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SIMPLICAL STRUCTURES OF KNOT COMPLEMENTS

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Abstract. It was shown in [7] that there exists an explicit bound for the number of Pachner moves needed to connect any two triangulation of any Haken 3-manifold which contains no fibred sub-manifolds as strongly simple pieces of its JSJ-decomposition. In this paper we prove a generalisation of that result to all knot complements. The explicit formula for the bound is in terms of the numbers of tetrahedra in the two triangulations. This gives a conceptually trivial algorithm for recognising any knot complement among all 3-manifolds.

1. Introduction

The main aim of this paper is to understand the relation between any two triangulations of a given knot complement. In order to alter one simplicial structure (throughout the paper the term *simplicial structure* will be used as a synonym for a triangulation) to another we need the following definition.

Definition. Let T be a triangulation of a compact PL n-manifold M. Suppose D is a combinatorial n-disc which is a sub-complex both of T and of the boundary of a standard $(n+1)$-simplex Δ^{n+1}. A *Pachner move* consists of changing T by removing the sub-complex D and inserting $\partial\Delta^{n+1} - \mathrm{int}(D)$ (for n equals 3, see figure 1).

It is an immediate consequence of this definition that there are precisely $(n+1)$ possible Pachner moves in dimension n. If our n-manifold M has non-empty boundary, then the moves from this definition do not alter the induced triangulation of ∂M. But changing the simplicial structure of the boundary with an $(n-1)$-dimensional Pachner move can be achieved by gluing onto (or removing from) our manifold M the standard n-simplex Δ^n that exists by the definition of the move.

The moves from the above definition are in some sense completely general. It was proved by Pachner in [8] that any two triangulations of the same PL n-manifold are related by a finite sequence of Pachner moves and simplicial isomorphisms. It is well known (see proposition 1.3 in [5]) that in case of a fixed 3-manifold M a computable function, depending only on the number of tetrahedra in the triangulations of M, bounding the length of the sequence from Pachner's theorem, gives an algorithm for recognising M among all 3-manifolds. The following theorem gives an explicit formula for such a bound in case M is a

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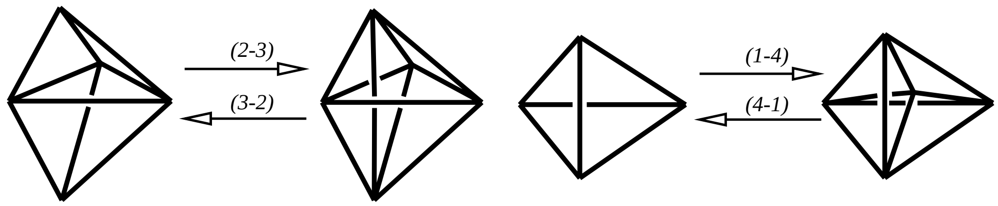

FIGURE 1. Three dimensional Pachner moves.

knot complement.

Theorem 1.1. *Let P and Q be two triangulations of a knot complement M which contain p and q tetrahedra respectively. Then there exists a sequence of Pachner moves of length at most $e^{2^{ap}}(p) + e^{2^{aq}}(q)$ which transforms P into a triangulation isomorphic to Q. The constant a is bounded above by 200. The homeomorphism of M that realizes this simplicial isomorphism is supported in the characteristic sub-manifold of M.*

The triangulations appearing in theorem 1.1 are allowed to be non-combinatorial, which means that the simplices are not (necessarily) uniquely determined by their vertices. The exponent in the above expression containing the exponential function $e(x) = 2^x$ stands for the composition of the function with itself rather than for multiplication. Since the formula in theorem 1.1 is explicit, it gives a conceptually trivial algorithm to recognise any knot complement among all 3-manifolds (just make all possible sequences of Pachner moves whose length is smaller than the bound!).

In addition to the recognition problem for all knot complements, theorem 1.1 gives a simple procedure that can be used to decide if any knot, represented by a knot diagram, is the same as our given knot. This procedure will be described in section 2.

Theorem 1.1 can be obtained as a direct consequence of theorem 2.1 which is contained in the next section. The proof uses the main result from [7]. This however is not enough because we also need to deal with fibred knot complements. It has been know for a long time that an algorithm capable of deciding whether two homeomorphisms of the fibre are conjugate can be used to recognise fibred manifolds. Our approach, however, is completely different. We avoid this problem by applying the crucial proposition 4.2 which ensures that the first surface in the canonical hierarchy is not a fibre. In other words this means that the procedure used to prove theorem 3.1 in [7] also works in this setting and implies theorem 1.1. The proof of proposition 4.2 depends heavily on the existence of a separating incompressible surface which was established by Culler and Shalen in [2].

In section 2 we give a precise description of the class of 3-manifolds which appear in theorem 2.1. After stating the main theorem we also outline its proof.

Section 3 contains the relevant results from [2] and their application to our setting. In the last section we prove the key propositions 4.1 and 4.2.

2. The main theorem

Let's start by recalling some well-known definitions. A 3-manifold M is *irreducible* if every embedded 2-sphere in it bounds a 3-ball. A properly embedded surface F in an irreducible 3-manifold M is *injective* if the homomorphism $\pi_1(F) \to \pi_1(M)$, induced by the inclusion of F into M, is a monomorphism. A surface F is said to be *incompressible* if no component of it is a 2-sphere or a disc and if for every disc D in M with $D \cap S = \partial D$, there is a disc D' in S with $\partial D = \partial D'$.

There is also a relative notion of incompressibility which we will need to consider. A surface F is *∂-incompressible* if for each disc D in M, such that ∂D splits into two arcs α and β meeting only at their common endpoints with $D \cap F = \alpha$ and $D \cap \partial M = \beta$, there is a disc D' in F with $\alpha \subset \partial D'$ and $\partial D' - \alpha \subset \partial F$.

A *horizontal boundary* of an I-bundle over a surface is the part of the boundary corresponding to the ∂I-bundle. The *vertical boundary* is the complement of the horizontal boundary and consists of annuli that fibre over the bounding circles of the base surface. It is a well-known fact that a properly embedded one-sided surface in M is injective if and only if the horizontal boundary of its regular neighbourhood is incompressible. An irreducible 3-manifold M with possibly empty incompressible boundary is *Haken* if it contains an injective surface different from a disc or a 2-sphere. A torus (resp. annulus) in M that is incompressible and is not boundary parallel is sometimes referred to as an *essential torus* (resp. *essential annulus*). Notice that in a Haken 3-manifold an essential annulus can not be ∂-compressible.

Before stating theorem 2.1 we need to recall some more standard terminology. An orientable *surface bundle* with an orientable fibre S is just a mapping torus, i.e. a quotient $S \times I/(x, 0) \sim (\varphi(x), 1)$, for some orientation preserving surface automorphism $\varphi : S \to S$. Since S is orientable this construction gives an orientable 3-manifold. But for a non-orientable surface R with a non-trivial two-sheeted covering $S \to R$, the mapping cylinder of the covering projection is an orientable twisted I-bundle over R. Gluing two such I-bundles together along their horizontal boundaries by an automorphism of S gives a 3-manifold N which is foliated by parallel copies of S and the two copies of R. The leaves of this foliation are the "fibres" of a natural projection map $N \to I$, where the two copies of R are the pre-images of the endpoints of the interval I. Such a 3-manifold N will be called a *semi-bundle* (with fibre S) over an interval I. The surfaces S and R can be either closed or bounded.

Manifolds which are homeomorphic to semi-bundles do sometimes arise naturally. The simplest example is a connected sum of two projective spaces

$\mathbb{R}P^3 \# \mathbb{R}P^3$ where the fibre is a 2-sphere. On the other hand a semi-bundle structure can never arise in a knot complement. This is because the boundary circles of the two non-orientable leaves would be disjoint curves in the boundary torus and could therefore be capped off by the annuli they bound in the torus. This would then give a closed non-orientable surface in S^3. This observation will be crucial for us because it will insure that theorem 2.1 implies theorem 1.1.

The JSJ-decomposition of a Haken 3-manifold consists of strongly simple pieces, I-bundles and Seifert fibred spaces (for precise definitions see section 2 in [7]). The strongly simple pieces are the ones that contain all the interesting topological information about M but also have the crucial property of being both atoroidal and an-annular (i.e. all incompressible annuli and tori in them are boundary parallel). Loosely speaking the union of all the components of the JSJ-decomposition that are either homeomorphic to I-bundles or to Seifert fibred spaces constitute the characteristic sub-manifold Σ of M. Before we state theorem 2.1, we should remind ourselves that the exponent in the formula below, containing the exponential function $e(x) = 2^x$, stands for the composition of the function with itself rather than for multiplication.

Theorem 2.1. *Let M be an orientable Haken 3-manifold. Assume that the strongly simple pieces of its JSJ-decomposition are not homeomorphic to any of the following types of 3-manifolds: a closed surface semi-bundle which is a rational homology 3-sphere, a closed surface bundle with the first Betti number equal to one or a surface bundle with a single boundary component which contains no closed injective surfaces (other than the boundary torus) and which is at the same time homeomorphic to a surface semi-bundle. Let P and Q be two triangulations of M that contain p and q tetrahedra respectively. Then there exists a sequence of Pachner moves of length at most $e^{2^{ap}}(p) + e^{2^{aq}}(q)$ which transforms P into a triangulation isomorphic to Q. The constant a is bounded above by 200. The homeomorphism of M, that realizes this simplicial isomorphism, is supported in the characteristic sub-manifold Σ of M and it does not permute the components of ∂M.*

Since knot complements satisfy the hypothesis of theorem 2.1 we get a conceptually trivial algorithm for determining whether any 3-manifold is homeomorphic to a complement of a given knot in S^3. Moreover theorem 2.1 also gives a simple procedure to determine whether any knot diagram represents a knot which is isotopic to our given knot. It is enough to establish whether their respective complements are homeomorphic (which we already know how to do) and, if they are, to determine whether the homeomorphism maps the meridian of one onto the meridian of the other. If the boundary torus of the knot complement is not contained in the characteristic sub-manifold, then the homeomorphism from theorem 2.1 equals the identity on the boundary. If on the other hand the bounding torus is contained in Σ, then we first make sure that the simplicial structures on the boundary of both knot complements coincide. It follows from the proof of theorem 3.1 in [7] that this is enough to make our homeomorphism equal to the identity on the boundary torus. So in this way, using theorem 2.1, we can solve the recognition problem for any knot.

The proof of theorem 2.1 follows the same lines as the proof of the main theorem in [7]. The main tool for probing the topology of the simple pieces of the JSJ-decomposition of M is the canonical hierarchy (see section 4 in [7]). This hierarchy is based on Haken's original recognition algorithm for non-fibred 3-manifolds which are sufficiently large. The last step of Haken's program for the classification of sufficiently large 3-manifolds is the solution of the recognition problem for surface bundles. Haken knew that an algorithm capable of deciding whether two automorphisms of the fibre are conjugate would suffice. The algorithmic solution of the conjugacy problem in the mapping class group of the fibre was first proved by Hemion in [4] and has been reproved many time since. We, however, take a completely different approach in the fibred case situation.

The proof of theorem 3.1 in [7] starts by constructing the canonical hierarchy (section 4 of [7]) in M. The first surface S_1 in the hierarchy consists of the JSJ-system (i.e. canonical tori and annuli, see section 2 in [7]) and of the closed two-sided injective surfaces in the strongly simple pieces of the JSJ-decomposition of M. The surface S_1 is defined so that the complement $M - \mathrm{int}(\mathcal{N}(S_1))$ contains no closed orientable incompressible surfaces which are not boundary parallel. In each component of $M - \mathrm{int}(\mathcal{N}(S_1))$, which is disjoint from the characteristic submanifold Σ, we take the surface S_2 to be a bounded two-sided incompressible surface with the largest Euler characteristic in that piece. It was shown in section 4 of [7] that the components of $M - \mathrm{int}(\mathcal{N}(S_1 \cup S_2))$ are topologically equivalent to I-bundles, handlebodies and compression bodies. We continue by cutting these complementary regions using step 3 of the canonical hierarchy. The key lemma 4.2 of [7] tells us that the canonical hierarchy decomposes M in a manageable way if and only if no component of S_1 or S_2 is a fibre in a bundle structure or a semi-bundle structure of a simple piece in the JSJ-decomposition of M. In [7] we made sure that this was not the case by hypothesising away all 3-manifolds that contain strongly simple pieces which support bundle and semi-bundle structures.

In theorem 2.1 we allow for many of the 3-manifolds from the "fibred" family. We avoid problems by making sure that the crucial components of S_1 and S_2 are not fibres. Once we show that such surfaces exist and that they have bounded normal complexity, everything works in exactly the same way as in the proof of theorem 3.1 in [7]. If a fibred strongly simple piece of M has boundary, we use propositions 4.1 and 4.2. Their proofs are not based on the solution of the conjugacy problem. Instead they use a different deep (geometric) fact from [2] which says that our surface bundle has to contain a separating incompressible surface. The same philosophy of looking for a surface which is not a fibre can be applied to all closed atoroidal 3-manifolds that have enough homology. The existence of such a surface is guaranteed by the work of Thurston in [9]. A bound on the normal complexity of such a surface can be obtained directly from the results of Wang and Tollefson in [10]. A more detailed description of this procedure will be given in the last section.

Once we find the surfaces that are not fibres, we apply the canonical hierarchy techniques (section 4 in [7]) together with theorem 1.2 of [5] to all strongly simple pieces of M. This makes it possible to connect any two triangulations of the simple sub-manifolds by a sequence of Pachner moves. The subdivision of the original triangulation in the characteristic sub-manifold can be altered directly by applying the main theorem of [6].

The reason why our strategy fails for some surface bundles with a single boundary component is the following. There seems to be no way of ensuring that the component of the surface S_2 (we are trying to construct) is neither a fibre in the bundle structure nor in the semi-bundle structure of the piece. Since 3-manifolds supporting both of these fibred structures exist, we have to exclude them by hypothesis.

It seems that dealing with triangulations of closed fibred manifolds which do not satisfy the assumptions of theorem 2.1 requires solving the conjugacy problem in the mapping class group of the fibre. On the other hand theorem 2.1 can be used to solve the conjugacy problem for the elements in the mapping class group of a surface with at least two punctures, which fix the boundary circles. This is because any two orientation preserving homeomorphisms of a surface are conjugate if and only if the two associated mapping tori are homeomorphic via a homeomorphism which maps fibres to fibres. If the surface bundle has at least two boundary components we can check, using theorem 2.1, if the mapping tori are homeomorphic. While we are changing one of the triangulations using Pachner moves, we can at the same time keep track of the original fibre in normal form. This is because at the beginning the fibre can be easily isotoped into normal form with respect to the starting triangulation. It follows directly from the definition of Pachner moves that we can keep it in normal form after we make each move. If in the end the two surface bundles are homeomorphic, we must check whether the fibres we have been keeping track of are isotopic. Since this can be done algorithmically, because both surfaces are represented by their normal forms, we can use this to solve the conjugacy problem.

3. Separating incompressible surfaces

The following amazing result is one of the main theorems of [2]. We will use it to prove the existence of surfaces which are not fibres in bounded 3-manifolds.

Theorem 3.1 (Culler, Shalen). *Let M be a compact connected orientable 3-manifold whose boundary is a non-empty union of tori. Suppose that*

$$H_1(\partial M; \mathbb{Q}) \to H_1(M; \mathbb{Q})$$

is surjective and that M is not homeomorphic to a solid torus or (torus)$\times I$*. Then for each component B of ∂M, there is a separating connected incompressible* *surface in M, which is not ∂-parallel and whose boundary is not empty and is contained in B.*

This is a deep fact indeed. The starting point of its proof is Thurston's geometrization of simple bounded 3-manifolds. It then applies some algebro-geometric techniques to analyze a certain complex curve in the set of characters of representations of $\pi_1(M)$ in $SL_2(\mathbb{C})$. An "ideal point" point on this curve gives rise to a "splitting" of the fundamental group which in turn can be used to produce an incompressible surface in M that is not boundary parallel. We will now apply theorem 3.1 to our setting.

Corollary 3.2. *Let M be a compact connected orientable irreducible atoroidal 3-manifold with boundary that is a non-empty collection of tori. Assume also that M is not a Seifert fibred space and that it does not contain a closed injective surface which is not boundary parallel. Then the following holds.*

(a) Assume that ∂M is disconnected and that B is one of its components. Then M contains a connected orientable separating incompressible ∂-incompressible surface with non-empty boundary which is neither a fibre in a bundle structure over a circle nor in a semi-bundle structure over an interval. Also the boundary of this surface is contained in B.

(b) Assume that ∂M is a single torus and that M is not homeomorphic to a surface semi-bundle. Then M contains a connected orientable separating incompressible ∂-incompressible surface with non-empty boundary which is not a fibre in any bundle structure over a circle.

This corollary can be applied directly to any strongly simple piece in the JSJ-decomposition of our manifold which contains no components of the surface S_1 after step 1 of the canonical hierarchy and whose boundary consist of tori. We saw in the previous section that knot complements in S^3 do not admit a semi-bundle structure. So corollary 3.2 applies to all knot complements.

Proof. Let us start by showing that the surjectivity assumption of theorem 3.1 is satisfied. Consider the subsequence

$$H_1(\partial M;\mathbb{Q}) \to H_1(M;\mathbb{Q}) \to H_1(M,\partial M;\mathbb{Q}) \xrightarrow{\varphi} H_0(\partial M;\mathbb{Q})$$

of the long exact homology sequence of the pair $(M, \partial M)$. It is well-known that every non-trivial element in $H_2(M;\mathbb{Q})$ can be represented by a closed embedded orientable surface in M. Each component of such a surfaces must, by our assumptions, be parallel to a component of the boundary of M. This, together with Poincaré duality, implies that the vector space $H_1(M, \partial M;\mathbb{Q})$ has a basis consisting of properly embedded arcs in M connecting a fixed component C_0 of ∂M with each of the components of $(\partial M) - C_0$. The homomorphism φ of the long exact homology sequence above maps this basis into non-trivial linearly independent vectors and is therefore injective. This fact and exactness of the homology sequence imply that the homomorphism from theorem 3.1 must be surjective.

If ∂M is disconnected, we obtain a separating orientable connected incompressible surface F in M which is not boundary parallel and which is disjoint from at least one component of ∂M. Also ∂F is not empty and is contained in B. The surface F can not be a fibre because it is disjoint from $(\partial M) - B$. We need to show that it is ∂-incompressible. Since F is separating it contains at least two boundary circles in B. The bounding circle of a ∂-compression disc for F is a union of two arcs: one in F and one in B. The arc in B either runs between two distinct components of ∂F or hits a single circle in ∂F twice from the same side. In the former case we can construct, using the annulus in B between the two circles of $B \cap F$, a genuine compression disc for F. Since F is incompressible and M is irreducible this would imply that F is ∂-parallel. In the latter case it is even easier to construct a compression disc for F which again leads into contradiction. So F must be ∂-incompressible.

If ∂M consists of a single torus, then theorem 3.1 gives us a surface with the same properties as F, which can not be a fibre in a bundle structure over a circle because it is separating. □

4. Non-fibres in normal form

In this section we are going to describe how to construct an incompressible surface, which is not a fibre, in a triangulated (semi-)bundle M that satisfies the assumptions of corollary 3.2. There are two cases depending on whether ∂M is connected or not. We will first deal with the latter case which is easier and then go on to discuss the former case. At the end of this section we will discuss the closed case thus completing the proof of theorem 2.1. Let's start by recalling some normal surface theory.

Let T be a triangulation of a 3-manifold M. An arc in a 2-simplex of T is *normal* if its ends lie in different sides of a 2-simplex. A simple closed curve in the 2-skeleton of T is a *normal curve* if it intersects each 2-simplex of T in normal arcs. A properly embedded surface F in M is in *normal form* with respect to T if it intersects each tetrahedron in T in a collection of discs all of whose boundaries are normal curves consisting of 3 or 4 normal arcs, i.e. triangles and quadrilaterals. A *normal disc* is a triangle or a quadrilateral. There are precisely seven normal disc types in any tetrahedron of T. An isotopy of M is called a *normal isotopy* with respect to T if it leaves all simplices of T invariant. In particular this implies that it is fixed on the vertices of T.

A normal surface is determined, up to normal isotopy, by the number of normal disc types in which it meets the tetrahedra of T. It therefore defines a vector with $7t$ coordinates. Each coordinate represents the number of copies of normal disc types that are contained in the surface (t is the number of tetrahedra in T). It turns out that there is a certain restricted linear system that such a vector is a solution of. Moreover there is a one to one correspondence between the solutions of that restricted linear system and normal surface in M. If the sum of two vector solutions of this system satisfies the restrictions on the system, then it represents a normal surface in M. On the other hand there is a geometric process called *regular alteration* (see figure 2 in [1]) which can be carried out on the normal surfaces representing the summands and which yields the normal surface corresponding to the sum. It follows directly from the definition of regular alteration that the Euler characteristic is additive over normal addition. We can define the *weight* $w(F)$ of a surface F, which is transverse to the 1-skeleton of T, to be the number of points of intersection between the surface and the 1-skeleton. Since regular alteration only changes the surfaces involved away from the 1-skeleton, the weight too is additive over normal addition.

A normal surface is called *fundamental* if the vector corresponding to it is not a sum of two integral solutions of the restricted linear system. The solution space of the restricted linear system projects down to a compact convex linear cell which is called the *projective solution space*. A *vertex surface* is a connected two-sided normal surface that projects onto a vertex of the projective solution space (see [10] for a more detailed description). The normal sum $F = F_1 + F_2$ is in *reduced form* if the number of components of $F_1 \cap F_2$ is minimal among all normal surfaces F_1' and F_2' isotopic to F_1 and F_2 respectively such that $F = F_1' + F_2'$.

Before we proceed we need to define two (very simple) kinds of complexities of the surfaces embedded in our triangulated 3-manifold M. First there is the *normal complexity*, i.e. the number of normal pieces a minimal weight representative in the isotopy class of the surfaces consist of. Second there is the *topological complexity* of a surface which is defined in terms of its components in the following way. To each component we assign its negative Euler characteristic and then define the complexity to be the sum over all of the components. Since there are no 2-spheres, discs or projective planes among the surfaces we are trying to construct in this paper, their topological complexity will coincide with the Thurston complexity as defined in [9]. Now we can state the following proposition.

Proposition 4.1. *Let M be a triangulated 3-manifold which satisfies the assumptions of corollary 3.2 (a) and which is also an-annular. Let B be a component of ∂M. Let F be a two-sided connected incompressible ∂-incompressible surface in M with non-empty boundary lying in B, which minimises the topological complexity among all such surfaces. Then we can isotope F into normal form so that it is a sum of at most two fundamental surfaces.*

Proof. Corollary 3.2 guarantees the existence of at least one surface with the properties from the proposition. That surface is also separating while our F might not be. Assume that F is in normal form and that it has minimal weight in its isotopy class. Now express F as a sum of fundamental surfaces: $F = k_1F_1 + \ldots + k_nF_n$. By theorem 2.3 from [6] we can conclude that each F_i is incompressible and ∂-incompressible. In fact all the summands are injective because we can apply the same theorem to $2F$ (which also minimises the weight in its isotopy class since F is two-sided).

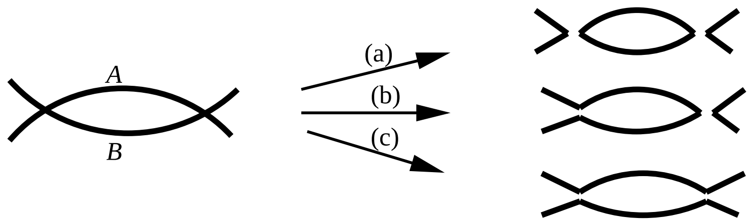

FIGURE 2. Possible normal alterations.

If some F_i were closed, then it would have to be a ∂-parallel torus by our assumption on M. We would then get $F = F_i + S$ where S is some normal surface in M. We can assume that the sum $F = F_i + S$ is in reduced form. Lemma 2.2 from [6] then implies that no component of the surface $F - (F_i \cap S)$ is a disc. So the space $F_i \cap S$ is a 1-manifold that is homeomorphic to a disjoint union of non-trivial parallel simple closed curves in the torus F_i. Let X be the (torus) $\times I$ region between F_i and the toral boundary component of M. Then the components of the surface $S \cap X$ must be injective in X simply because the patches of $F = F_i + S$ are injective by lemma 2.2 from [6] and ∂X is incompressible in M. So, since $S \cap X$ contains no closed components, it consists of incompressible annuli that are either disjoint from the torus $X \cap \partial M$ or are spanning annuli in X. Let B be an outermost annular component of $S \cap X$ which is disjoint from $X \cap \partial M$ and let A be the annulus in F_i that is parallel to B. There are three possible (essentially different) ways normal alteration can act on ∂A. They are depicted by figure 3.

They all lead to contradiction. Case (a) produces a disconnected sum. In case (b) we can isotope the union of the patches A and B over the solid torus that they bound, to reduce the weight of F. If both A and B had zero weight, then there would exist a normal isotopy that would reduce the number of components in $F_i \cap S$. This contradicts the reduced form assumption. Case (c) contradicts it as well, because the surfaces we obtain after we do the normal alterations along ∂A, are isotopic to F_i and S, but have fewer components of intersection.

So the only possible components of $S \cap X$ are the spanning annuli, i.e. the ones that are vertical in the product structure of X. There are essentially only two different ways of doing normal alterations along all the simple closed curves in $F_i \cap S$ if we want to obtain a connected surface. They lead to contradiction because the surface F we get in both cases is isotopic to S. This contradicts the assumption that F has minimal weight in its isotopy class.

We have just shown that ∂F_i is not empty and that it is contained in B for every $i = 1, \ldots, n$. Since M is an-annular, no F_i can be an annulus or a Moebius band. Since all F_i are connected, none of them can be two-sided (unless the sum has only one summand). But if we have at least three one-sided surfaces in the sum, then the double of one of them is going to satisfy all the necessary conditions and will have its topological complexity smaller than that of F. Hence the proposition follows. $\square$

Notice that the assumption that M is an-annular does not create any problems for us because the JSJ-pieces we are interested in have this property by lemma 4.2 in [7]. We are now going to deal with the remaining case when our manifold has only one boundary component.

Proposition 4.2. *Let M be a triangulated 3-manifold which satisfies the assumptions of corollary 3.2 (b) and which is also an-annular. Let t be the number of tetrahedra in M. Assume further that M is a surface bundle over a circle. Let F be a connected two-sided separating incompressible ∂-incompressible surface with non-empty boundary that has the smallest topological complexity among all such surfaces. Then F can be isotoped into normal form so that it consists of not more than 2^{40t} normal discs.*

It is clear that the surface F from proposition 4.2 can not be a fibre in the bundle structure of M because it is separating. Also, by applying proposition 4.2 while constructing components of the surface S_2 in step 2 of the canonical hierarchy (see section 4 of [7] for more details), we can make sure we never adjoin a fibre. So if our 3-manifold has a non-trivial JSJ-decomposition, the only case we can not deal with is if there is a strongly simple piece with a single toral boundary component which is a semi-bundle and a bundle at the same time. We can thus use propositions 4.1 and 4.2, combined with the proof of theorem 3.1 in [7], to show that theorem 2.1 holds.

Proof. The hypotheses on the manifold M, together with Poincaré duality, imply that the vector spaces $H_1(M, \partial M; \mathbb{Q})$ and $H_2(M; \mathbb{Q})$ are trivial. The exact homology sequence of the pair $(M, \partial M)$ yields a short exact sequence

$$0 \to H_2(M, \partial M; \mathbb{Q}) \to H_1(\partial M; \mathbb{Q}) \to H_1(M; \mathbb{Q}) \to 0$$

of vector spaces over the field $\mathbb{Q}$. Since $H_2(M, \partial M; \mathbb{Q})$ and $H_1(M; \mathbb{Q})$ are isomorphic by Poincaré duality and the dimension of $H_1(\partial M; \mathbb{Q}))$ equals 2, we get that the first Betti number $\beta_1(M; \mathbb{Q})$ must be 1. Applying Poincaré duality once again we get that $H_2(M, \partial M; \mathbb{Z})$ is isomorphic to $H^1(M; \mathbb{Z})$ and is hence torsion-free. In other words we have shown that $H_2(M, \partial M; \mathbb{Z})$ is isomorphic to $\mathbb{Z}$, which will be very useful later on.

Let's now isotope F into normal form so that it minimises the weight in its isotopy class. We can now express it as a sum of fundamental surfaces in the usual way: $F = k_1 F_1 + \ldots + k_n F_n$. Like in the proof of proposition 4.1 we can conclude that each F_i is an injective ∂-incompressible bounded surface with $\chi(F_i) < 0$. The Euler characteristic inequality comes from the fact that M is an-annular and can therefore not contain non-trivial annuli and Moebius bands. The same argument shows that any normal surface which appears as a summand of F has to be an injective ∂-incompressible surface with non-empty boundary and strictly negative Euler characteristic. Clearly no F_i can be both two-sided and separating (unless F itself is fundamental). But if a surface F_j is one-sided then its double $2F_j$ is a bounded connected two-sided separating incompressible ∂-incompressible surface with negative Euler characteristic that satisfies the inequality

$$2\chi(F_j) = \chi(2F_j) \leq \chi(F) = k_1\chi(F_1) + \ldots + k_n\chi(F_n) \leq -\sum_{i=1}^{n} k_i.$$

The first inequality comes from the fact that F minimises topological complexity in the family of surfaces which contains $2F_j$ and the second one is simply saying that no F_i is an annulus or a Moebius band. The well-known bound on the number of triangles and quadrilaterals in a fundamental normal surface (see lemma 3.2 in [3]) implies the inequality $-2\chi(F_j) < 2^{20t}$. So it follows that the sum representing F has at most 2^{20t} fundamental summands. This proves the proposition in case one of the surfaces F_i is one-sided. The following claim is fundamental for the rest of the proof.

Claim. Let the surface S be a fibre of the bundle M. Then any connected two-sided incompressible ∂-incompressible surface R in M, which is not separating, must be isotopic to S.

It is clear that R represents a non-trivial element in $H_2(M, \partial M; \mathbb{Z}) = \mathbb{Z}$. This element is primitive because R is connected (see lemma 1 in [9]). So if we choose our orientations correctly, we get that the surfaces R and S represent the same class in $H_2(M, \partial M; \mathbb{Z})$. Now we want to lift R to the infinite cyclic cover of M which corresponds to the element $[S]$ in $H^1(M; \mathbb{Z})$. This covering space is clearly homeomorphic to $S \times \mathbb{R}$. The surface R lifts if and only if every element in $\pi_1(R)$, when viewed as a homology class, has trivial algebraic intersection with $[S]$. This condition is satisfied because the surfaces R and S are homologous and R has trivial algebraic intersection with any closed loop it contains (since it is orientable). Once we lift R to the cover $S \times \mathbb{R}$, we can use incompressibility and ∂-incompressibility of R to finish the proof of the claim.

Using this claim we can prove the proposition. The conclusion clearly holds if F has at most two summands. Now we can assume that every surface F_i is a fibre of M and that there are more than two summands in the whole expression.

Let F_j and F_k be two summands in $F = k_1F_1 + \ldots + k_nF_n$ that have non-trivial intersection. After making regular alterations along all the curves in $F_j \cap F_k$ we get $F_j + F_k = A + B$ where A and B are disjoint connected normal surfaces that are isotopic to the fibre of M. We can see this in the following way. Each component D of the surface $F_k + F_j$ appears as a summand in some normal sum representing F. It is therefore injective ∂-incompressible and homologically non-trivial in $H_2(M, \partial M; \mathbb{Z}_2)$ (the surface D can be neither separating because $\chi(F) < \chi(D) < 0$ nor can it be closed since it is injective). Think of F_k and F_j as non-trivial elements of $H_2(M, \partial M; \mathbb{Z}_2)$. Their normal sum is therefore zero in $H_2(M, \partial M; \mathbb{Z}_2)$ which means that $F_j + F_k$ has an even number of components. If there are four or more, then at most one is a fibre because $\chi(F_j + F_k) = 2\chi(S)$, where S is a fibre of M. In that case at least one component is a bounded one-sided surface with its Euler characteristic strictly larger than $\frac{1}{2}\chi(S)$. This is a contradiction since $\chi(F) < \chi(F_j) + \chi(F_k) = 2\chi(S)$ and therefore the double of that one-sided surface would satisfy all the conditions from the proposition with its Euler characteristic strictly larger than that of F. So we can conclude that $F_j + F_k = A' \cup B'$, where A' and B' are disjoint connected homologically non-trivial surfaces. We also have $\chi(A') + \chi(B') = 2\chi(S)$. If both A' and B' are one-sided, then the double of the component with the larger Euler characteristic implies $2\chi(S) \leq \chi(F)$, which is a contradiction. If only one of them is one-sided, then the other one is isotopic to the fibre by the claim. So the Euler characteristic of the double of the one-sided component equals $2\chi(S)$ which leads to contradiction as before. So in the end we get that both components are two-sided and hence fibres by the claim.

If F is a sum of at least four fibres (there can not be three because F is trivial in $H_2(M, \partial M; \mathbb{Z}_2)$) then, by what we have just proved, we never reduce the number of fibres in the sum by doing regular alterations. This is a contradiction because F is connected. So the proposition follows. $\square$

The only classes of 3-manifolds we need to think about now in order to finish the proof of theorem 2.1 are: closed atoroidal surface semi-bundles which are not rational homology 3-spheres, closed atoroidal surface bundles with first Betti number at least 2 and an-annular semi-bundles with a single boundary component which contain no closed injective surfaces other than the boundary torus and which are not homeomorphic to surface bundles. In all these manifolds we can find a homologically non-trivial surface which is not a fibre in any fibration of the manifold. It follows from [9] that any connected incompressible ∂-incompressible surface whose homology class is carried by a vertex in the boundary of the unit ball for the Thurston norm on $H_2(M, \partial M; \mathbb{R})$ can not be a fibre. If we pick one such surface which minimises the topological complexity in its homology class, then its Euler characteristic is bounded by corollary 5.8 in [10]. Since all 3-manifolds on the above list are both atoroidal and an-annular, we can use this bound and the techniques developed in subsection 4.2 of [7] to control the normal complexity of our surface. This completes the proof of theorem 2.1.

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