Math. Res. Lett. **13** (2006), *no.* 1, 59–69

ON EXTREME X-HARMONIC FUNCTIONS

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Abstract. All positive harmonic functions in an arbitrary domain E of a Euclidean space can be decomposed in a unique way into extreme functions. The latter can be obtained by a passage to the limit from $k^y(x) = \frac{g(x,y)}{g(a,y)}$ where $g(x,y)$ is the Green function of the Laplacian and a is a fixed point of E. Our goal is to get similar results for a class of positive functions on a space of measures. These functions are associated with a superdiffusion X and we call them X-harmonic. Denote $\mathcal{M}_c(E)$ the set of all finite measures μ supported by compact subsets of E. X-harmonic functions are functions on $\mathcal{M}_c(E)$ characterized by a mean value property formulated in terms of exit measures of a superdiffusion. Extreme X-harmonic functions play the same role as their classical counterpart. We describe a limit process for getting these functions. Instead of the ratio $\frac{g(x,y)}{g(a,y)}$ we use a Radon-Nikodym derivative of the probability distribution of an exit measure of X with respect to the probability distribution of another such measure.

1. Introduction

1.1. ***X*-harmonic functions.** Suppose that L is a second order uniformly elliptic operator in a domain E of $\mathbb{R}^d$. An L-diffusion is a continuous strong Markov process $\xi = (\xi_t, \Pi_x)$ in E with the generator L. A function h in a domain E is called *ξ-harmonic* (or *L-harmonic*) if, for every domain $D \Subset E$, [1]

$$\Pi_x h(\xi_{\tau_D}) = h(x) \quad \text{for all } x \in D.$$

Here τ_D is the first exit time of ξ from D. This condition is satisfied if and only if $Lh = 0$ in E.

Let ψ be a function from $E \times \mathbb{R}_+$ to $\mathbb{R}_+$ where $\mathbb{R}_+ = [0, \infty)$. An (L, ψ)-superdiffusion is a model of an evolution of a random cloud. It is described by a family of random measures (X_D, P_μ) where $D \subset E$ and μ is a finite measure on E. [2] If μ is concentrated on D, then X_D is concentrated on ∂D. We call X_D the *exit measure from* D. Heuristically, it describes the mass distribution on an absorbing barrier placed on ∂D.

We put $\mu \in \mathcal{M}_c(D)$ if μ is a finite measure concentrated on a compact subset of D. We say that a function $H : \mathcal{M}_c(E) \to \mathbb{R}_+$ is *X-harmonic* and we write $H \in \mathbb{H}(X)$ if, for every $D \Subset E$ and every $\mu \in \mathcal{M}_c(D)$,

$$P_\mu H(X_D) = H(\mu). \tag{1.1}$$

Received by the editors January 9, 2005.
The author is partially supported by National Science Foundation Grant DMS-0204237.

[1] We write $D \Subset E$ if D is a bounded domain such that the closure $\bar{D}$ of D is contained in E.

[2] Assumptions about these random measures are formulated in Section 1.2.

For every domain $D \subset E$ we have an inclusion $\mathcal{M}_c(D) \subset \mathcal{M}_c(E)$. We say that H is *X-harmonic in D* if

$$P_\mu H(X_O) = H(\mu) \quad \text{for all } O \Subset D, \mu \in \mathcal{M}_c(O). \tag{1.2}$$

An element H of $\mathbb{H}(X)$ is called *extreme* if the conditions $\tilde{H} \le H, \tilde{H} \in \mathbb{H}(X)$ imply that $\tilde{H} = \text{const.}\, H$.

Fix $a \in E$ and denote by $\mathbb{H}(X, a)$ the class of all positive X-harmonic functions H such that $H(\delta_a) = 1$ [δ_a is the unit mass concentrated at a]. Let $\mathbb{H}_e(X, a)$ stand for the set of all extreme elements that belong to $\mathbb{H}(X, a)$. According to Theorem 3.1 in [Dyn04a], the formula

$$H(\mu) = \int \hat{H}(\mu)\nu(d\hat{H}) \tag{1.3}$$

establishes a 1-1 correspondence between $H \in \mathbb{H}(X, a)$ and probability measures ν on $\mathbb{H}_e(X, a)$. [3]

1.2. Superdiffusions. We write $f \in \mathcal{B}$ if f is a positive $\mathcal{B}$-measurable function. We denote by $\mathcal{B}(E)$ the class of all Borel subsets of E and by $\mathcal{M}(E)$ the set of all finite measures on $\mathcal{B}(E)$.

Suppose that to every open set $D \subset E$ and every $\mu \in \mathcal{M}(E)$ there corresponds a random measure (X_D, P_μ) on $\mathbb{R}^d$ [4] such that, for every $f \in \mathcal{B}(E)$,

$$P_\mu e^{-\langle f, X_D \rangle} = e^{-\langle V_D(f), \mu \rangle} \tag{1.4}$$

where $u = V_D(f)$ satisfies the equation [5]

$$u + G_D \psi(u) = K_D f. \tag{1.5}$$

Here

$$\begin{aligned} G_D f(x) &= \Pi_x \int_0^{\tau_D} f(\xi_s)\, ds, \\ K_D f(x) &= \Pi_x 1_{\tau_D < \infty} f(\xi_{\tau_D}) \end{aligned} \tag{1.6}$$

are the *Green operator* and the *Poisson operator* of ξ in D. We call the family $X = (X_D, P_\mu)$ an *(L, ψ)-superdiffusion* if, besides (1.4)-(1.5) it satisfies the following condition.

1.2.A. [Markov property] For every $\mu \in \mathcal{M}_c(E)$ and every $D \Subset E$,

$$P_\mu Y Z = P_\mu(Y P_{X_D} Z)$$

if $Y \ge 0$ is measurable with respect to the σ-algebra $\mathcal{F}_{\subset D}$ generated by $X_O, O \subset D$ and $Z \ge 0$ is measurable with respect to the σ-algebra $\mathcal{F}_{\supset D}$ generated by $X_{O'}, O' \supset D$.

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[3] In Section 4.1 we deduce the representation (1.3) from a result in [Dyn78].

[4] A random measure on a measurable space $(S, \mathcal{B}_S)$ is a pair (X, P) where $X(\omega, B)$ is a kernel from an auxiliary measurable space $(\Omega, \mathcal{F})$ to $(S, \mathcal{B}_S)$ and P is a probability measure on $\mathcal{F}$. [We say that $p(x, B), x \in E, B \in \mathcal{B}'$ is a kernel from a measurable space $(E, \mathcal{B})$ to a measurable space $(E', \mathcal{B}')$ if it is a $\mathcal{B}$-measurable function in x and a finite measure in B.]

[5] $\psi(u)$ is a short writing for $\psi(x, u(x))$.

The existence of a (L, ψ)-superdiffusion is proved for a convex class of functions ψ which contains functions $\psi(x,u) = b(x)u^\alpha$ with bounded positive Borel b and $1 < \alpha \le 2$. [See, e. g., Chapter 4 in [Dyn02].] It follows from (1.6)–(1.4) that

$$P_\mu\{X_D(D) = 0\} = 1 \tag{1.7}$$

and

$$P_\mu\{X_D = \mu\} = 1 \quad \text{if } \mu(D) = 0. \tag{1.8}$$

Let $\mathcal{F}$ stand for the σ-algebra in Ω generated by $X_D(B)$ where $D \Subset E$ and $B \in \mathcal{B}(E)$. Denote by $\mathfrak{M}$ the σ-algebra in $\mathcal{M}_c(E)$ generated by the functions $F(\mu) = \mu(B)$ with $B \in \mathcal{B}(E)$. If $\mu \in \mathcal{M}_c(E)$ and $D \Subset E$, then, P_μ-a.s., $X_D \in \mathcal{M}_c(E)$ and X_D is a measurable mapping from $(\Omega, \mathcal{F})$ to $(\mathcal{M}_c(E), \mathfrak{M})$. Moreover, if $\mu \in \mathcal{M}_c(D)$, then, P_μ-a.s., $X_D \in \mathcal{M}(\partial D)$. It follows from (1.4) that $H(\mu) = P_\mu Y$ is $\mathfrak{M}$-measurable for every $\mathcal{F}$-measurable $Y \ge 0$.

We have:

1.2.B. [Absolute continuity property] For every set $C \in \mathcal{F}_{\supset D}$ either $P_\mu(C) = 0$ for all $\mu \in \mathcal{M}_c(D)$ or $P_\mu(C) > 0$ for all $\mu \in \mathcal{M}_c(D)$.

A proof of this property can be found in [Dyn04b],Theorem 5.3.2.

1.3. ***H*-transform.** Let $X = (X_D, P_\mu)$ be a superdiffusion in E and let E be the union of U_k such that $U_1 \Subset U_2 \Subset \dots U_k \Subset \dots$. Put $\mathcal{M} = \mathcal{M}_c(E)$ and denote by $\mathcal{O}_k$ the class of all open sets $D \Subset U_k$.

The space $(\mathcal{M}, \mathfrak{M})$ is a measurable Luzin space. [6] Therefore Kolmogorov's extension theorem is applicable to $\mathcal{M}^{\mathcal{O}_k}$. Fix $a \in E$ and $H \in \mathbb{H}(X, a)$. Put $P_a = P_{\delta_a}$ and consider a family

$$M_{n,k}(D_1, C_1; \dots; D_n, C_n) = P_a\{X_{D_1} \in C_1, \dots, X_{D_n} \in C_n; H(X_{U_k})\}$$

where $n = 1, 2, \dots,$ $D_1, \dots, D_n \in \mathcal{O}_k$ and $C_1, \dots, C_n \in \mathcal{M}$. [7] Note that for $n > 1$

$$M_{n,k}(D_1, C_1; \dots; D_{n-1}, C_{n-1}; D_n, \mathcal{M}) = M_{n-1,k}(D_1, C_1; \dots; D_{n-1}, C_{n-1})$$

Since $H \in \mathbb{H}(X, a)$, $M_{1,k}(D, \mathcal{M}) = 1$ if $a \in U_k$. By (1.8), this is true also if $a \notin U_k$. By Kolmogorov's theorem, there exists a probability measure $P^H_{a,k}$ on $\mathcal{M}^{\mathcal{O}_k}$ such that, for all $D_1, \dots, D_n \in \mathcal{O}_k$ and $C_1, \dots, C_n \in \mathcal{M}$

$$\begin{aligned} P^H_{a,k}\{X_{D_1} \in C_1, \dots, X_{D_n} \in C_n\} &= M_{n,k}(D_1, C_1; \dots; D_n, C_n) \\ &= P_a\{X_{D_1} \in C_1, \dots, X_{D_n} \in C_n; H(X_{U_k})\}. \end{aligned} \tag{1.9}$$

This implies

$$P^H_{a,k} Y = P_a[Y H(X_{U_k})] \tag{1.10}$$

[6] That is there exists a 1-1 mapping from $\mathcal{M}$ onto a Borel subset $\tilde{\mathcal{M}}$ of a compact metric space such that elements of $\mathfrak{M}$ correspond to Borel subsets of $\tilde{\mathcal{M}}$.

[7] Writing $P\{A; f\}$ means $\int_A f dP$.

for all k and all $Y \in \mathcal{F}_{\subset U_k}$. Indeed, by (1.9), formula (1.10) holds for $Y = 1_{C_1}(X_{D_1}) \dots 1_{C_n}(X_{D_n})$ where $D_1, \dots, D_n \in \mathcal{O}_k$ and these functions generate $\mathcal{F}_{\subset U_k}$. By 1.2.A, $P_a[YH(X_{U_k})] = P_a[YH(X_{U_\ell})]$ for $k < \ell$ and $Y \in \mathcal{F}_{\subset U_k}$. Since $\mathcal{O}_k \uparrow \mathcal{O}(E)$, there exists a measure P_a^H on $\mathcal{M}^{\mathcal{O}(E)}$ which coincides with $P_{a,k}^H$ on $\mathcal{M}^{\mathcal{O}_k}$. Clearly,

$$P_a^H Y = P_a[YH(X_U)] \quad \text{for all } U \Subset E, Y \in \mathcal{F}_{\subset U}.$$

The measure P_a^H is called the *H-transform of P_a*. [8]

1.4. Main results. We denote by $\mathcal{P}_D(\mu, \cdot)$ the probability distribution of X_D under P_μ, that is

$$\mathcal{P}_D(\mu, A) = P_\mu\{X_D \in A\} \quad \text{for } A \in \mathfrak{M}.$$

Fix a reference point $a \in E$ and put $\mathcal{P}_D(A) = \mathcal{P}_D(\delta_a, A)$. By 1.2.B, there exists a Radon-Nikodym derivative

$$H_D^\nu(\mu) = \frac{\mathcal{P}_D(\mu, d\nu)}{\mathcal{P}_D(d\nu)}. \tag{1.11}$$

For every $\mu \in \mathcal{M}_c(D)$, this is a function of $\nu \in \mathcal{M}(\partial D)$ defined up to $\mathcal{P}_D$-equivalence. We continue it to $\mathcal{M}(E) \times \mathcal{M}(E)$ by setting $H_D^\nu(\mu) = 0$ off $\mathcal{M}_c(D) \times \mathcal{M}(\partial D)$.

Theorem 1.1. *There exists a version of $H_D^\nu(\mu)$ which is $\mathfrak{M} \times \mathfrak{M}$-measurable and X-harmonic in μ in the domain D for every $\nu \in \mathcal{M}(\partial D)$.*

In Theorems 1.2 and 1.3, $H_D^\nu(\mu)$ is the version of the Radon-Nikodym derivative (1.11) described in Theorem 1.1.

We say that a sequence D_k *exhausts* E if $D_1 \Subset D_2 \Subset \dots D_k \Subset \dots$ and E is the union of D_k.

Theorem 1.2. *If $H \in \mathbb{H}_e(X, a)$, then, for every $\gamma \in \mathcal{M}_c(E)$ and for every sequence D_k exhausting E,*

$$H(\gamma) = \lim_{k \to \infty} H_{D_k}^{X_{D_k}}(\gamma) \quad P_a^H\text{-a. s.} \tag{1.12}$$

Theorem 1.3. *Let H and D_k be the same as in Theorem 1.2 and let $M_k^\mu(\cdot) = \mathcal{P}_{D_k}(\mu, \cdot)$. There exists a sequence $\nu_n \in \partial D_n$ such that, for every $\mu \in \mathcal{M}_c(E)$ and for every k,*

$$M_k^\mu\{\gamma : H_{D_n}^{\nu_n}(\gamma) \to H(\gamma) \quad \text{as } n \to \infty\} = 1. \tag{1.13}$$

[8] J. L. Doob introduced h-transforms associated with excessive functions h for a Markov process. This is an important tool in the probabilistic analysis.

1.5. Comparison with the Martin boundary theory. In 1941 Martin studied positive harmonic functions in an arbitrary domain $E \subset \mathbb{R}^d$. Denote by $\mathcal{H}(a)$ the set of all such functions equal to 1 at $a \in E$. Martin proved that every $h \in \mathcal{H}(a)$ has a unique integral representation

$$h(x) = \int \hat{h}(x)\mu(d\hat{h})$$

where μ is a probability measure on the set $\mathcal{H}_e(a)$ of all exreme points of $\mathcal{H}(a)$. Formula (1.3) provides a counterpart of this result for X-harmonic functions.

A central role in the Martin theory is played by the function $k^y(x) = \frac{g(x,y)}{g(a,y)}$ (g is the Green function of the Laplacian in E). In terms of the Brownian motion (ξ_t, Π_x) it can be expressed by the formula

$$k^y(x) = \frac{g(x, dy)}{g(a, dy)} \tag{1.14}$$

where

$$g(x, B) = \Pi_x \int_0^{\tau_E} 1_B(\xi_t)dt.$$

Note an obvious similarity between (1.14) and (1.11). If $D \Subset E$, then $k^y(x)$ is harmonic in D for every $y \in \partial D$. Theorem 1.1 establishes a similar property of $H_D^\nu(\mu)$.

Theorem 1.2 is a counterpart of the following proposition: [9]

1.5.A. If D_n exhaust E and if $h \in \mathcal{H}_e(a)$, then

$$h(x) = \lim k^{\xi_{\tau_n}}(x) \quad \Pi_a^h\text{-a.s.}.$$

Here Π_a^h is the h-transform of Π_a and τ_n is the first exit time from D_n.

Proposition 1.5.A in combination with the Harnack's inequality implies:

1.5.B. If D_n exhaust E and if $h \in \mathcal{H}_e(a)$, then there exist $y_n \in \partial D_n$ such that

$$h(x) = \lim k^{y_n}(x) \quad \text{for all } x \in E.$$

We would like to prove that, if $H \in \mathbb{H}_e(X, a)$, then there exist $\nu_n \in \mathcal{M}(\partial D_n)$ such that

$$H(\mu) = \lim H_{D_n}^{\nu_n}(\mu) \quad \text{for all } \mu \in \mathcal{M}_c(E). \tag{1.15}$$

Theorem 1.3 is a weaker statement. It implies only that (1.15) holds if, for some k, the functions $H_n = H_{D_n}^{\nu_n}, n > k$ were uniformly M_k^μ-integrable. Indeed, since H_n is X-harmonic in D_n, we have

$$\int H_n(\gamma)\mathcal{M}_k^\mu(d\gamma) = P_\mu H_n(X_{D_k}) = H_n(\mu) \quad \text{for every } n > k. \tag{1.16}$$

[9] We refer for the proof to [Dyn02], Chapter 7.

By (1.13) and (1.16),

$$H(\mu) = P_\mu H(X_{D_k}) = \int H(\gamma) M_k^\mu(d\gamma) = \int \lim H_n(\gamma) M_k^\mu(d\gamma)$$
$$= \lim \int H_n(\gamma) M_k^\mu(d\gamma) = \lim H_n(\mu).$$

However we do not know, if the condition of the uniform integrability of H_n is satisfied.

The set $\mathcal{H}_e(a)$ can be interpreted as the exit space for ξ. Proposition 1.5.B is applicable to a wide class of Markov processes ξ and it allows to describe the exit spaces for a number of interesting processes. [See, for instance, [Dyn64] and [Dyn66].] To apply, in a similar way, Theorem 1.3, it is necessary to learn more about the functions $H_D^\nu(\mu)$.

2. Proof of Theorem 1.1

The σ-algebra $\mathfrak{M}$ is countably generated. [10] The existence of $\mathfrak{M} \times \mathfrak{M}$-measurable version of $H_D^\nu(\mu)$ follows from Theorem A.1 in the Appendix. Let us prove that this version can be chosen to be X-harmonic in μ in the domain D.

First we prove that, if $H_D^\nu(\mu)$ is $\mathfrak{M} \times \mathfrak{M}$-measurable, then

$$P_\mu H_D^\nu(X_D) = H_D^\nu(\mu) \quad \text{for } \mathcal{P}_D\text{-almost all } \nu. \tag{2.1}$$

Indeed, for every $A \in \mathfrak{M}$,

$$\mathcal{P}_D(\mu, A) = \int_A \mathcal{P}_D(d\nu) H_D^\nu(\mu), \quad \mathcal{P}_D(X_O, A) = \int_A \mathcal{P}_D(d\nu) H_D^\nu(X_O).$$

Therefore, by 1.2.A,

$$P_\mu \int_A \mathcal{P}_D(d\nu) H_D^\nu(X_O) = P_\mu \mathcal{P}_D(X_O, A) \tag{2.2}$$
$$= P_\mu P_{X_O}\{X_D \in A\} = P_\mu\{X_D \in A\} = \mathcal{P}_D(\mu, A) = \int_A \mathcal{P}_D(d\nu) H_D^\nu(\mu).$$

The function $H_D^\nu(X_O(\omega))$ is $\mathfrak{M} \times \mathcal{F}$-measurable. By Fubini's theorem, it follows from (2.2) that

$$\int_A \mathcal{P}_D(d\nu) P_\mu H_D^\nu(X_O) = \int_A \mathcal{P}_D(d\nu) H_D^\nu(\mu)$$

for all $A \in \mathfrak{M}$ which implies (2.1).

To prove Theorem 1.1, we consider any $\mathfrak{M} \times \mathfrak{M}$ measurable version $\tilde{H}_D^\nu$ of the Radon-Nikodym derivative (1.6) and we put

$$H_D^\nu(\mu) = P_\mu \tilde{H}_D^\nu(X_D).$$

[10] Every uncountable Luzin measurable space is isomorphic to the unit interval $[0, 1]$ with the Borel σ-algebra. This is proved, for instance, in [DY79], Appendix 1.

This function is $\mathfrak{M} \times \mathfrak{M}$-measurable. By (2.1) it coincides, for $\mathcal{P}_D$-almost all ν with $\tilde{H}^\nu_D(\mu)$. By the Markov property 1.2.A,

$$P_\mu H^\nu_D(X_O) = P_\mu P_{X_O} \tilde{H}^\nu_D(X_D) = P_\mu \tilde{H}^\nu_D(X_D) = H^\nu_D(\mu)$$

for every $O \Subset D, \mu \in \mathcal{M}_c(O)$. □

To prove Theorem 1.2 we need some preparations.

3. Exit laws for Markov chains

3.1. Markov chains. Suppose $(E_n, \mathcal{B}_n)$, $n = 0, 1, 2, \dots$ is a sequence of measurable spaces. A Markov transition function is a family of kernels $p(r, x; n, B)$, $0 \le r < n$ from $(E_r, \mathcal{B}_r)$ to $(E_n, \mathcal{B}_n)$ such that

$$p(r, x; n, E_n) = 1 \quad \text{for all } r < n, x \in E_r$$

and

$$\int_{E_k} p(r, x; k, dy) p(k, y; n, B) = p(r, x : n, B)$$

for all $r < k < n$ and all $x \in E_r, B \in \mathcal{B}_n$.

A sequence $\omega = \{x_o, x_1, \dots x_n, \dots\}$ where $x_n \in E_n$ is called a path. Consider the space Ω of all paths and denote by $\mathcal{F}_{\le r}$ $[\mathcal{F}_{\ge r}]$ the σ-algebra in Ω generated by $\{X_n(\omega) \in B_n\}$ with $B_n \in \mathcal{B}_n$ and $n \le r$ $[n \ge r]$. By Kolmogorov's theorem, to every $x \in E_0$ there corresponds a probability measure $\mathbb{P}_x$ on $\mathcal{F}_{\ge 0}$ such that

$$\mathbb{P}_x\{X_0 = x\} = 1$$

and

$$\mathbb{P}_x\{X_0 = r, X_1 \in B_1, \dots, X_n \in B_n\} = \int_{B_1} \cdots \int_{B_n} p(0, x; 1, dy_1) \dots p(n-1, y_{n-1}; n, dy_n)$$

for all $n > 0$ and all $B_1 \in \mathcal{B}_1, \dots, B_n \in \mathcal{B}_n$. The family $(X_n, \mathbb{P}_x)$ is a Markov chain with the transition function p.

3.2. Exit laws. A sequence of positive measurable functions $F^n(x), x \in E_n$ is called a p-*exit law* if

$$\int_{E_n} p(m, x; n, dy) F^n(y) = F^m(x) \quad \text{for all } m < n, x \in E_r.$$

We denote $\mathcal{E}(p)$ the set of all p-exit laws and we put $F \in \mathcal{E}(p, a)$ if $F \in \mathcal{E}(p)$ and $F^0(a) = 1$.

Suppose that p satisfies the condition:

3.2.A. If $p(0, a; n, B) = 0$, then $p(m, x; n, B) = 0$ for all x and all $m < n$.[11]

Let $\mathcal{E}_e(p,a)$ stand for the set of all extreme elements of $\mathcal{E}(p,a)$. The formula

$$F^n(x) = \int \hat{F}^n(x)\mu(d\hat{F}) \tag{3.1}$$

establishes a 1-1 correspondence between $F \in \mathcal{E}(p,a)$ and probability measures μ on $\mathcal{F}_e(p,a)$. This was proved in [Dyn78], Section 10.2.

Put

$$\nu_n(A) = p(0,a;n,A).$$

The Radon-Nikodym derivative

$$\rho(m,x;n,y) = \frac{p(m,x;n,dy)}{\nu_n(dy)}$$

can be chosen to satisfy equation

$$\int_{E_k} \rho(r,x;k,y)\nu_k(dy)\rho(k,y;n,z) = \rho(r,x;n,z)$$

for all $r < k < n$ and all $x \in E_r, z \in E_n$.

3.3. ***F*-transform.** Suppose that $F \in \mathcal{E}(p,c)$. By Kolmogorov's theorem, there exists a probability measure $\mathbb{P}_x^F$ on the path space Ω such that

$$\begin{aligned} \mathbb{P}_x^F\{X_0 = x, X_1 \in B_1, \dots, X_n \in B_n\} \\ = P_x\{X_0 = x, X_1 \in B_1, \dots, X_n \in B_n; F^n(X_n)\} \end{aligned} \tag{3.2}$$

for all $n > 0$ and all $B_1 \in \mathcal{B}_1, \dots, B_n \in \mathcal{B}_n$.

The measure $\mathbb{P}_x^F$ is called the *F-transform of* $\mathbb{P}_x$. We have

$$\mathbb{P}_x^F Y = P_x Y F^n(X_n)$$

for every $Y \in \mathcal{F}_{\geq n}$. It is proved in [Dyn78], Section 10 that, if F is an extreme element of $\mathcal{E}(p,a)$ and if $F^r(x) < \infty$, then

$$F^r(x) = \lim_{n\to\infty} \rho(r,x;n,X_n) \quad \mathbb{P}_a^F\text{-a.s.} \tag{3.3}$$

4. Proof of Theorems 1.2 and 1.3

4.1. Markov chains associated with superdiffusions. To construct such chains we fix a sequence $D_0, D_1, \dots$ exhausting E and we put

$$\mathcal{M}_0 = \mathcal{M}_c(D_0), X_0 = \mu \in \mathcal{M}_0,$$
$$\mathcal{M}_n = \mathcal{M}(\partial D_n), X_n = X_{D_n} \quad \text{for } n \geq 1.$$

The Markov property 1.2.A of a superdiffusion implies that (X_n, P_μ) is a Markov chain with the transition function

$$\mathcal{P}(r,\mu;n,A) = P_\mu(X_n \in A), \quad 0 \leq r \leq n, \mu \in \mathcal{M}_r, A \subset \mathcal{M}_n. \tag{4.1}$$

We call it the *chain associated with the superdiffusion* (X_D, P_μ).

[11] This property can be thought of as a probabilistic statement of the strong minimum principle.

If H is X-harmonic and if F^n is the restriction of H to $\mathcal{M}_n$, then F is a $\mathcal{P}$-exit law. If $H \in \mathbb{H}(X,a)$ and if $a \in D_0$, then $F \in \mathcal{E}(\mathcal{P}, \delta_a)$. This way we define a mapping $j : \mathbb{H}(X,a) \to \mathcal{E}(\mathcal{P}, \delta_a)$. On the other hand, if $r < n$ and if $\mu \in \mathcal{M}_c(D_n)$, then, by the Markov property 1.2.A, $P_\mu F^n(X_n) = P_\mu P_{X_r} F^n(X_n) = P_\mu F^r(X_r)$ and therefore $P_\mu F^n(X_n)$ does not depend on $n \geq r$. We define $H = i(F)$ by the formula

$$H(\mu) = P_\mu F^n(X_n) \quad \text{for } \mu \in \mathcal{M}_c(D_n).$$

Every $D \Subset E$ is contained in D_n for sufficiently large n, and, by 1.2.A,

$$P_\mu H(X_D) = P_\mu P_{X_D} F^n(X_n) = P_\mu F^n(X_n) = H(\mu).$$

Hence, $H \in \mathbb{H}(X,a)$ and we have a map $i : \mathcal{E}(\mathcal{P}, \delta_a) \to \mathbb{H}(X,a)$. Clearly, i is the inverse for j and both mappings preserve the convex structure. It follows from the Absolute continuity property 1.2.B that $\mathcal{P}$ satisfies the condition 3.2.A and therefore the integral representation (3.1) of exit laws implies the integral representation (1.3) of X-harmonic functions.

4.2. Proof of Theorem 1.2. If $\mu \in \mathcal{M}_r, A \subset \mathcal{M}_n$, then

$$\mathcal{P}(r,\mu;n,A) = \mathcal{P}_{D_n}(\mu, A), \quad \mathcal{P}(0,\delta_a;n,A) = \mathcal{P}_{D_n}(\delta_a, A)$$

and therefore

$$\frac{\mathcal{P}(r,\mu;n,d\nu)}{\mathcal{P}(0,\delta_a;n,d\nu)} = H^\nu_{D_n}(\mu). \tag{4.2}$$

On the other hand, by comparing (1.9) and (3.2), we get

$$P^H_a = \mathbb{P}^F_a. \tag{4.3}$$

If $\mu \in \mathcal{M}_c(E)$, then $\mu \in \mathcal{M}_c(D_0)$ for some $D_0 \subset E$. Consider a sequence $D_0, D_1, \dots$ exhausting E. By applying (4.2) and (4.3), we get (1.12) from (3.3). □

4.3. Proof of Theorem 1.3. Every function $H^{X_{D_n}(\omega)}_{D_n}(\gamma)$ is $\mathfrak{M}\times\mathcal{F}$-measurable in (γ,ω) and therefore the set

$$\mathcal{W} = \{(\gamma,\omega) : H^{X_{D_n}(\omega)}_{D_n}(\gamma) \to H(\gamma) \quad \text{as } n \to \infty\}$$

belongs to $\mathfrak{M} \times \mathcal{F}$. Put

$$\Omega_\gamma = \{\omega : (\gamma,\omega) \in \mathcal{W}\}, \quad \mathcal{M}^\omega = \{\gamma : (\gamma,\omega) \in \mathcal{W}\}$$

and $P = P^H_a$. By Theorem 1.2, $P(\Omega_\gamma) = 1$ and, by Fubini's theorem,

$$\int_\Omega M^\mu_k(\mathcal{M}^\omega) P(d\omega) = \int_{\mathfrak{M}} P(\Omega_\gamma) M^\mu_k(d\gamma) = 1.$$

Hence the measure P is concentrated on each of sets $\{M^\mu_k(\mathcal{M}^\omega) = 1\}$ and therefore it is concentrated on their intersection. Since $P(\Omega) = 1$, this intersection is not empty. □

Appendix

We need the following result

Theorem A.1. *Suppose that $Q(x,dy)$ is a kernel from a measurable space $(X,\mathcal{A}_X)$ to a Luzin measurable space $(Y,\mathcal{A}_Y)$ and that $\mathcal{A}_Y$ is countable generated. Let P be a finite measure on $(Y,\mathcal{A}_Y)$. If $Q(x,\cdot)\prec P(\cdot)$* [12] *for all x, then there exists a $\mathcal{A}_X\times\mathcal{A}_Y$-measurable version of the Radon-Nikodym derivative $\frac{Q(x,dy)}{P(dy)}$.*

Proof. 1°. First, we note that, if a σ-algebra $\mathcal{A}$ is generated by the union of σ-algebras $\mathcal{A}_1\subset\mathcal{A}_2\cdots\subset\mathcal{A}_n\dots$ if $\rho(y)=\frac{Q(dy)}{P(dy)}$, then the conditional mathematical expectation $\rho_n=P\{\rho|\mathcal{A}_n\}$ is equal to $\frac{Q_n(dx)}{P_n(dx)}$ where P_n and Q_n are the restrictions of P and Q to $\mathcal{A}_n$. Indeed, for every $A\in\mathcal{A}_n$, $\int_A\rho dP=Q(A)=\int_A\rho_n dP$. Therefore $\rho_n\to\rho$ off a set $C\in\mathcal{A}$ such that $P(C)=0$. A version of $\frac{Q(dx)}{P(dx)}$ can be defined as $\lim\rho_n$ off C and a constant on C.

2°. If a σ-algebra $\mathcal{A}$ in Y is countably generated, then it is generated by a sequence of finite partitions of Y into disjoint sets. Moreover, we can choose this partitions to generate a monotone increasing sequence of σ-algebras $\mathcal{A}_n$.

3°. If $\mathcal{A}$ is generated by a partition $Y=Y_1\cup\dots\cup Y_n$ and if $Q\prec P$, then

$$\frac{Q(dy)}{P(dy)}=\frac{Q(Y_k)}{P(Y_k)}\quad\text{on }Y_k.$$

4°. It is suffiient to prove our theorem for the case $Q(x,Y)=1$ for all x. We apply 1° and 2° to the σ-algebra $\mathcal{A}_Y$. By 3° , ρ_n are $\mathcal{A}_X\times\mathcal{A}_Y$-measurable and an $\mathcal{A}_X\times\mathcal{A}_Y$-measurable version of $\frac{Q(x,dy)}{P(dy)}$ can be defined as in 1°. □

Acknowledgements

The proof of Theorem A.1 was suggested by D.W.Stroock. It is more natural and much simpler than my original proof based on theorems of Lebesgue-Vitaly by De Possel. I am indebted to Dan Stroock for this suggestion and for other remarks that contributed to an improvement of the presentation.

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[12] We write $Q\prec P$ if $P(A)=0$ implies $Q(A)=0$.

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