathcal{E}^\vee[-1] \boxtimes \mathcal{E}) \longrightarrow k_*\mathcal{O}_{\mathcal{X}}.$$

The morphism ξ itself is constructed as a composition as follows. Consider first the trace map $\text{tr} : \mathcal{E}^\vee[-1] \boxtimes \mathcal{E} \rightarrow \iota_{0*}\mathcal{O}_X[-1]$ and take its image under f_* . Then use the short exact sequence $0 \rightarrow \mathcal{O}_{\mathcal{X}} \rightarrow \mathcal{O}_{\mathcal{X}}(X) \rightarrow j_*\mathcal{O}_X \rightarrow 0$ and the induced boundary map $j_*\mathcal{O}_X[-1] \rightarrow \mathcal{O}_{\mathcal{X}}$. Using $k \circ j = f \circ \iota_0$, the image of the latter under k_* can be composed with $f_*(\text{tr})$. This yields

$$(1) \quad \xi : f_*(\mathcal{E}^\vee[-1] \boxtimes \mathcal{E}) \longrightarrow f_*\iota_{0*}\mathcal{O}_X[-1] \cong k_*j_*\mathcal{O}_X[-1] \longrightarrow k_*\mathcal{O}_{\mathcal{X}}.$$

Its cone $\mathcal{L} := \mathbb{C}(\xi)$ can be alternatively described as

$$\begin{aligned} \ell_*\mathcal{L} &\cong \ell_*\mathbb{C}(f_*(\mathcal{E}^\vee[-1] \boxtimes \mathcal{E}) \xrightarrow{\xi} k_*\mathcal{O}_{\mathcal{X}}) \\ &\cong \mathbb{C}(\ell_*f_*(\mathcal{E}^\vee[-1] \boxtimes \mathcal{E}) \xrightarrow{\ell_*\xi} \ell_*k_*\mathcal{O}_{\mathcal{X}}) \\ &\cong \mathbb{C}((j_*\mathcal{E})^\vee \boxtimes j_*\mathcal{E} \longrightarrow \iota_*\mathcal{O}_{\mathcal{X}}), \end{aligned}$$

where we use duality for the isomorphism $j_*(\mathcal{E}^\vee[-1]) \cong (j_*\mathcal{E})^\vee$. Also observe that the functorial properties of duality imply that indeed $\ell_*\xi = \text{tr}$ and hence $\ell_*\mathcal{L} \cong \mathcal{P}_{j_*\mathcal{E}}$.

To show $f^*\mathcal{L} \cong \mathcal{Q}_{\mathcal{E}}$ one first observes that $f^*k_*\mathcal{O}_{\mathcal{X}} \cong \iota_{0*}\mathcal{O}_X$, because the intersection of $f(X \times X)$ and $k(\mathcal{X})$ (inside $\mathcal{X} \times_C \mathcal{X}$) is transversal. Next we use the existence of the distinguished triangle, for an object on the divisor $f : X \times X \hookrightarrow \mathcal{X} \times_C \mathcal{X}$,

$$(\mathcal{E}^\vee \boxtimes \mathcal{E})[-2] \xrightarrow{\delta} \mathcal{E}^\vee \boxtimes \mathcal{E} \longrightarrow f^*f_*(\mathcal{E}^\vee[-1] \boxtimes \mathcal{E}) \longrightarrow (\mathcal{E}^\vee \boxtimes \mathcal{E})[-1],$$

with the boundary map δ given by the cup-product with the obstruction class $A(\mathcal{E}^\vee \boxtimes \mathcal{E}) \cdot \kappa(\mathcal{X} \times_C \mathcal{X}) \in \text{Ext}_{X \times X}^2(\mathcal{E}^\vee \boxtimes \mathcal{E}, \mathcal{E}^\vee \boxtimes \mathcal{E})$ (see the appendix).

As we may assume that $A(\mathcal{E}) \cdot \kappa(\mathcal{X}) = \bar{h}$ and passing from a bundle to its dual changes the Atiyah class by a sign, one finds $A(\mathcal{E}^\vee \boxtimes \mathcal{E}) \cdot \kappa(\mathcal{X} \times_C \mathcal{X}) = \text{id} \boxtimes \bar{h} - \bar{h}^\vee \boxtimes \text{id}$. Hence $f^*f_*(\mathcal{E}^\vee[-1] \boxtimes \mathcal{E}) \cong \mathcal{H}$.

The last thing one has to check before concluding $f^*\mathcal{L} \cong \mathcal{Q}_\mathcal{E}$ is the commutativity of

$$\begin{array}{ccc} \mathcal{E}^\vee \boxtimes \mathcal{E} & \longrightarrow & f^*f_*(\mathcal{E}^\vee[-1] \boxtimes \mathcal{E}) \\ & \searrow \text{tr} & \downarrow f^*\xi \\ & & \iota_{0*}\mathcal{O}_X, \end{array}$$

i.e. that the following diagram, whose vertical part is (1), commutes

$$(2) \quad \begin{array}{ccc} \mathcal{E}^\vee \boxtimes \mathcal{E} & \longrightarrow & f^*f_*(\mathcal{E}^\vee[-1] \boxtimes \mathcal{E}) \\ \text{tr} \downarrow & & \downarrow f^*f_*\text{tr} \\ \iota_{0*}\mathcal{O}_X & \longrightarrow & f^*f_*\iota_{0*}\mathcal{O}_X[-1] \\ & \searrow \cong & \downarrow \\ & & f^*k_*\mathcal{O}_\mathcal{X}. \end{array} \quad \begin{array}{c} \curvearrowright \\ f^*\xi \end{array}$$

Commutativity of the rectangle follows from the functoriality of the distinguished triangle constructed in Proposition 3.1 in the appendix, i.e. from the commutativity of (9) applied to $\mathcal{E}_1 = \mathcal{E}^\vee \boxtimes \mathcal{E}$, $\mathcal{E}_2 = \iota_{0*}\mathcal{O}_X \in \mathbf{D}^b(X \times X)$ and the divisor $f : X \times X \rightarrow \mathcal{X} \times_C \mathcal{X}$.

To show commutativity of the triangle, we apply f^* to $0 \rightarrow k_*\mathcal{O}_\mathcal{X} \rightarrow k_*\mathcal{O}_\mathcal{X} \rightarrow f_*\iota_{0*}\mathcal{O}_X \rightarrow 0$. This gives an exact triangle

$$\iota_{0*}\mathcal{O}_X \rightarrow f^*f_*\iota_{0*}\mathcal{O}_X \xrightarrow{e} \iota_{0*}\mathcal{O}_X[1] \rightarrow \iota_{0*}\mathcal{O}_X[1],$$

whose boundary morphism is zero (since $L_0f^*f_*\iota_{0*}\mathcal{O}_X \cong \iota_{0*}\mathcal{O}_X$, for instance). Thus $f^*f_*\iota_{0*}\mathcal{O}_X \cong \iota_{0*}\mathcal{O}_X \oplus \iota_{0*}\mathcal{O}_X[1]$, to which the only morphisms from $\iota_{0*}\mathcal{O}_X[1]$ are multiples of $(0, \text{id})$, since $\iota_{0*}\mathcal{O}_X$ is a simple *sheaf*. Therefore in our exact triangle of Proposition 3.1

$$\iota_{0*}\mathcal{O}_X[1] \rightarrow f^*f_*\iota_{0*}\mathcal{O}_X \rightarrow \iota_{0*}\mathcal{O}_X \rightarrow \iota_{0*}\mathcal{O}_X[2].$$

the first arrow must be a nonzero multiple of $(0, \text{id})$, and so the above morphism $e : f^*f_*\iota_{0*}\mathcal{O}_X \rightarrow \iota_{0*}\mathcal{O}_X[1]$ splits the exact triangle.

If we show that this morphism e is the vertical arrow of the triangle (2) then we have shown that the composition of the horizontal and vertical morphisms in the triangle is the natural identification between $\iota_{0*}\mathcal{O}_X$ and $f^*k_*\mathcal{O}_\mathcal{X}$, i.e. the triangle is commutative.

So finally we must check in the definition of ξ that

$$f^*k_*(j_*\mathcal{O}_X[-1] \rightarrow \mathcal{O}_\mathcal{X}) = f^*(k_*\mathcal{O}_\mathcal{X} \otimes [f_*\mathcal{O}_{X \times X}[-1] \rightarrow \mathcal{O}_{\mathcal{X} \times_C \mathcal{X}}]).$$

This follows from the fact that the natural map between the short exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{O}_{\mathcal{X} \times_C \mathcal{X}} & \longrightarrow & \mathcal{O}_{\mathcal{X} \times_C \mathcal{X}} & \longrightarrow & f_*\mathcal{O}_{X \times X} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & k_*\mathcal{O}_\mathcal{X} & \longrightarrow & k_*\mathcal{O}_\mathcal{X} & \longrightarrow & k_*j_*\mathcal{O}_X \longrightarrow 0 \end{array}$$

factorizes over the tensor product of the first one with $k_*\mathcal{O}_{\mathcal{X}}$, which stays exact due to the transversality of the intersection of $f(X \times X)$ and $k(\mathcal{X})$. \square

Remark 2.8. The restriction of the spherical kernel of $j_*\mathcal{E}$ to a fibre \mathcal{X}_t with $t \neq 0$ is isomorphic to the diagonal and the induced equivalence $\mathbf{D}^b(\mathcal{X}_t) \cong \mathbf{D}^b(\mathcal{X}_t)$ is the identity. In this sense, a projective twist associated to a \mathbb{P}^n -object \mathcal{E} with $A(\mathcal{E}) \cdot \kappa(\mathcal{X}) \neq 0$ can be deformed to the identity.

This is the derived version of the fact that any birational correspondence between holomorphic symplectic varieties can be deformed to an isomorphism (see [6]). It is also mirror to what Seidel calls ‘fragility’ of his Dehn twists about lagrangian \mathbb{P}^n submanifolds, at least in the case $n = 1$ (see [11]).

Let us conclude with a discussion of the two-dimensional situation, where spherical and \mathbb{P}^1 -twist are related more directly.

Proposition 2.9. *Let $\mathcal{E} \in \mathbf{D}^b(X)$ be a \mathbb{P}^1 -object (thus $\dim(X) = 2$). Then*

$$T_{\mathcal{E}}^2 \cong P_{\mathcal{E}}.$$

Proof. In order to compare the Fourier-Mukai kernel

$$(3) \quad \mathbf{C} \left(\mathbf{C}(\mathcal{E}^\vee \boxtimes \mathcal{E}[-2] \xrightarrow{H} \mathcal{E}^\vee \boxtimes \mathcal{E}) \rightarrow \mathcal{O}_\Delta \right)$$

of the \mathbb{P}^1 -twist $P_{\mathcal{E}}$ with the kernel \mathcal{K} of the square $T_{\mathcal{E}}^2$, we will compute the latter explicitly by the standard product formula. Let $\mathbf{C} := \mathbf{C}(\mathcal{E}^\vee \boxtimes \mathcal{E} \rightarrow \mathcal{O}_\Delta)$. Then $\mathcal{K} \cong \pi_{13*}(\pi_{12}^*\mathbf{C} \otimes \pi_{23}^*\mathbf{C})$, with $\pi_{ij} : X \times X \times X \rightarrow X \times X$ denoting the usual projections.

The tensor product involves terms in degree $-2, -1$, and 0 . In degree -2 this is $\mathcal{L}_2 := \pi_{12}^*(\mathcal{E}^\vee \boxtimes \mathcal{E}) \otimes \pi_{23}^*(\mathcal{E}^\vee \boxtimes \mathcal{E})$, in degree -1 one finds $\mathcal{L}_1 := (\pi_{12}^*(\mathcal{E}^\vee \boxtimes \mathcal{E}) \otimes \pi_{23}^*\mathcal{O}_\Delta) \oplus (\pi_{12}^*\mathcal{O}_\Delta \otimes \pi_{23}^*(\mathcal{E}^\vee \boxtimes \mathcal{E}))$ and in degree 0 simply $\mathcal{L}_0 := \pi_{12}^*\mathcal{O}_\Delta \otimes \pi_{23}^*\mathcal{O}_\Delta$.

Clearly, $\pi_{13*}\mathcal{L}_2 \cong (\mathcal{E}^\vee \boxtimes \mathcal{E}) \otimes \text{End}^*(\mathcal{E})$, $\pi_{13*}\mathcal{L}_1 \cong (\mathcal{E}^\vee \boxtimes \mathcal{E}) \oplus (\mathcal{E}^\vee \boxtimes \mathcal{E})$, and $\pi_{13*}\mathcal{L}_0 \cong \mathcal{O}_\Delta$. Moreover, $\pi_{13*}\mathcal{L}_1 \rightarrow \pi_{13*}\mathcal{L}_0$ is $\text{tr} \oplus \text{tr}$ and $\pi_{13*}\mathcal{L}_2 \cong (\mathcal{E}^\vee \boxtimes \mathcal{E}) \oplus (\mathcal{E}^\vee \boxtimes \mathcal{E})[-2] \rightarrow \pi_{13*}\mathcal{L}_1$ is the diagonal on the first summand and $h^\vee \boxtimes \text{id} \oplus \text{id} \boxtimes h$ on the second.

To conclude, embed the complex $\mathcal{E}^\vee \boxtimes \mathcal{E} \xrightarrow{\text{id}} \mathcal{E}^\vee \boxtimes \mathcal{E}$ into $\pi_{13*}\mathcal{L}_2 \rightarrow \pi_{13*}\mathcal{L}_1$ via $(1, 0) : \mathcal{E}^\vee \boxtimes \mathcal{E} \rightarrow \pi_{13*}\mathcal{L}_2 = (\mathcal{E}^\vee \boxtimes \mathcal{E}) \oplus (\mathcal{E}^\vee \boxtimes \mathcal{E}[-2])$ and $(1, 1) : \mathcal{E}^\vee \boxtimes \mathcal{E} \rightarrow \pi_{13*}\mathcal{L}_1$ respectively. The cokernel of this map is identified with $\mathcal{E}^\vee \boxtimes \mathcal{E}[-2] \xrightarrow{H} \mathcal{E}^\vee \boxtimes \mathcal{E}$ by means of the second projection $\pi_{13*}\mathcal{L}_2 \rightarrow \mathcal{E}^\vee \boxtimes \mathcal{E}[-2]$ and $(1, -1) : \pi_{13*}\mathcal{L}_1 = (\mathcal{E}^\vee \boxtimes \mathcal{E}) \oplus (\mathcal{E}^\vee \boxtimes \mathcal{E}) \rightarrow \mathcal{E}^\vee \boxtimes \mathcal{E}$. Thus \mathcal{K} is isomorphic to the kernel (3). \square

3. Appendix

Let $\mathcal{X} \rightarrow C$ be a smooth projective morphism over a smooth curve C with parameter t . The central fibre will be called $X = \mathcal{X}_0$ and its inclusion $j : X \hookrightarrow \mathcal{X}$.

The family \mathcal{X} viewed as a deformation of X induces the Kodaira-Spencer class $\kappa(\mathcal{X}) \in H^1(X, \mathcal{T}_X)$, which is by definition the extension class of the normal bundle sequence

$$(4) \quad 0 \longrightarrow \mathcal{T}_X \longrightarrow \mathcal{T}_{\mathcal{X}}|_X \longrightarrow \mathcal{O}_X \longrightarrow 0.$$

(Of course multiplication by t induces a trivialization of the normal bundle: $\mathcal{O}_X \cong \mathcal{O}_X(X)$.) The sequence can be dualized to yield

$$(5) \quad 0 \longrightarrow \mathcal{O}_X \longrightarrow \Omega_{\mathcal{X}|X} \longrightarrow \Omega_X \longrightarrow 0,$$

and the Kodaira-Spencer class will be viewed as its boundary morphism $\kappa(\mathcal{X}) : \Omega_X \rightarrow \mathcal{O}_X[1]$.

For $\mathcal{E} \in \mathbf{D}^b(X)$ we denote by $J(\mathcal{E})$ its first jet space, i.e.

$$J(\mathcal{E}) = \pi_{2*}(\pi_1^* \mathcal{E} \otimes \mathcal{O}_{2\Delta}),$$

where $\Delta \subset X \times X$ is the diagonal, 2Δ is its double: $\mathcal{I}_{2\Delta} := \mathcal{I}_\Delta^2$, and π_i is the projection onto the i th factor X .

As $\mathcal{O}_{2\Delta}$ sits in the short exact sequence

$$0 \longrightarrow \Omega_\Delta \longrightarrow \mathcal{O}_{2\Delta} \longrightarrow \mathcal{O}_\Delta \longrightarrow 0,$$

the jet space $J(\mathcal{E})$ sits in a distinguished triangle of the form

$$(6) \quad \mathcal{E} \otimes \Omega_X \longrightarrow J(\mathcal{E}) \longrightarrow \mathcal{E} \longrightarrow \mathcal{E} \otimes \Omega_X[1].$$

The extension class, i.e. the boundary morphism $\mathcal{E} \rightarrow \mathcal{E} \otimes \Omega_X[1]$, is by definition the Atiyah class $A(\mathcal{E}) \in \text{Ext}^1(\mathcal{E}, \mathcal{E} \otimes \Omega_X)$.

The product $A(\mathcal{E}) \cdot \kappa(\mathcal{X}) \in \text{Ext}^2(\mathcal{E}, \mathcal{E}) = \text{Hom}(\mathcal{E}[-1], \mathcal{E}[1])$ can be described as the composition of $A(\mathcal{E})[-1] : \mathcal{E}[-1] \rightarrow \mathcal{E} \otimes \Omega_X$ with $\text{id}_{\mathcal{E}} \otimes \kappa(\mathcal{X}) : \mathcal{E} \otimes \Omega_X \rightarrow \mathcal{E}[1]$. In particular, there exists a distinguished triangle

$$(7) \quad \mathcal{E}[1] \longrightarrow \mathbf{C}(\mathcal{E} \otimes \Omega_{\mathcal{X}|X} \rightarrow J(\mathcal{E})) \longrightarrow \mathcal{E} \xrightarrow{A \cdot \kappa} \mathcal{E}[2].$$

Proposition 3.1. *Let $\mathcal{E}[-1] \rightarrow \mathcal{E}[1]$ be the morphism given by $A(\mathcal{E}) \cdot \kappa(\mathcal{X})$ as above. Then there exists a functorial (in \mathcal{E}) isomorphism $\mathbf{C}(\mathcal{E}[-1] \rightarrow \mathcal{E}[1]) \cong j^* j_* \mathcal{E}$.*

Note that in the general situation of a divisor $j : X \hookrightarrow \mathcal{X}$ there always exists a distinguished triangle of the form (see e.g. [5, Ch. 11])

$$(8) \quad \mathcal{E} \otimes \mathcal{O}_X(-X)[1] \longrightarrow j^* j_* \mathcal{E} \longrightarrow \mathcal{E} \longrightarrow \mathcal{E} \otimes \mathcal{O}_X(-X)[2].$$

and the proposition asserts that in the special case of a family $\mathcal{X} \rightarrow C$ there exists such a triangle with the boundary morphism given by $A(\mathcal{E}) \cdot \kappa(\mathcal{X})$. (Of course the assertion holds true also in the general situation, with $\kappa(\mathcal{X})$ defined appropriately as a class in $\text{Ext}^1(\Omega_X, \mathcal{O}_X(-X))$, but we won't need this.)

In the following we shall denote by \mathcal{I} the ideal sheaf of the diagonal $\iota : \Delta \hookrightarrow X \times X$ and by \mathcal{J} the ideal sheaf of Δ as a subvariety of $X \times \mathcal{X}$ via the closed embedding $i := \text{id} \times j : X \times X \hookrightarrow X \times \mathcal{X}$. Then $\iota_* \Omega_X \cong \mathcal{I}/\mathcal{I}^2$, $\iota_*(\Omega_{\mathcal{X}|X}) \cong i^*(\mathcal{J}/\mathcal{J}^2)$, and the conormal bundle sequence of $X \hookrightarrow \mathcal{X}$ is induced by the natural map $i^*(\mathcal{J}/\mathcal{J}^2) \rightarrow \mathcal{I}/\mathcal{I}^2$. The latter can be composed with $\mathcal{I}/\mathcal{I}^2 \rightarrow \mathcal{O}_{X \times X}/\mathcal{I}^2 = \mathcal{O}_{2\Delta}$ to yield $\eta : i^*(\mathcal{J}/\mathcal{J}^2) \rightarrow \mathcal{O}_{2\Delta}$. The cone of this morphism $\iota_*(\Omega_{\mathcal{X}|X}) \xrightarrow{\eta} \mathcal{O}_{2\Delta}$ is described by the following lemma.

Lemma 3.2. *There is a natural isomorphism $\mathbf{C}(\eta) \cong i^* i_*(\iota_* \mathcal{O}_\Delta)$.*

Proof. Pulling-back the short exact sequence

$$0 \longrightarrow \mathcal{J} \longrightarrow \mathcal{O}_{X \times \mathcal{X}} \longrightarrow i_* \iota_* \mathcal{O}_\Delta \longrightarrow 0$$

via i yields a distinguished triangle

$$i^* \mathcal{J} \longrightarrow \mathcal{O}_{X \times X} \longrightarrow i^* i_* (\iota_* \mathcal{O}_\Delta) \longrightarrow i^* \mathcal{J}[1].$$

(Note that $i^* \mathcal{J}$ need not be derived; it is the normal pullback. This follows, for instance, from the fact that $i^* i_* (\iota_* \mathcal{O}_\Delta)$ has cohomology only in degrees 0 and -1 , by (8).)

Then use the diagram

$$\begin{array}{ccccc} \mathcal{K} & \longrightarrow & \mathcal{I}^2 & & \\ \downarrow & & \downarrow & & \\ i^* \mathcal{J} & \longrightarrow & \mathcal{O}_{X \times X} & \longrightarrow & i^* i_* (\iota_* \mathcal{O}_\Delta) \\ \downarrow & & \downarrow & & \\ i^* (\mathcal{J}/\mathcal{J}^2) & \longrightarrow & \mathcal{O}_{2\Delta} & & \end{array}$$

and the fact, which can be easily verified by a local calculation, that the natural map

$$\mathcal{K} := \ker(i^* \mathcal{J} \rightarrow i^* (\mathcal{J}/\mathcal{J}^2)) \rightarrow \mathcal{I}^2$$

is an isomorphism. Hence,

$$i^* i_* (\iota_* \mathcal{O}_\Delta) \cong \mathbf{C}(i^* \mathcal{J} \rightarrow \mathcal{O}_{X \times X}) \cong \mathbf{C}(i^* (\mathcal{J}/\mathcal{J}^2) \rightarrow \mathcal{O}_{2\Delta}).$$

□

Proof of proposition. Denote by p_i the projection from $X \times \mathcal{X}$ onto the i th factor. Then use $p_2 \circ i = j \circ \pi_2$ to conclude

$$j^* j_* \mathcal{E} \cong j^* j_* \pi_{2*} (\pi_1^* \mathcal{E} \otimes \iota_* \mathcal{O}_\Delta) \cong j^* p_{2*} i_* (\pi_1^* \mathcal{E} \otimes \iota_* \mathcal{O}_\Delta).$$

Use basechange for $p_2 \circ i = j \circ \pi_2$ (see [2, Lemma 1.3]) to identify this with

$$\pi_{2*} i^* i_* (\pi_1^* \mathcal{E} \otimes \iota_* \mathcal{O}_\Delta) \cong \pi_{2*} i^* (p_1^* \mathcal{E} \otimes i_* (\iota_* \mathcal{O}_\Delta)).$$

Now $p_1 \circ i = \pi_1$ and tensor product commutes with pullback, so this is

$$\pi_{2*} ((\pi_1^* \mathcal{E}) \otimes i^* i_* (\iota_* \mathcal{O}_\Delta)).$$

So by the lemma,

$$\begin{aligned} j^* j_* \mathcal{E} &\cong \mathbf{C}(\pi_{2*} (\pi_1^* \mathcal{E} \otimes i^* (\mathcal{J}/\mathcal{J}^2)) \rightarrow \pi_{2*} (\pi_1^* \mathcal{E} \otimes \mathcal{O}_{2\Delta})) \\ &\cong \mathbf{C}(\mathcal{E} \otimes \Omega_{\mathcal{X}|X} \rightarrow J(\mathcal{E})). \end{aligned}$$

Functoriality is clear from the functoriality of pushforwards and pullbacks, which is all that we have used. That is a morphism $\varphi : \mathcal{E}_1 \rightarrow \mathcal{E}_2$ in $\mathbf{D}^b(X)$ induces a

commutative diagram

$$(9) \quad \begin{array}{ccccccc} \mathcal{E}_1[1] & \longrightarrow & j^* j_* \mathcal{E}_1 & \longrightarrow & \mathcal{E}_1 & \xrightarrow{A(\mathcal{E}_1) \cdot \kappa(\mathcal{X})} & \mathcal{E}_1[2] \\ \varphi[1] \downarrow & & j^* j_* \varphi \downarrow & & \downarrow \varphi & & \downarrow \varphi[2] \\ \mathcal{E}_2[1] & \longrightarrow & j^* j_* \mathcal{E}_2 & \longrightarrow & \mathcal{E}_2 & \xrightarrow{A(\mathcal{E}_2) \cdot \kappa(\mathcal{X})} & \mathcal{E}_2[2]. \end{array}$$

□

Acknowledgements

We would like to thank Paul Seidel for a useful conversation, Paul Horja, David Ploog and Balázs Szendrői for comments on the text, and Institut de Mathématiques de Jussieu for support for both authors. The second author is supported by a Royal Society university research fellowship.

References

- [1] T. Bridgeland, *Equivalences of triangulated categories and Fourier-Mukai transforms*, Bull. London Math. Soc. **31** (1999) 25–34.
- [2] A. Bondal and D. Orlov, *Semiorthogonal decomposition for algebraic varieties*, alg-geom/9506012.
- [3] R. Buchweitz and H. Flenner, *The Atiyah-Chern Character yields the semi-regularity map as well as the infinitesimal Abel–Jacobi map*, The arithmetic and geometry of algebraic cycles (Banff, AB, 1998), 33–46, CRM Proc. Lecture Notes, 24, Amer. Math. Soc., Providence, RI, 2000.
- [4] J.-C. Chen, *Flops and equivalences of derived categories for threefolds with only Gorenstein singularities*, J. Differential Geom. **61** (2002) 227–261.
- [5] D. Huybrechts, *Fourier–Mukai transforms in algebraic geometry*, to appear in Oxford Monographs.
- [6] ———, *Birational symplectic manifolds and their deformations*, J. Differential Geom. **45** (1997) 488–513.
- [7] L. Illusie, *Complexe cotangent et déformations I*, (French) Lecture Notes in Mathematics, Vol. 239. Springer-Verlag, Berlin–New York, 1971.
- [8] D. Matsushita, *Higher direct images of dualizing sheaves of Lagrangian fibrations*, Amer. J. Math. **127** (2005) 243–259.
- [9] D. Ploog, *Autoequivalences of derived categories of smooth projective varieties*, PhD thesis, Berlin (2005).
- [10] P. Seidel, *Graded lagrangian submanifolds*, Bull. Soc. Math. France **128** (2000) 103–146.
- [11] ———, *Lectures on four-dimensional Dehn twists*, math.SG/0309012.
- [12] P. Seidel and R. Thomas, *Braid group actions on derived categories of coherent sheaves*, Duke Math. J. **108** (2001) 37–108.

MATHEMATISCHES INSTITUT, UNIVERSITÄT BONN, BERINGSTR 1, D-53115 BONN, GERMANY

E-mail address: huybrech@math.uni-bonn.de

DEPARTMENT OF MATHEMATICS, SOUTH KENSINGTON CAMPUS, IMPERIAL COLLEGE LONDON, LONDON, SW7 2AZ, UNITED KINGDOM

E-mail address: rpwt@ic.ac.uk