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THE FREDHOLM INDEX OF QUOTIENT HILBERT MODULES

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ABSTRACT. We show that the (multivariable) Fredholm index of a broad class of quotient Hilbert modules can be calculated by the Samuel multiplicity. These quotient modules include the Hardy, Bergman, or symmetric Fock spaces in several variables modulo submodules generated by multipliers. Our calculation is based on Gleason, Richter, Sundberg's results on the Fredholm index of the corresponding submodules.

However, our main result (Theorem 2) establishes a formula with independent interests in a broader context. It relates the fibre dimension of a submodule, an analytic notion, to the Samuel multiplicity of the quotients module, an algebraic notion.

When applied to multivariable Fredholm theory, we establish the following general principle which yields the above calculation of indices as a special case: *The Fredholm index of a submodule is equal to its fibre dimension if and only if the index of the quotient module is equal to its Samuel multiplicity.*

0. Introduction

In this paper we show that for a broad class of quotient Hilbert modules the multivariable Fredholm index, in the sense of Joseph L. Taylor, can be calculated by the Samuel multiplicity (see Definition 1), which originally was an important invariant in commutative algebra, but recently has found many applications to operator theory [9], [10], [11], [12]. As for background on the multivariable Fredholm index we refer readers to look at [2], [4], [8], and [20].

Typical examples of the quotient modules under our consideration include the following: let H^2 be the Hardy space or the Bergman space over the polydisc or the unit ball in $\mathbb{C}^n$, let $\mathcal{M} \subset H^2 \otimes \mathbb{C}^N$ $(N \in \mathbb{N})$ be a closed invariant subspace generated by *bounded functions*; or let H^2 be the symmetric Fock space, and let $\mathcal{M} \subset H^2 \otimes \mathbb{C}^N$ be any invariant subspace, then the quotient module is obtained by the quotient Hilbert space $H^2 \otimes \mathbb{C}^N/\mathcal{M}$, with the Hilbert module structure endowed by the multiplication by coordinate functions. Note that for these examples the invariant subspaces are all assumed to be generated by multipliers.

Our calculation of the Fredholm index is build upon the work of J. Gleason, S. Richter, and C. Sundberg [13], which also explains why our calculation is

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restricted to the quotient modules corresponding to submodules generated by multipliers. However, our main result (Theorem 2), which enables us to carry out the calculation, establishes a formula in more general situations, and is of independent interests besides its application to multivariable Fredholm theory.

A special case of Theorem 2 over the symmetric Fock space has appeared in [12], where the theory of Nevannlinna-Pick kernels, especially the multipliers in McCullough-Trent's [15], plays an essential role in the proof. In this paper we are able to avoid the N-P arguments, hence the main result is applicable to quite general spaces. A consequence in [12] is that Arveson's curvature invariant is always equal to the Samuel multiplicity for an *arbitrary* pure d-contraction with finite defect rank, thus providing an intrinsic formula for the curvature. In particular, it follows that the curvature is invariant under similarity.

1. Preliminaries

Now we provide some terminology and background information. By a *Hilbert module* [5] we mean a Hilbert space H, together with a module structure over the polynomial ring $A = \mathbb{C}[z_1, \cdots, z_n]$ in n complex variables, such that the action by each coordinate function z_i induces a bounded operator on H. Conversely, if H is a Hilbert space together with an n tuple of commuting operators $(T_1, \cdots, T_n)$ acting on it, then we can equip H with a Hilbert module structure by sending $(z_i, h) \mapsto T_i h$ for all $h \in H$ and $i = 1, \cdots, n$. In this paper, if H is a Hilbert module consisting of holomorphic functions over a domain in $\mathbb{C}^n$, then the module action on H by z_i is assumed to be induced by direct multiplication. Using J.L. Taylor's definition of the joint spectrum for a tuple of commuting operators [18], [19], a multivariable Fredholm theory can be naturally formulated, see Curto [4], Eschmeier-Putinar [8], Vasilescu [20], and the references therein. We say that a Hilbert module H is *Fredholm* if the n tuple of commuting operators determined by the module actions of $z_1, \cdots, z_n$ is Fredholm. The index of the n tuple is defined to be the index of the Hilbert module, denoted by $index(H)$.

Multivariable Fredholm theory has many important connections with various areas in analysis, geometry, and topology, but its development has proved to be quite resistent so far. In particular, the calculation of the multivariable Fredholm index is usually very difficult, and examples whose indices can be calculated are scant. For past results we suggest the readers to look at the references [4], [8], and [20], where more extensive bibliographies can be found.

In this paper we are mainly interested in examples furnished by submodules of a "nice" Hilbert module, and their corresponding quotient modules. In this direction, Gleason, Richter, and Sundberg [13] recently proved that the Fredholm index of a large class of submodules can be calculated by their fibre dimension. For example, let $\mathcal{M} \subset H^2 \otimes \mathbb{C}^N$ be a submodule described in the first paragraph, let $\Omega \subset \mathbb{C}^n$ be the underlying domain, and let $\mathcal{M}(\lambda) = \{f(\lambda) : f \in \mathcal{M}\} \subset \mathbb{C}^N$

for any $\lambda \in \Omega$, then we define its fibre dimension by

$$fd(\mathcal{M}) = sup\ dim\ \mathcal{M}(\lambda), \quad \lambda \in \Omega. \tag{1.1}$$

Note that the fibre dimension is in fact achieved at almost every $\lambda \in \Omega$. The results of [13] imply that if $\mathcal{M}$, equipped with the tuple of multiplication by coordinate functions, is Fredholm, then its index is given by

$$index(\mathcal{M}) = fd(\mathcal{M}). \tag{1.2}$$

Now we look at the corresponding quotient modules $H = H^2 \otimes \mathbb{C}^N/\mathcal{M}$. It is a basic fact that H^2 is Fredholm with $index(H^2) = 1$. Hence by the fundamental theorem of homological algebra, $\mathcal{M}$ is Fredholm if and only if H is Fredholm, and, when Fredholm, their indices naturally add up to N, that is,

$$index(\mathcal{M}) + index(H) = index(H^2 \otimes \mathbb{C}^N) = N. \tag{1.3}$$

It follows the formula

$$index(H) = N - fd(\mathcal{M}). \tag{1.4}$$

So on the surface, the index problem on quotient modules can be solved by looking at the difference between N and $fd(\mathcal{M})$. However, this is not very satisfactory in some aspects, since *Equation (1.4) is not an intrinsic formula.* On one hand, given the Hilbert module H, the dependence on N in formula (1.4) requires one to know how to construct a dilation model for H. Moreover, the fibre dimension $fd(\mathcal{M})$ defined on the submodule $\mathcal{M}$ appears to have no direct connection with the corresponding quotient module H, hence it is hard to calculate in terms of H. This phenomenon has also been pointed out by Arveson in [3]. On the other hand, there is a deeper reason to search for an intrinsic formula: the quotient modules usually serve as the general model for abstract contraction operators, as illustrated by Sz.-Nagy and Foias' dilation theory [17], and Arveson's [1]. These considerations bring up the central problem in this paper: *"we aim at calculating the index of H in terms of the Hilbert module H itself, independent of its dilation model."*

Now we consider two methods of calculating the multivariable Fredholm index which are of interests to us. Note that the method represented by formula (1.2) is essentially analytic: both $\mathcal{M}$ and $fd(\mathcal{M})$ naturally concern holomorphic functions. On the other hand, there is a very different approach through commutative and homological algebra: recall that the multivariable Fredholm index is defined as the Euler characteristic of a Koszul complex over Hilbert modules. Then a seminal result of J.-P. Serre in local algebra ([16], page 57, Theorem 1) expresses the Euler characteristic of a Koszul complex over algebra modules in terms of a Samuel multiplicity. Although Serre's theorem has no direct generalization to Hilbert modules, some work along this line has been done in [10], [12].

In this paper, we show that at least for the examples we have considered, a version of Serre's theorem is true; namely, we show that the Fredholm index of the quotient modules is calculated by the Samuel multiplicity, which is an algebraic invariant, and is intrinsically defined on the quotient modules (see Definition 1). In fact, we establish the following correspondence principle between the analytic approach and the algebraic approach:

> *"The Fredholm index of a submodule $\mathcal{M}$ is equal to its fibre dimension if and only if the Fredholm index of the quotient module is equal to its Samuel multiplicity."*

2. Main result

Next we state our results more precisely.

Definition 1. *Let H be a Hilbert module over $A = \mathbb{C}[z_1, \cdots, z_n]$, and let $I = (z_1, \cdots, z_n)$ be the maximal ideal of A at the origin. If dim $H/IH < \infty$, then dim $H/I^kH < \infty$ for all $k \in \mathbb{N}$, and the limit*

$$e(H) = n! \lim_{k\to\infty} \frac{dim\ H/I^kH}{k^n}$$

exists, and is an integer, called the Samuel multiplicity of H.

For more information on the Samuel multiplicity over Hilbert modules, see [6], [9], [10], [11], [12].

Due to the natural additivity of Fredholm index, that is

$$index(\mathcal{M}) + index(H) = N$$

when they exist, in order to prove the above correspondence principle between the analytic and algebraic approaches, we just need to show the following:

> *"If $\mathcal{M}$ and H are Fredholm, then $fd(\mathcal{M}) + e(H) = N$."*

However, observe that both $fd(\mathcal{M})$ and $e(H)$ may exist even when $\mathcal{M}$ and H are not Fredholm. This leads us to conjecture that these two invariants will always naturally add up. In fact, we have

Theorem 2. *Let H^2 be a Hilbert module over $A = \mathbb{C}[z_1, \cdots, z_n]$ obtained by completing A in an inner product such that*

(i) *H^2 consists of holomorphic functions on a domain $\Omega \subset \mathbb{C}^n$ around the origin, and*
(ii) *$z_1H^2 + \cdots + z_nH^2$ is closed.*

Let $\mathcal{M} \subset H^2 \otimes \mathbb{C}^N$ ($N \in \mathbb{N}$) be an arbitrary submodule, and let $H = H^2 \otimes \mathbb{C}^N/\mathcal{M}$ be the quotient module. Then $e(H^2) = 1$, and

$$fd(\mathcal{M}) + e(H) = N. \tag{2.5}$$

A novelty in formula (2.5) is that $fd(\mathcal{M})$ is an analytic notion, while $e(H)$ is obviously an algebraic notion. Also in some sense the conditions in Theorem 2 are optimal: Part (i) is required in order to define the fibre dimension $fd(\mathcal{M}) =$

sup dim $\mathcal{M}(\lambda)$ for $\lambda \in \Omega$ as in Equation (1.1). Part (ii) is needed before we can define the Samuel multiplicities $e(H^2)$ and $e(H)$.

From now on we use H^2 to denote the space defined in Theorem 2, instead of the Hardy, Bergman, or symmetric Fock spaces considered in examples at the beginning of the paper.

A few more remarks are in order now.

(i) Equation (2.5) is true for an arbitrary submodule. This is somehow rare in multivariable operator theory, because submodules, say those of the bidisk Hardy module $H^2(\mathbb{D}^2)$, can be very complicated in general.

(ii) To experts in algebra, Equation (2.5) is reminiscent of the well known additivity of Samuel multiplicity in algebraic geometry. But in operator theory for many situations of interests, $e(\mathcal{M})$ can not be defined. Because of the important role this type of additivity plays in algebraic geometry, we seek to find an invariant defined $\mathcal{M}$, and another invariant on H, such that *(a)* they are always defined, and *(b)* they naturally add up to N. This is in fact part of our motivation, and is achieved by Theorem 2.

(iii) Finally, $e(H)$ admits an interesting interpretation if we look at it in terms of its dilation model. Consider $H = \mathcal{M}^\perp$ as the orthogonal complement of $\mathcal{M}$ in $H^2 \otimes \mathbb{C}^N$. Then as isomorphisms between finite dimensional vector spaces, we have

$$H/I^kH \cong H^2 \otimes \mathbb{C}^N/(I^kH^2 \otimes \mathbb{C}^N + \mathcal{M}) \cong \mathcal{M}^\perp \cap \mathcal{P}_{k-1}.$$

Here $\mathcal{P}_{k-1}$ denotes the polynomials with degrees at most $k-1$. *So in some sense* $e(H) = e(\mathcal{M}^\perp)$ *measures how many polynomials* $\mathcal{M}^\perp$ *contains.*

Using the results of Gleason, Richter, and Sundberg [13] and the above Theorem 2, we can calculate the Fredholm index of the quotient modules whose corresponding submodules are considered in [13]. Since a full statement of the various conditions formulated in [13] is somehow lengthy and technical, we only offer the following special case for some familiar spaces

Corollary 3. *Let* $\mathcal{M} \subset H^2 \otimes \mathbb{C}^N$ $(N \in \mathbb{N})$ *be a submodule and let* $H = H^2 \otimes \mathbb{C}^N/\mathcal{M}$ *be the quotient module. If we assume that* $\mathcal{M} \subset H^2 \otimes \mathbb{C}^N$ *has one of the following form*

- H^2 *is the Hardy space or the Bergman space over the polydisk or the unit ball in* $\mathbb{C}^n$*, and* $\mathcal{M}$ *is generated by bounded functions; or*
- H^2 *is the symmetric Fock space in* n *variables,*

and we assume further that $\mathcal{M}$ *is Fredholm, then* H *is also Fredholm,* $e(H)$ *exists, and the index of* H *is given by*

$$index(H) = e(H).$$

In particular, if T *is a Fredholm pure d-contraction of finite defect rank, then* $index(T)$ *is equal to the associated Samuel multiplicity.*

3. Proof of Theorem 2

We first give the algebraic part of the proof, which is in fact based on quite standard arguments usually used to obtain additive invariants in commutative algebra. Then modulo these largely algebraic arguments the proof of Theorem 2 is reduced to Lemma 4, which is the main analytic part of the proof.

We start with the natural short exact sequence of Hilbert modules

$$0 \to \mathcal{M} \to H^2 \otimes \mathbb{C}^N \to H \to 0. \tag{3.1}$$

Applying the tensor product functor $\cdot \otimes_A A/I$, which is right half-exact, one has the following sequence to be exact

$$\cdots \to \frac{\mathcal{M}}{I\mathcal{M}} \to \frac{H^2 \otimes \mathbb{C}^N}{IH^2 \otimes \mathbb{C}^N} \to \frac{H}{IH} \to 0. \tag{3.2}$$

Of course the exactness of (3.2) can be verified directly without employing the half-exactness of the functor $\cdot \otimes_A A/I$. But we feel that in this setting direct verification is somehow unnatural. This will be more obvious when we have to do similar things again in more involved situation. On the other hand, we feel that basic algebraic machinery will eventually be indispensable in multivariable operator theory, and a little effort from the algebraic side can often streamline the presentation greatly.

We first verify that the second and third terms in (3.2) are finite dimensional, which is necessary in order to define the Samuel multiplicity. We claim that $\mathbb{C} + IH^2 = H^2$, and $\mathbb{C} \cap IH^2 = \{0\}$. Here the second equation is because elements of IH^2 are homomorphic functions vanishing at the origin. For the first equation, recall that IH^2 is closed by the condition (ii) in Theorem 2. So $\mathbb{C} + IH^2$ is also closed. Since it contains all polynomials, it has to be H^2. Hence $dim\ H^2/IH^2 = 1$. The middle term in the above sequence (3.2) is isomorphic to $(H^2/IH^2) \otimes \mathbb{C}^N$, which is finite dimensional. It follows that H/IH is finite dimensional. Consequently, we are in an algebraic situation, and the Samuel multiplicity $e(H) = n!\ \lim_{k\to\infty} \frac{dim\ H/I^k \cdot H}{k^n}$ is a well defined integer. For the convenience of readers we give some details here. For any Hilbert module K, we can form an associated graded module over A

$$gr(K) = K/IK \oplus IK/I^2K \oplus I^2K/I^3K \oplus \cdots .$$

Then $gr(K)$ is generated by its first component K/IK. Note, however, that K as a Hilbert module is usually *not* generated by $K \ominus IK$. If $dim\ K/IK < \infty$, then $gr(K)$ is finitely generated over the Noetherian ring A, hence each component $I^{k-1}K/I^kK$ is finite dimensional. So K/I^kK is also finite dimensional. By counting the dimension of the first k components, and the existence of Hilbert polynomials, we know that the function $k \mapsto dim\ K/I^kK$ is a numerical polynomial when $k >> 0$. Then it follows from basic facts on numerical polynomials that the Samuel multiplicity is an integer. For more information on numerical polynomials, see [6], [7], [9], [12], and [14] .

Now applying the right exact functors $\cdot \otimes_A A/I^k$ ($k \in \mathbb{N}$) to the short exact sequence (3.1), one has the following exact sequence

$$\cdots \to \frac{\mathcal{M}}{I^k\mathcal{M}} \to \frac{H^2 \otimes \mathbb{C}^N}{I^kH^2 \otimes \mathbb{C}^N} \to \frac{H}{I^kH} \to 0. \tag{3.3}$$

We complete the above sequence (3.3) by looking at the image of the second arrow. It follows the following exact sequence

$$0 \to \frac{\mathcal{M} + I^kH^2 \otimes \mathbb{C}^N}{I^kH^2 \otimes \mathbb{C}^N} \to \frac{H^2 \otimes \mathbb{C}^N}{I^kH^2 \otimes \mathbb{C}^N} \to \frac{H}{I^kH} \to 0. \tag{3.4}$$

We first examine the middle term in the above sequence, which is straightforward. Note that the above algebraic consideration implies that H^2/I^kH^2 is finite dimensional, hence I^kH^2 is always closed. Let $\mathcal{P}_k \subset A$ denote the collection of polynomials of degree at most k. Then similar to the case $\mathcal{P}_0 = \mathbb{C}$ we have $\mathcal{P}_{k-1} + I^kH^2 = H^2$, and $\mathcal{P}_{k-1} \cap I^kH^2 = \{0\}$. In particular, note that I^kH^2 consists of those functions in H^2 which vanish at the origin of order at least k, and $dim\ H^2/I^kH^2 = dim\ \mathcal{P}_{k-1}$. So

$$e(H^2 \otimes \mathbb{C}^N) = n! \lim_{k\to\infty} \frac{dim\ \frac{H^2\otimes\mathbb{C}^N}{I^kH^2\otimes\mathbb{C}^N}}{k^n} = N \cdot n! \lim_{k\to\infty} \frac{dim\ \mathcal{P}_{k-1}}{k^n} = N. \tag{3.5}$$

Now we look at the second term in sequence (3.4). For any (vector-valued) holomorphic function f around the origin and for any $k \geq 0$, let T_k denote the linear mapping sending f to its Taylor polynomial of degree k. Then we claim that

$$dim\ \frac{\mathcal{M} + I^kH^2 \otimes \mathbb{C}^N}{I^kH^2 \otimes \mathbb{C}^N} = dim\ T_{k-1}\mathcal{M}. \tag{3.6}$$

It is easy to see: for any $f \in \mathcal{M}$, $T_{k-1}f \in H^2 \otimes \mathbb{C}^N$ since it is a polynomial. So $f - T_{k-1}f \in H^2 \otimes \mathbb{C}^N$. Observe that $f - T_{k-1}f$ vanishes at the origin up to order k. By the discussions before Equation (3.5), $f - T_{k-1}f \in I^kH^2 \otimes \mathbb{C}^N$. Now Equation (3.6) follows.

From sequence (3.4) and Equation (3.6), one concludes that the function

$$k \mapsto dim\ T_k\mathcal{M}, \qquad k \in \mathbb{N}$$

becomes a polynomial of degree at most n when $k >> 0$, and the limit

$$n! \lim_{k\to\infty} \frac{dim\ T_k\mathcal{M}}{k^n} \tag{3.7}$$

is an integer. Moreover, it naturally adds up with $e(H)$, that is,

$$e(H) + n! \lim_{k\to\infty} \frac{dim\ T_k\mathcal{M}}{k^n} = e(H^2 \otimes \mathbb{C}^N) = N. \tag{3.8}$$

So in order to prove Theorem 2, it suffices to prove

Lemma 4. *For any submodule $\mathcal{M} \subset H^2 \otimes \mathbb{C}^N$,*

$$fd(\mathcal{M}) = n! \lim_{k\to\infty} \frac{dim\ T_k\mathcal{M}}{k^n}. \tag{3.9}$$

The rest of the paper is devoted to the proof of Lemma 4.

For simplicity, let $\delta = fd(\mathcal{M})$. We fix an orthonormal basis $e_1, \cdots, e_N$ of $\mathbb{C}^N$. For any $f \in H^2 \otimes \mathbb{C}^N$, write $f = (f_1, \cdots, f_N)$ with respect to the basis $\{e_i\}$. By the definition of δ, we can choose δ elements $h^1, \cdots, h^\delta$ from $\mathcal{M}$, such that at some point $\lambda \in \Omega$, the δ vectors $h^1(\lambda), \cdots, h^\delta(\lambda)$ are linearly independent in $\mathbb{C}^N$. Then, without loss of generality, we assume that the determinant $det(\Theta)$ of the $\delta \times \delta$ matrix $\Theta = (h^i_j)^\delta_{i,j=1}$ is a nonzero holomorphic function over Ω, since it is non-vanishing at λ.

Let Q_δ be the projection from $H^2 \otimes \mathbb{C}^N$ onto $H^2 \otimes span\{e_1, \cdots, e_\delta\}$, that is, the first δ copies of H^2 in $H^2 \otimes \mathbb{C}^N$. For a function f holomorphic at the origin, define its order to be the vanishing order at the origin, denoted by $ord(f)$. That is, if $f = f_m + f_{m+1} + \cdots$ is the homogeneous expansion at the origin, and f_m is the first nonzero term, then $ord(f) = m$. In particular, we let

$$ord(det(\Theta)) = c < \infty \tag{3.10}$$

since $det(\Theta)$ is a nonzero holomorphic function.

Firstly we show $\delta \leq n! \lim_{k\to\infty} \frac{dim\ T_k\mathcal{M}}{k^n}$. Recall that the inverse matrix of Θ is given by $\frac{1}{det(\Theta)}(A_{i,j})^\delta_{i,j=1}$, here $A_{i,j}$ is the $(\delta-1)\times(\delta-1)$ minor of Θ associated with the entry h^i_j. It follows

$$\Theta \cdot (A_{i,j}) = det(\Theta) \cdot I_\delta \tag{3.11}$$

at the level of matrix multiplication.

Then for any $k \in \mathbb{N}$, we have the following equation in terms of matrices

$$\Theta \cdot (T_k A_{ij}) = det(\Theta) \cdot I_\delta - (*), \tag{3.12}$$

here the entries of the matrix $(*)$ consist of functions of order $\geq k+1$.

Now multiply polynomials of degree at most $k-c$ to both sides of Equation (3.12), that is, for $p \in \mathcal{P}_{k-c}$

$$\Theta \cdot p \cdot (T_k A_{ij}) = p \cdot det(\Theta) \cdot I_\delta - p \cdot (*). \tag{3.13}$$

Note that entries of $p \cdot det(\Theta) \cdot I_\delta$ have order at most k, while entries of $p \cdot (*)$ have order at least $k+1$, which will be annihilated by T_k.

Now apply T_k to Equation (3.13): note that the column vectors of Θ, that is $Q_\delta h^i$, are elements of $Q_\delta\mathcal{M}$, and $p \cdot (T_k A_{ij})$ has polynomial entries, so the column vectors of $\Theta \cdot p \cdot (T_k A_{ij})$ still belong to $Q_\delta\mathcal{M}$ since $\mathcal{M}$ is invariant under polynomial multiplication. We conclude

$$T_k(p \cdot det(\Theta) \cdot e_i) \in T_k Q_\delta(\mathcal{M}) \tag{3.14}$$

for all $p \in \mathcal{P}_{k-c}$, and $i = 1, \cdots, \delta$.

By looking at the order, the linear map

$$p \to p \cdot det(\Theta) \cdot e_i \to T_k(p \cdot det(\Theta) \cdot e_i) \in T_k Q_\delta(\mathcal{M})$$

is seen to be an isomorphism for each i and all $p \in \mathcal{P}_{k-c}$. It follows that

$$\begin{aligned} dim\ T_k\mathcal{M} &\geq dim\ T_k Q_\delta(\mathcal{M}) \\ &\geq \delta \cdot dim\ \mathcal{P}_{k-c}, \end{aligned}$$

which implies $\delta \leq n! \lim_{k\to\infty} \frac{dim\ T_k(\mathcal{M})}{k^n}$.

Now we look at the reverse inequality. Pick any $f = (f_1, \cdots, f_N) \in \mathcal{M}$, the determinant of the $(\delta+1)\times(\delta+1)$ matrix

$$\begin{pmatrix} h_1^1 & h_1^2 & \cdots & h_1^\delta & f_1 \\ h_2^1 & h_2^2 & \cdots & h_2^\delta & f_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ h_\delta^1 & h_\delta^2 & \cdots & h_\delta^\delta & f_\delta \\ h_i^1 & h_i^2 & \cdots & h_i^\delta & f_i \end{pmatrix}$$

is identically zero for any fixed $i = \delta+1, \cdots, N$. Now by expanding the above determinant according to the last column, one finds

$$g_1 f_1 + g_2 f_2 + \cdots + g_\delta f_\delta + g_i f_i = 0. \tag{3.15}$$

Here g_j are holomorphic functions over Ω, not necessarily in H^2, for $j = 1, \cdots, \delta$, and $g_i = det(\Theta)$. In particular, g_i is independent of i.

Now for $k \in \mathbb{N}$, we consider the following natural map induced by Q_δ

$$J_k : T_k(\mathcal{M}) \to Q_\delta T_k(\mathcal{M}),$$

and its kernel. If $\xi = T_k(f) \in ker(J_k)$, that is,

$$T_k(f_1 \otimes e_1) = \cdots = T_k(f_\delta \otimes e_\delta) = 0,$$

or, $ord(f_i) \geq k+1$ for $i = 1, \cdots, \delta$, then by Equation (3.15)

$$0 = T_k(g_1 f_1 + \cdots + g_\delta f_\delta) = -T_k(g_i f_i).$$

Hence $ord(g_i f_i) \geq k+1$. Note that $ord(g_i) = ord(det(\Theta)) = c$, it follows that $ord(f_i) \geq k - c + 1$, that is, we have

$$T_{k-c}(f_i) = 0, \quad i = \delta+1, \cdots, N. \tag{3.16}$$

By considering T_k as a map defined on $H^2 \otimes \mathbb{C}^N$, Equation (3.16) implies that the kernel $ker(J_k)$ is contained in the range of

$$(\mathbf{1}_{H^2\otimes\mathbb{C}^N} - Q_\delta)(T_k - T_{k-c}),$$

whose rank is

$$(N-\delta)(\begin{pmatrix} n+k \\ n \end{pmatrix} - \begin{pmatrix} n+k-c \\ n \end{pmatrix}),$$

which is a polynomial of degree $n-1$. That is, it makes no contribution when we look at the asymptotic limits. Hence

$$\begin{aligned} n! \lim_{k\to\infty} \frac{dim\ T_k(\mathcal{M})}{k^n} &= n! \lim_{k\to\infty} \frac{dim\ Q_\delta T_k(\mathcal{M})}{k^n} \\ &\leq n! \lim_{k\to\infty} \frac{rank\ Q_\delta T_k}{k^n} \\ &= \delta. \end{aligned}$$

□

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