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GAPS IN HOCHSCHILD COHOMOLOGY IMPLY SMOOTHNESS FOR COMMUTATIVE ALGEBRAS

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Abstract. The paper concerns Hochschild cohomology of a commutative algebra S, which is essentially of finite type over a commutative noetherian ring K and projective as a K-module. For a finite S-module M it is proved that vanishing of $\mathrm{HH}^n(S\,|\,K;M)$ in sufficiently long intervals imply the smoothness of $S_{\mathfrak q}$ over K for all prime ideals $\mathfrak q$ in the support of M. In particular, S is smooth if $\mathrm{HH}^n(S\,|\,K;S)=0$ for $(\dim S+2)$ consecutive $n\geq 0$.

Introduction

Let K be a commutative noetherian ring, S a commutative K-algebra, and M an S-module. We let $\mathrm{HH}_*(S\,|\,K;M)$ and $\mathrm{HH}^*(S\,|\,K;M)$ denote, respectively, the Hochschild homology and the Hochschild cohomology of the K-algebra S with coefficients in M. For each $n\in\mathbb{Z}$ there are canonical homomorphisms

$$\lambda_n^M\colon (\wedge_S^n\Omega_{S\,|\,K})\otimes_S M \longrightarrow \mathrm{HH}_n(S\,|\,K;M)$$
$$\lambda_M^n\colon\ \mathrm{HH}^n(S\,|\,K;M)\longrightarrow \mathrm{Hom}_S(\wedge^n\Omega_{S\,|\,K},M)$$

of S-modules, where $\Omega_{S\,|\,K}$ is the S-module of K-linear Kähler differentials of S. Other concepts appearing the next result are defined following its statement.

Main Theorem. *Let K be a commutative noetherian ring and S a commutative K-algebra essentially of finite type, flat as a K-module. For a prime ideal $\mathfrak q$ in S and a finite S-module M with $M_{\mathfrak q}\neq 0$ the following conditions are equivalent*:

(i) *The K-algebra $S_{\mathfrak q}$ is smooth.*

(ii$_*$) *Each map $(\lambda_n^M)_{\mathfrak q}$ is bijective and the $S_{\mathfrak q}$-module $\Omega_{S_{\mathfrak q}\,|\,K}$ is projective.*

(iii$_*$) *There exist non-negative integers t,u of different parity satisfying*

$$\mathrm{HH}_t(S\,|\,K;M)_{\mathfrak q}=0=\mathrm{HH}_u(S\,|\,K;M)_{\mathfrak q}$$

When the K-module S is projective they are also equivalent to:

(ii*) *Each map $(\lambda_M^n)_{\mathfrak q}$ is bijective.*

(iii*) *There exist non-negative integers t,u of different parity satisfying*

$$\mathrm{HH}^{t+i}(S\,|\,K;M)_{\mathfrak q}=0=\mathrm{HH}^{u+i}(S\,|\,K;M)_{\mathfrak q}\quad \textit{for}\quad 0\leq i\leq \dim_{S_{\mathfrak q}} M_{\mathfrak q}$$

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Recall that one says that S is *essentially of finite type* if it is a localization of a finitely generated K-algebra. A flat K-algebra S essentially of finite type is *smooth* if the structure map $K \to S$ has geometrically regular fibers. Equivalently, for every homomorphism of rings $K \to \ell$, where ℓ is field, the ring $S \otimes_K \ell$ has finite global dimension. We say that an S-module M is *finite* if it is finitely generated, and let $\dim_S M$ denote the Krull dimension of M.

The theorem incorporates several known results, discussed below. There are two new aspects to our characterizations of smoothness: the use of *cohomology* (with a couple of exceptions, earlier results used vanishing of homology) and the introduction of *coefficients* (all earlier results dealt with the case $M = S$). A special case of the theorem relates to a question of Happel [16, (1.4)]:

> For a (not necessarily commutative) algebra A over a field K, with $\operatorname{rank}_K A$ finite, does $\mathrm{HH}^n(A|K;A) = 0$ for $n \gg 0$ imply finite global dimension?

The next corollary provides a strong affirmative answer in the commutative case. This is in sharp contrast to the general situation, where the answer is negative: see the companion paper [11] by Buchweitz, Green, Madsen, and Solberg.

Corollary. *Let K be a field let S be a commutative K-algebra, finite dimensional as a K-vector space. If $\mathrm{HH}^n(S|K;S) = 0$ for two non-negative values of n of different parity, then S is a product of separable field extensions of K.*

Proof. The hypothesis $\operatorname{rank}_K S < \infty$ implies that $\dim S$ is 0, and that S is smooth precisely when it is a product of finite separable field extensions of K. □

We place our result in the context of earlier work relating vanishing of Hochschild (co)homology and smoothness. As always, $\operatorname{Spec} S$ denotes the set of prime ideal of S; its subset $\operatorname{Supp}_S M = \{\mathfrak{q} \in \operatorname{Spec} S | M_{\mathfrak{q}} \neq 0\}$ is the *support* of M.

Antecedents. Let S be a K-algebra S essentially of finite type, flat as a K-module. When citing results, a roman numeral in *italic font* font indicates the variant of the correspondingly numbered condition in the Main Theorem, where the hypothesis is assumed to hold for $M = S$ and for all $\mathfrak{q} \in \operatorname{Spec} S$.

The HKR Theorem. Hochschild, Kostant, and Rosenberg [18] (when K is a perfect field) and André [1] (in general) proved (i) $\implies$ (ii$_*$) & (ii*). As S is essentially of finite type, the S-module $\Omega_{S|K}$ is finite, so $\wedge^n_S \Omega_{S|K} = 0$ holds for all $n \gg 0$, hence one always has (ii$_*$) $\implies$ (iii$_*$) and (ii*) $\implies$ (iii*).

Homological converses to the HKR Theorem. André [1] proved $(ii_*) \implies (i)$.

$(iii_*) \implies (i)$ was proved by Avramov and Vigué-Poirrier [6] when K is a field; by Campillo, Guccione, Guccione, Redondo, Solotar, and Villamayor [7] when, in addition, $\operatorname{char}(K) = 0$; by Rodicio [22] in general.

Cohomological converses to the HKR Theorem. Assume S is projective over K.

For a Gorenstein ring S Blanco and Majadas [8] proved that $\mathrm{HH}^n(S|K;S) = 0$ for $(\dim S + 2)$ consecutive values of $n \geq 0$ implies S is smooth over K; this is subsumed in the implication (iii*) $\implies$ (i) of the Main Theorem. In joint work with Rodicio [9] they showed that if S is locally complete intersection over K, then $\mathrm{HH}^{2n}(S|K;S)=0$ or $\mathrm{Ker}(\lambda_S^{2n})=0$ for a single $n\geq 0$ implies S is smooth.

Generalizations. The Main Theorem is a special case of a much more general result, Theorem (4.2), concerning gaps in $\mathrm{Tor}_*^R(S,M)$ and $\mathrm{Ext}_R^*(S,N)$ when R is a noetherian ring, S is an algebra retract of R, and M is a *complex* of S-modules. For $\mathrm{Tor}_*^R(S,S)$ that result is due to Rodicio [22]. However, to prove (iii*) $\implies$ (i) in the Main Theorem, even for $M=S$, we do need to use complexes.

Conventions. In the rest of this article all rings are assumed to be commutative. A *local ring* is a noetherian ring that has a unique maximal ideal. A *local homomorphism* is a homomorphism of rings, whose source and target are local and which maps maximal ideal into maximal ideal.

1. Closed homomorphisms

In this section $\varphi\colon (R,\mathfrak{m},k)\to S$ is a surjective local homomorphism.

We recall a general construction due to Tate [24]. More details about Tate resolutions and acyclic closures can be found in the original paper, in the book of Gulliksen and Levin [14, Chapter I], or in the survey [2, Chapter 6].

1.1. Tate resolutions. For each positive integer n let X_n denote a free graded R-module concentrated in degree n; furthermore, $R\langle X_n\rangle$ denotes the exterior algebra on X_n if n is odd, and the divided powers algebra on X_n if n is even; in the latter case, $x^{(i)}$ denotes the ith divided power of $x\in X_n$.

A *Tate resolution* of φ is a DG (= differential graded) algebra G having a system of divided powers compatible with the action of the differential and a filtration $\{G^{(p)}\}_{p\geq 0}$ by DG subalgebras with divided powers, such that

(0) $G^{(0)}=R$ and $G^{(p-1)}\subseteq G^{(p)}$, for $p\geq 1$.
(1) $G^{(p)}=G^{(p-1)}\otimes_R R\langle X_p\rangle$ as graded R-modules, for $p\geq 1$.
(2) $\partial(x^{(i)})=\partial(x)x^{(i-1)}$ for all $i\geq 1$ when $|x|$ is even and positive.
(3) $\mathrm{H}_0(G^{(p)})=S$ for $p\geq 1$.
(4) $\mathrm{H}_i(G^{(p)})=0$ for $1\leq i<p$.
(5) $G=\bigcup_{p\geqslant 0} G^{(p)}$.

Forgetting the multiplicative structures, G is a free resolution of R over S. A Tate resolution always exists: form DG algebras satisfying conditions (0) through (4) by induction on p, then use (5) to define G; see [24,§2]. Control may be exercised at each step of the process.

As starting point, one may choose any surjective R-linear map

$$\delta_1\colon X_1\longrightarrow \mathrm{Ker}(\varphi)$$

and define the differential on $G^{(1)}$ so that its restriction to X_1 is the composition of δ_1 with the inclusion $\mathrm{Ker}(\varphi)\subseteq R=G_0^{(0)}$. If $\boldsymbol{e}_1$ is a basis for X_1, then $\delta_1(\boldsymbol{e}_1)$ generates the ideal $\mathrm{Ker}(\varphi)$ and $G^{(1)}$ is the Koszul complex on $\delta(\boldsymbol{e}_1)$.

For each $p\geq 2$ one may choose any surjective R-linear map

$$\delta_p\colon X_p\to \mathrm{H}_{p-1}(G^{(p-1)})$$

lift it to a homomorphism $\widetilde{\delta}_p\colon X_p \to \mathrm{Z}_{p-1}(G^{(p-1)})$, and define a differential on $G^{(p)}$, which on X_p is the composition of $\widetilde{\delta}_p$ with $\mathrm{Z}_{p-1}(G^{(p-1)}) \subseteq G^{(p-1)}_{p-1} = G^{(p)}_{p-1}$.

1.2. Acyclic closures. An *acyclic closure* of φ is a Tate resolution obtained by choosing for each $p \geq 1$ the map δ_p in (1.1) to be a projective cover.

Let G be an acyclic closure of φ and let G' be a Tate resolution of φ. There exists then a morphism $\gamma\colon G \to G'$ of DG R-algebras with divided powers, and for any such morphism the homomorphism of R-modules $\gamma_n\colon G_n \to G'_n$ is a split injection. If G' is also an acyclic closure of φ, then γ is an isomorphism, and it induces an isomorphism $G^{(p)} \to G'^{(p)}$ for each $p \geq 0$.

In particular, the pth *stage* $G^{(p)}$ of an acyclic closure G of φ is independent, up to isomorphism, of the choice of G.

The next remark is immediate from the construction of acyclic closures.

1.2.1. $G^{(1)}$ is the Koszul complex E on a minimal generating set for $\mathrm{Ker}(\varphi)$.

We introduce two numerical invariants of φ, for use throughout the paper. Letting $\nu_S(N)$ denote the minimal number of generators an S-module N, we set

$$\varepsilon_2(\varphi) = \nu_S(\mathrm{Ker}(\varphi)) \quad \text{and} \quad \varepsilon_3(\varphi) = \nu_S(\mathrm{H}_1(E))$$

These are part of the *deviations* of φ; see [5, (2.5)]. The first assertion below is clear; the second one is a standard characterization of regular sequences.

1.2.2. $\varepsilon_2(\varphi) = 0$ if and only if $\varphi = \mathrm{id}^R$.

1.2.3. $\varepsilon_3(\varphi) = 0$ if and only if φ is generated by a regular sequence.

A complex F of finite free R-modules is said to be *minimal* if $\partial(F) \subseteq \mathfrak{m}F$. For each integer $p \geq 1$, the construction of the pth stage $F_{\leqslant p}$ of a minimal resolution of S adds to $F_{\leqslant p-1}$ a single new free module in degree p.

In contrast, the construction of the pth stage $G^{(p)}$ of an acyclic closure of φ adds to $G^{(p-1)}$ shifts of *every* free module present in it: finitely many shifts appear when p is odd, and infinitely many when p is even. Thus, when the resolution of S over R provided by an acyclic closure is minimal, one has a certain control of the growth of that resolution.

This explains our interest in the class of maps described below.

1.3. Closed homomorphisms. We say that the homomorphism φ is *closed* if some acyclic closure G of φ is a minimal resolution of S over R.

A celebrated result of Gulliksen [13] and Schoeller [23] can be read as follows:

1.3.1. The canonical surjection $R \to k$ is closed for every R.

To state an extension, we recall that the homomorphism φ is *large* if the map

$$\mathrm{Tor}^{\varphi}_n(k,k)\colon \mathrm{Tor}^R_n(k,k) \to \mathrm{Tor}^S_n(k,k)$$

is surjective for each n. The notion was introduced by Levin [20]. The following theorem of Avramov and Rahbar-Rochandel, see [20, (2.5)], provides a significant supply of closed homomorphisms.

1.3.2. Every large homomorphism is closed.

The last result will be applied through the following observation:

1.3.3. If there is a homomorphism of rings $\psi\colon S \to R$ with $\varphi \circ \psi = \mathrm{id}^S$, then

$$\mathrm{Tor}_n^{\varphi}(k,k) \circ \mathrm{Tor}_n^{\psi}(k,k) = \mathrm{Tor}_n^{\varphi\circ\psi}(k,k) = \mathrm{Tor}_n^{\mathrm{id}^S}(k,k) = \mathrm{id}^{\mathrm{Tor}_n^S(k,k)}$$

by functoriality; thus, $\mathrm{Tor}_n^{\varphi}(k,k)$ is surjective, hence φ is large, and so closed.

In this paper we are mostly interested in obtaining lower bounds on the sizes of the S-modules $\mathrm{Tor}_n^R(S,M)$ and $\mathrm{Ext}_R^n(S,M)$. For that purpose we use properties of φ that are weaker than closure.

1.4. Partly closed homomorphisms. Let $G^{(p)}$ be as in (1.2) for some $p \geq 1$ and F be a minimal free resolution of the R-module S. As $\mathrm{H}_0(G^{(p)}) = S$ and each $G_n^{(p)}$ is R-free, the augmentation $G^{(p)} \to S$ lifts to a *comparison morphism*

$$\gamma^{(p)}\colon G^{(p)} \to F$$

We say that φ is *p-closed* if $\gamma_n^{(p)}$ has an R-linear left inverse for each $n \in \mathbb{Z}$.

A homomorphism γ of free R-modules of finite rank has a left inverse if and only if the map $k \otimes_R \gamma$ is injective. This yields an alternative description:

1.4.1. The homomorphism φ is p-closed if and only if $G^{(p)}$ is minimal and the induced map $\mathrm{H}(k \otimes_R \gamma^{(p)})\colon k \otimes_R G^{(p)} \to \mathrm{Tor}^R(k,S)$ is injective.

1.4.2. If the homomorphism φ is p-closed, G' an acyclic closure of φ, F' is a resolution of S by finite free R-modules, and $\gamma'^{(p)}\colon G'^{(p)} \to F'$ is a comparison morphism, then $\gamma'^{(p)}_n$ has a left inverse for each $n \in \mathbb{Z}$.

Indeed, (1.2) yields an isomorphism $\alpha\colon G^{(p)} \to G'^{(p)}$ of DG algebras over R, so $G'^{(p)}$ is minimal by (1.4.1). For any comparison morphism $\beta\colon F' \to F$, the morphisms $\gamma^{(p)}$ and $\beta\gamma'^{(p)}\alpha$ are homotopic. Thus, $\mathrm{H}(k \otimes_R \gamma^{(p)})$ factors as

$$k \otimes_R G^{(p)} \xrightarrow{k\otimes_R\alpha} k \otimes_R G'^{(p)} \xrightarrow{\mathrm{H}(k\otimes_R\gamma'^{(p)})} \mathrm{H}(k \otimes_R F') \xrightarrow{\mathrm{H}(k\otimes_R\beta)} k \otimes_R F$$

It follows that $\mathrm{H}(k \otimes_R \gamma'^{(p)})$ is injective, see (1.4.1), hence $\gamma'^{(p)}_n$ is split injective.

1.4.3. Let R' be a local ring, let $\rho\colon R \to R'$ be a faithfully flat local homomorphism, set $S' = R' \otimes_R S$, and let $\rho'\colon R' \to S'$ denote the induced homomorphism. The map φ is p-closed if and only if so is φ'.

Indeed, $R' \otimes_R G$ is an acyclic closure of φ' if G is one of φ; see [14, (1.9.8)].

Next we place the properties discussed above in a familiar context, focusing on the case $p \leq 2$ because these are the classes of maps important for this paper.

1.5. Comparisons. The homomorphism φ is said to be *complete intersection* (or *c.i.*, for short) if $\operatorname{Ker}(\varphi)$ is generated by a regular sequence. Clearly, one has

$$\text{c.i.} \implies \text{closed} \implies \text{2-closed} \implies \text{1-closed}$$

1.5.1. The first implication is obviously strict; see for instance (1.3.1).

1.5.2. The canonical map from $R = k[[x,y]]/(x^2, xy)$ to $S = R/(y^2)$ is 1-closed, but not 2-closed: apply (1.4.2) to the commutative diagram

$$\begin{array}{ccccccccccc}
\cdots & \xrightarrow{\;x\;} & R & \xrightarrow{\;y^2\;} & R & \xrightarrow{\;x\;} & R & \xrightarrow{\;y^2\;} & R & \longrightarrow & 0 \\
 & & {\scriptstyle \left[\begin{smallmatrix}0\\ y\end{smallmatrix}\right]}\big\downarrow & & \big\| & & \big\| & & \big\| & & \\
\cdots & \longrightarrow & R^2 & \xrightarrow{\;[x\ y]\;} & R & \xrightarrow{\;x\;} & R & \xrightarrow{\;y^2\;} & R & \longrightarrow & 0
\end{array}$$

whose top row is the beginning of the second stage of an acyclic closure of φ and whose bottom row is the beginning of a minimal resolution of S over R.

1.5.3. We do not whether every 2-closed homomorphisms is actually closed.

Except for the name, 1-closed homomorphisms have appeared in literature.

1.6. One-closed homomorphisms. As defined, p-closure requires $\gamma_n^{(p)}$ to be split injective for each n. However, 1-closure can be detected in a single degree.

Lemma 1.6.1. *Let E be the Koszul complex on a minimal generating set for the ideal* $\operatorname{Ker}(\varphi)$ *and let $\gamma\colon E \to F$ be a comparison morphism to a minimal free resolution of S over R. The following conditions are equivalent:*

(1) *The homomorphism φ is* 1*-closed.*
(2) *For $c = \varepsilon_2(\varphi)$ the map $(k \otimes_R \gamma_c)\colon (k \otimes_R E_c) \to \operatorname{Tor}_c^R(k, S)$ is injective.*

Proof. Observation (1.2.1) and property (1.4.1) show that (1) implies (2).

For the converse, note that the isomorphism $k \otimes_R E \cong \wedge k^c$ of graded k-algebras shows that the socle of $k \otimes_R E$ is $k \otimes_R E_c$. As $k \otimes_R \gamma$ is a homomorphism of graded k-algebras, when $k \otimes_R \gamma_c$ is injective so is $k \otimes_R \gamma$; now use (1.4.2). □

The preceding description brings to light a connection between 1-closure for parameter ideals and Hochster's Canonical Element Conjecture, see [19].

1.6.2. Let R be a local ring. The following conditions are equivalent:

(1) The Canonical Element Conjecture holds for R.
(2) For each system of parameters $\boldsymbol{p}$ of R the map $\varphi\colon R \to R/(\boldsymbol{p})$ is 1-closed.

Indeed, Roberts [21] has proved that the Canonical Element Conjecture holds for R if and only if for each free resolution F of $R/(\boldsymbol{p})$ over R and each comparison morphism $\kappa\colon E \to F$, the induced map $(k \otimes_R E_c) \to \mathrm{H}_c(k \otimes_R F)$ is injective; another proof of his result is given by Huneke and Koh [17, (1.3)]. Thus, the desired equivalence is contained in (1.4.2) and Lemma (1.6.1).

The hypothesis on R in the next theorem reflects the use in its proof of a result of Bruns [10], which in turn relies on the Improved New Intersection Theorem.

Theorem 1.6.3. *Let $\varphi\colon R \to S$ be a 1-closed homomorphism and assume R contains a field as a subring. If $\operatorname{pd}_R S$ is finite, then φ is complete intersection.*

Proof. Set $I = \operatorname{Ker}\varphi$ and $c = \varepsilon_2(\varphi)$. The Koszul complex $E = G^{(1)}$ on a minimal generating set of I yields an injection $\kappa\colon k \otimes_R E \to \operatorname{Tor}^R(k, S)$, see (1.4.1).

Since $\operatorname{pd}_R S$ is finite, [10, Lemma 2] yields $\kappa_i = 0$ for $i > \operatorname{height} I$. By construction one has $\operatorname{rank}_R E_1 = c$, and this implies $c \leq \operatorname{height} I$. The reverse inequality always holds, due to the Principal Ideal Theorem, hence one gets $\operatorname{height} I = c$. On the other hand, $\operatorname{pd}_R S < \infty$ implies that $\operatorname{height} I$ equals the maximal length of an R-regular sequence in I, see [3, (2.5)]. We concude that I can be generated by an R-regular sequence, as desired. □

2. Bounds on homology

The main result of this section is a condition for a 2-closed homomorphism to be c.i. When φ admits a section and $M = S$ it specializes to a result of Rodicio, [22, Theorem 1]. The reason for dealing with complexes, rather than just with modules, will become apparent in the proof of Theorem (3.1).

Theorem 2.1. *Let $\varphi\colon R \to S$ be a 2-closed local homomorphism and M a complex of S-modules with $\operatorname{H}(M)$ degreewise finite and bounded below.*

If there exist integers $t, u \geq \inf \operatorname{H}(M)$ of different parity, such that

$$\operatorname{Tor}_t^R(S, M) = 0 = \operatorname{Tor}_u^R(S, M)$$

then the homomorphism φ is complete intersection.

We comment on notions and notation used in the theorem and its proof.

2.2. For definitions of Tor and Ext for complexes we refer to [25]. When their arguments are modules (modules are always identified with complexes concentrated in degree 0) these are the classical derived functors. We set

$$\inf \operatorname{H}(M) = \inf\{n \mid \operatorname{H}_n(M) \neq 0\}$$
$$\sup \operatorname{H}(M) = \sup\{n \mid \operatorname{H}_n(M) \neq 0\}$$

When $\inf \operatorname{H}(M)$ (respectively, $\sup \operatorname{H}(M)$) is finite we say that $\operatorname{H}(M)$ is *bounded below* (respectively, *above*). If $H(M)$ is bounded on either side, then $\operatorname{H}(M) \neq 0$, because $\operatorname{H}(M) = 0$ is equivalent to $\inf \operatorname{H}(M) = \infty$, and also to $\sup \operatorname{H}(M) = -\infty$.

For each integer j a complex $\Sigma^j M$ is defined by

$$\Sigma^j(M)_n = M_{n-j} \qquad \text{and} \qquad \partial_n^{\Sigma^j M} = (-1)^{|j|}\partial_{n-j}^M$$

Morphisms of complexes are chain maps of degree 0. A *quasiisomorphism* is a morphism that induces isomorphisms in homology in all degrees; we tag quasiisomorphisms with the symbol $\simeq$, and isomorphisms with $\cong$.

We deduce Theorem (2.1) from the following, much stronger, result.

Theorem 2.3. *Let $\varphi\colon R \to S$ be a 2-closed local homomorphism, set $c = \varepsilon_2(\varphi)$ and $d = \varepsilon_3(\varphi)$. If M is a complex of S-modules with $\mathrm{H}(M)$ degreewise finite and bounded below, then for $i = \inf \mathrm{H}(M)$ and $m = \nu_S(\mathrm{H}_i(M))$ one has inequalities*

$$\nu_S\big(\operatorname{Tor}^R_{n+i}(S, M)\big) \geq m \cdot \binom{c}{n} \qquad \textit{for} \quad 0 \leq n \leq c$$

$$\nu_S\big(\operatorname{Tor}^R_{2n+i+c}(S, M)\big) \geq m \cdot \binom{n+d-1}{d-1} \qquad \textit{for} \quad 1 \leq n$$

The proof uses a general lemma in homological algebra, presented below.

2.4. Let T be a covariant additive functor from the category of complexes of S-modules to the category of graded S-modules; for each complex M of S-modules we write $T_n(M)$ for the component in degree n of the graded S-module $T(M)$. Assume, furthermore, that T has the following properties:

(a) T preserves quasiisomorphisms.
(b) T commutes with shifts: $T_n(\Sigma^j M) = T_{n-j}(M)$ for each $n \in \mathbb{Z}$.
(c) $T(\mu^M_s) = \mu^{T(M)}_s$ for each $s \in S$, where μ_s denotes multiplication by s.

For the maximal ideal $\mathfrak{n}$ of S property (c) implies:

2.4.1. One has $\mathfrak{n} \cdot T(k) = 0$, so $T(k)$ is naturally a graded k-vector space.

Lemma 2.4.2. *Let $\varphi\colon R \to S$ be a surjective homomorphism and $\epsilon\colon S \to k$ the canonical surjection. If M is a complex of S-modules as in Theorem* (2.3)*, then*

$$\nu_S(T_{n+i}(M)) \geq m \cdot \operatorname{rank}_k \operatorname{Im}(T_n(\epsilon))$$

Proof. First we simplify M. The inclusion into M of the subcomplex

$$M' := \quad \cdots \longrightarrow M_{i+2} \xrightarrow{\partial_{i+2}} M_{i+1} \longrightarrow \operatorname{Ker}(\partial_i) \longrightarrow 0$$

is a quasiisomorphism. By (a) one has $T_n(M) \cong T_n(M')$, so we may assume $M = M'$. Set $H = \mathrm{H}_i(M)$, choose a surjection $H \to k^m$ and let π denote the composition $M \to \Sigma^i H \to \Sigma^i k^m$ of morphisms of complexes of S-modules. Lifting $\Sigma^i \epsilon^m$ over π to a morphism $\rho\colon \Sigma^i S^m \to M$, we get a commutative diagram

$$\begin{array}{ccccc}
 & & T_{n+i}(\Sigma^i S^m) & \xrightarrow{\cong} & T_{n+i}(\Sigma^i S)^m \\
 & \swarrow^{T_{n+i}(\rho)} & \big\downarrow{\scriptstyle T_{n+i}(\Sigma^i \epsilon^m)} & & \big\downarrow{\scriptstyle T_{n+i}(\Sigma^i \epsilon)^m} \\
T_{n+i}(M) & & & & \\
 & \searrow_{T_{n+i}(\pi)} & & & \\
 & & T_{n+i}(\Sigma^i k^m) & \xrightarrow{\cong} & T_{n+i}(\Sigma^i k)^m
\end{array}$$

of homomorphism of S-modules. We can now write the relations below

$$\begin{aligned}\nu_S(T_{n+i}(M)) &\geq \operatorname{rank}_k \operatorname{Im}(T_{n+i}(\pi)) \\ &\geq \operatorname{rank}_k \operatorname{Im}(T_{n+i}(\Sigma^i \epsilon^m)) \\ &= m \cdot \operatorname{rank}_k \operatorname{Im}(T_{n+i}(\Sigma^i \epsilon)) \\ &= m \cdot \operatorname{rank}_k \operatorname{Im}(T_n(\epsilon))\end{aligned}$$

by using consecutively the following facts: the maximal ideal of S annihilates $T_{n+i}(\Sigma^i k)$; the diagram commutes; $T_{n+i}(\Sigma^i k)$ is isomorphic to $T_n(k)$. □

We need an explicit description of a subcomplex of an acyclic closure G of φ.

2.5. In the notation of (1.1), each R-module $G_n^{(2)}$ has a basis

$$\left\{ x_{\boldsymbol{i}} y_{\boldsymbol{j}} \;\middle|\; \boldsymbol{i} \subseteq [1,c]\,, \boldsymbol{j} = (j_1, \dots, j_d) \in \mathbb{N}^d\,,\ \operatorname{card} \boldsymbol{i} + 2\sum_{h=1}^{d} j_h = n \right\}$$

where $[1,c] = \{1, \dots, c\}$. Let $\{a_1, \dots, a_c\}$ be a minimal set of generators of $\operatorname{Ker}\varphi$ and E the Koszul complex on it. The differential of $G^{(2)}$ then has the form

$$\partial\big(x_{\boldsymbol{i}} y_{\boldsymbol{j}}\big) = \sum_{i \in \boldsymbol{i}} \pm a_i\, x_{\boldsymbol{i} \setminus \{i\}} y_{\boldsymbol{j}} + \sum_{i \in [1,c]} \sum_{j=1}^{d} \pm b_{ij}\, x_{\boldsymbol{i} \cup \{i\}} y_{\boldsymbol{j} - \boldsymbol{e}_j}$$

where $\boldsymbol{e}_j \in \mathbb{N}^d$ is the jth unit vector, and

$$z_j = \sum_{i=1}^{c} b_{ij}\, x_i \in G_1^{(2)} \qquad \text{for} \qquad j = 1, \dots, c$$

are cycles whose homology classes minimally generate $\mathrm{H}_1(E)$.

All the coefficients a_i and b_{ij} are in $\mathfrak{m}$: this is clear for the a_i; as they minally generate the relation $0 = \partial(z_j) = \sum_{i=1}^{c} b_{ij}\, a_i$ implies $b_{ij} \in \mathfrak{m}$.

Proof of Theorem (2.3). Let $\epsilon\colon S \to k$ be the canonical surjection. Lemma (2.4.2) applied to the functor T defined by $T_n(M) = \operatorname{Tor}_n^R(S, M)$ yields

$$\nu_S(\operatorname{Tor}_{n+i}^R(S, M)) \geq m \cdot \operatorname{rank}_k \operatorname{Im}(\operatorname{Tor}_n^R(S, \epsilon))$$

Next we estimate the rank on the right hand side. Let $G^{(2)}$ be the second stage in an acyclic closure of φ, F a free resolution of S over R, and let $\gamma\colon G^{(2)} \to F$ be a comparison morphism. The following diagram commmutes:

$$\begin{array}{ccccc}
\mathrm{H}_n(G^{(2)} \otimes_R S) & \xrightarrow{\mathrm{H}_n(\gamma \otimes_R S)} & \mathrm{H}_n(F \otimes_R S) & = & \operatorname{Tor}_n^R(S, S) \\
\big\downarrow{\scriptstyle \mathrm{H}_n(G^{(2)} \otimes_R \epsilon)} & & \big\downarrow{\scriptstyle \mathrm{H}_n(F \otimes_R \epsilon)} & & \big\downarrow{\scriptstyle \operatorname{Tor}_n^R(S, \epsilon)} \\
\mathrm{H}_n(G^{(2)} \otimes_R k) & \xrightarrow{\mathrm{H}_n(\gamma \otimes_R k)} & \mathrm{H}_n(F \otimes_R k) & = & \operatorname{Tor}_n^R(S, k)
\end{array}$$

As $\mathrm{H}_n(\gamma \otimes_R k)$ is injective by (1.4.1), for each n we get

$$\operatorname{rank}_k(\operatorname{Im}\operatorname{Tor}_n^R(S,\epsilon)) \geq \operatorname{rank}_k \operatorname{Im}(\mathrm{H}_n(G^{(2)} \otimes_R \epsilon))$$

From the description of $G^{(2)}$ in (2.5) one sees that the graded submodule

$$Z = \bigoplus_{\boldsymbol{i} \subseteq [1,c]} S(x_{\boldsymbol{i}} y_0 \otimes 1) \oplus \bigoplus_{\boldsymbol{j} \in \mathbb{N}^d \setminus 0} S(x_{[1,c]} y_{\boldsymbol{j}} \otimes 1) \subseteq G^{(2)} \otimes_R S$$

consists of cycles and the differential of $G^{(2)} \otimes_R k$ is trivial; thus the composition

$$Z \otimes_S k \longrightarrow k \otimes_S \mathrm{H}(G^{(2)} \otimes_R S) \longrightarrow \mathrm{H}(G^{(2)} \otimes_R k) = G^{(2)} \otimes_R k$$

is injective. Counting ranks over k one obtains inequalities

$$\operatorname{rank}_k \operatorname{Im}(\mathrm{H}_n(G^{(2)} \otimes_R \epsilon)) \geq \binom{c}{n} \qquad \text{for} \quad 0 \leq n \leq c$$

$$\operatorname{rank}_k \operatorname{Im}(\mathrm{H}_{2n+c}(G^{(2)} \otimes_R \epsilon)) \geq \binom{n+d-1}{n} \qquad \text{for} \quad 1 \leq n$$

To get the desired result, concatenate the (in)equalities established above. □

Proof of Theorem (2.1). By hypothesis, one has $\operatorname{Tor}_t^R(S,M) = 0 = \operatorname{Tor}_u^R(S,M)$ for integers t,u satisfying $t,u \geq \inf(\mathrm{H}(M)) = i > -\infty$ and $t \not\equiv u \pmod 2$. The first inequality established in Theorem (2.3) implies $t,u > i+c$ for $c = \varepsilon_2(\varphi)$. For $d = \varepsilon_3(\varphi)$ the second inequality in the theorem then yields $\binom{n+d-1}{d-1} = 0$ for some $n \geq 1$, forcing $d = 0$. Thus, φ is complete intersection by (1.2.3). □

3. Vanishing of cohomology

In this section we provide a cohomological criterion for a 2-closed homomorphism to be c.i. It uses a notion of *depth* of a complex M, defined by

$$\operatorname{depth}_S M = \inf\{n \in \mathbb{Z} \mid \operatorname{Ext}_S^n(k,M) \neq 0\}$$

This is the classical concept when M is a finite S-module.

Theorem 3.1. *Let $\varphi\colon R \to S$ be a 2-closed homomorphism and M a complex of S-modules with $\mathrm{H}(M)$ degreewise finite and bounded above.*

If there exist integers $t,u \geq \operatorname{depth}_S M - \dim S$, of different parity, such that

$$\operatorname{Ext}_R^{t+n}(S,M) = 0 = \operatorname{Ext}_R^{u+n}(S,M) \quad \textit{for} \quad 0 \leq n \leq \max\{\dim_S \mathrm{H}_n(M) \mid n \in \mathbb{Z}\}$$

then the homomorphism φ is complete intersection.

Remark. As one always has $\dim S - \operatorname{depth}_S M \geq \sup \mathrm{H}(M)$, see [12, (2.11.3)], the bound on t,u in the theorem may be replaced by $t,u \geq -\sup \mathrm{H}(M)$.

Theorem (3.1) is a cohomological counterpart to Theorem (2.1), which provides a main ingredient in its proof. Another component is the use of properties of dualizing complexes, reviewed below; we refer to Hartshorne [15] for details.

3.2. Dualizing complexes. A *dualizing complex* for $(S, \mathfrak{n}, k)$ is a complex

$$D = \ 0 \to D_0 \to D_{-1} \to \cdots \to D_{-\dim S} \to 0$$

of injective modules with $\mathrm{H}(D)$ degreewise finite and $\mathrm{Hom}_S(k, D) \simeq \Sigma^{-\dim S} k$.

Up to a quasiisomorphism of complexes, S has at most one dualizing complex. Such a complex exists when the local ring S is complete.

For each complex of S-modules M we set $M^\dagger = \mathrm{Hom}_S(M, D)$.

3.2.1. If $\mathrm{H}(M)$ is degreewise finite, then so is $\mathrm{H}(M^\dagger)$.

3.2.2. If $\mathrm{H}(M)$ is bounded on one side, then $\mathrm{H}(M^\dagger)$ is bounded on the other.

Lemma 3.2.3. *If* $\mathrm{H}(M)$ *is degreewise finite and bounded above, then*

$$\inf \mathrm{H}(M^\dagger) = \mathrm{depth}_S M - \dim S$$

Proof. The complex $\mathrm{H}(M^\dagger)$ is degreewise finite and bounded below, see (3.2). This implies the first equality below; the second one holds by definition:

$$\inf \mathrm{H}(M^\dagger) = \inf \mathrm{H}(k \otimes^{\mathbf{L}}_S M^\dagger) = \inf \mathrm{H}(k \otimes^{\mathbf{L}}_S \mathrm{Hom}_S(M, D))$$

To compute the right hand side we use a sequence of quasiisomorphisms:

$$\begin{aligned} k \otimes^{\mathbf{L}}_S \mathrm{Hom}_S(M, D) &\simeq \mathrm{Hom}_S(\mathbf{R}\mathrm{Hom}_S(k, M), D) \\ &\simeq \mathrm{Hom}_S(\mathrm{Ext}_S(k, M), D) \\ &\cong \mathrm{Hom}_k(\mathrm{Ext}_S(k, M), \mathrm{Hom}_S(k, D)) \\ &\simeq \mathrm{Hom}_k(\mathrm{Ext}_S(k, M), \Sigma^{-\dim S} k) \\ &\simeq \Sigma^{-\dim S} \mathrm{Hom}_k(\mathrm{Ext}_S(k, M), k) \end{aligned}$$

The first one holds because k has a resolution by finite free S-modules and D is a bounded complex of injectives. For the second, note that $\mathbf{R}\mathrm{Hom}_S(k, M)$ can be represented by a complex of S-modules annihilated by $\mathfrak{n}$, so it is quasiisomorphic to its own homology, namely, $\mathrm{Ext}_S(k, M)$. The third one holds because $\mathrm{Ext}_S(k, M)$ is a direct sum of copies of shifts of k. The fourth quasiisomorphism is induced by $\mathrm{Hom}_S(k, D) \simeq \Sigma^{-\dim S} k$; see (3.2). The last one is standard.

We now finish the computation of $\inf \mathrm{H}(M^\dagger)$ as follows:

$$\begin{aligned} \inf \mathrm{H}(M^\dagger) &= \inf \left(\Sigma^{-\dim S} \mathrm{Hom}_k(\mathrm{Ext}_S(k, M), k)\right) \\ &= \inf \mathrm{Hom}_k(\mathrm{Ext}_S(k, M), k) - \dim S \\ &= \mathrm{depth}_S M - \dim S \end{aligned}$$

□

3.2.4. For every finite S-module N one has

$$\mathrm{Ext}^n_S(N, D) = 0\,, \quad \text{unless} \quad \dim S - \dim_S N \le n \le \dim S - \mathrm{depth}_S N$$

3.3. The *support* of a complex M is defined to be the set

$$\mathrm{Supp}_S M = \{\mathfrak{q} \in \mathrm{Spec}\, S \mid \mathrm{H}(M_{\mathfrak{q}}) = 0\}$$

Let $\dim \mathrm{Supp}_S M$ denote the dimension of the space $\mathrm{Supp}_S M$ in the Zariski topology on $\mathrm{Spec}\, S$. It is not hard to see that if $\mathrm{H}(M)$ is degreewise finite, then

$$\dim \mathrm{Supp}_S M = \max\{\dim_S \mathrm{H}_n(M) \mid n \in \mathbb{Z}\}$$

Proof of Theorem (3.1). By (1.4.3), the map $\widehat{\varphi}\colon \widehat{R} \to \widehat{S}$ of maximal-ideal-adic completions induced by φ is 2-closed. For $\widehat{M} = M \otimes_S \widehat{S}$ and each $n \in \mathbb{Z}$ one has

$$\mathrm{H}_n(\widehat{M}) \cong \mathrm{H}_n(M) \otimes_R \widehat{R} \qquad \text{and} \qquad \mathrm{Ext}^n_{\widehat{R}}(\widehat{S}, \widehat{M}) \cong \mathrm{Ext}^n_R(S, M) \otimes_R \widehat{R}$$

where the first isomorphism is due to the flatness of $\widehat{R}$ over R, while the second uses, in addition, that S is finite over R and that $\mathrm{H}(M)$ is bounded above. In particular, one has $\dim_{\widehat{S}} \mathrm{H}_n(\widehat{M}) = \dim_S \mathrm{H}_n(M)$ for each $n \in \mathbb{Z}$. Thus, the hypotheses of the theorem do not change when R, S, M are replaced by $\widehat{R}$, $\widehat{S}$, $\widehat{M}$, respectively. Furthermore, if $\widehat{\varphi}$ is c.i., then so is φ. Thus, we may assume that the ring S is complete, and hence that it has a dualizing complex D. Set $m = \max\{\dim_S \mathrm{H}_n(M) \mid n \in \mathbb{Z}\}$.

As D is a bounded complex of injectives, there is a natural quasiisomorphism

$$(*) \qquad \mathrm{Hom}_S(\mathbf{R}\mathrm{Hom}_R(S, M), D) \simeq S \otimes^{\mathbf{L}}_R \mathrm{Hom}_S(M, D)$$

The composition of the functors on the left gives rise to a spectral sequence with

$${}_2\mathrm{E}^{p,q} = \mathrm{Ext}^{-p}_S(\mathrm{Ext}^q_R(S, M), D) \qquad \text{and} \qquad {}_r\mathrm{d}^{p,q}\colon {}_r\mathrm{E}^{p,q} \longrightarrow {}_r\mathrm{E}^{p-r,q+r-1}$$

As the R-module S is finite, one has $\mathrm{Ext}^q_R(S, M)_{\mathfrak{q}} \cong \mathrm{Ext}^q_R(S, M_{\mathfrak{q}})$ for each $\mathfrak{q} \in \operatorname{Spec} S$, so $\operatorname{Supp}_S \mathrm{Ext}^q_R(S, M) \subseteq \operatorname{Supp}_S M$, for each $q \in \mathbb{Z}$. Thus, one gets

$$\dim_{S_{\mathfrak{q}}}(\mathrm{Ext}^q_R(S, M)_{\mathfrak{q}}) \le \dim \operatorname{Supp}_S M = m$$

where the equality comes from (3.3). Now (3.2.4) yields

$${}_2\mathrm{E}^{p,q} = 0 \quad \text{for} \quad p \notin [-\dim S, -\dim S + m]$$

so the sequence converges. Formula $(*)$ shows that its abutment is equal to

$$\mathrm{H}(S \otimes^{\mathbf{L}}_R \mathrm{Hom}_S(M, D)) = \mathrm{Tor}^R(S, M^\dagger)$$

On the other hand, our hypothesis entails ${}_2\mathrm{E}^{p,q} = 0$ for

$$t \le q \le t + m \quad \text{and} \quad u \le q \le u + m$$

As a consequence, one obtains equalities

$${}_2\mathrm{E}^{p,q} = 0 \quad \text{whenever} \quad p + q = t \quad \text{or} \quad p + q = u$$

They imply ${}_\infty\mathrm{E}^{p,q} = 0$ if $p + q = t$ or $p + q = u$, so convergence yields

$$\mathrm{Tor}^R_t(S, M^\dagger) = 0 = \mathrm{Tor}^R_u(S, M^\dagger)$$

In view of Lemma (3.2.3) and our hypothesis, the complex $M^\dagger$ satisfies

$$\inf \mathrm{H}(M^\dagger) = \operatorname{depth}_S M^\dagger - \dim S \le \min\{t, u\}$$

Now Theorem (2.1), applied to $M^\dagger$, shows that φ is complete intersection. $\square$

4. (Co)homology of algebra retracts

Let $\varphi\colon R \to S$ be a homomorphism of noetherian rings.

A *section* of φ is a homomorphism of rings $\psi\colon S \to R$ such that $\psi \circ \varphi = \mathrm{id}^S$; when such a homomorphism exits S is said to be an *algebra retract* of R. Another way to describe this situation is to say that R is a *supplemented* algebra over S. The study of homological and cohomological properties of supplemented algebras is a central topic in the classical literature on homological algebra.

Each $\mathfrak{q} \in \operatorname{Spec} S$ defines a local homomorphism $\varphi_{\mathfrak{q}}\colon R_{\varphi^{-1}(\mathfrak{q})} \to S_{\mathfrak{q}}$. If ψ is a section of φ, the local homomorphism $\psi_{\mathfrak{p}}$, where $\mathfrak{p} = \varphi^{-1}(\mathfrak{q})$, is a section of $\varphi_{\mathfrak{q}}$. In particular, the homomorphism $\varphi_{\mathfrak{q}}$ is 2-closed; see (1.3.3).

Next we establish global versions of results from the preceding sections. To this end we recall the construction of certain canonical homomorphisms.

4.1. With $I = \operatorname{Ker}(\varphi)$, one has a canonical isomorphism of S-modules

$$I/I^2 \cong \operatorname{Tor}_1^R(S,S)$$

The graded S-module $\operatorname{Tor}_*^R(S,S)$ has a natural structure of a strictly commutative graded S-algebra, see [25, (2.7.8)], so there is a homomorphism of graded S-algebras: $\lambda^S\colon \wedge_S (I/I^2) \to \operatorname{Tor}^R(S,S)$. Define λ^M to be the composition

$$\wedge_S(I/I^2) \otimes_S \mathrm{H}(M) \xrightarrow{\lambda^S \otimes_S \mathrm{H}(M)} \operatorname{Tor}^R(S,S) \otimes_S \mathrm{H}(M) \longrightarrow \operatorname{Tor}^R(S,M)$$

where the second arrow is a Künneth map. Let λ_M denote the composition

$$\operatorname{Ext}_R(S,M) \longrightarrow \operatorname{Hom}_S(\operatorname{Tor}^R(S,S),\mathrm{H}(M)) \longrightarrow \operatorname{Hom}_S(\wedge_S(I/I^2),\mathrm{H}(M))$$

where the first arrow is a Künneth map, and the second is $\operatorname{Hom}_S(\lambda^S,\mathrm{H}(M))$

Theorem 4.2. *Let $\varphi\colon R \to S$ be a homomorphism of rings that admits a section, and let M be a complex of S-modules with $\mathrm{H}(M)$ finite. Set $I = \operatorname{Ker}(\varphi)$.*

For each prime ideal $\mathfrak{q} \in \operatorname{Supp}_S M$ the following conditions are equivalent.

(i) *The homomorphism $\varphi_{\mathfrak{q}}$ is complete intersection.*

(ii$_*$) *The map $(\lambda^M)_{\mathfrak{q}}$ is bijective and the $S_{\mathfrak{q}}$-module $(I/I^2)_{\mathfrak{q}}$ is projective.*

(iii$_*$) *For integers $t,u \geq \inf \mathrm{H}(M)_{\mathfrak{q}}$ of different parity one has*

$$\operatorname{Tor}_t^R(S,M)_{\mathfrak{q}} = 0 = \operatorname{Tor}_u^R(S,M)_{\mathfrak{q}}$$

(ii*) *The map $(\lambda_M)_{\mathfrak{q}}$ is bijective.*

(iii*) *For integers $t,u \geq \operatorname{depth}_{S_{\mathfrak{q}}} M_{\mathfrak{q}} - \dim S_{\mathfrak{q}}$ of different parity one has*

$$\operatorname{Ext}_R^{t+i}(S,M)_{\mathfrak{q}} = 0 = \operatorname{Ext}_R^{u+i}(S,M)_{\mathfrak{q}} \quad \textit{for} \quad i = 0,\dots,\dim \operatorname{Supp}_{S_{\mathfrak{q}}} M_{\mathfrak{q}}$$

Proof. Set $\mathfrak{p} = \varphi^{-1}(\mathfrak{q})$. The R-module S is finite, so for each $n \in \mathbb{Z}$ one has

$$\operatorname{Tor}_n^R(S,M)_{\mathfrak{q}} \cong \operatorname{Tor}_n^{R_{\mathfrak{p}}}(S_{\mathfrak{q}},M_{\mathfrak{q}}) \quad \text{and} \quad \operatorname{Ext}_R^n(S,M)_{\mathfrak{q}} \cong \operatorname{Ext}_{R_{\mathfrak{p}}}^n(S_{\mathfrak{q}},M_{\mathfrak{q}})$$

Therefore, each of the conditions listed above is local; moreover, any section of φ localizes to a section of $\varphi_{\mathfrak{q}}$. Thus, changing notation we may assume that φ is a local homomorphism and that $\mathfrak{q}$ is the maximal ideal of S.

(i) $\implies$ (ii$_*$) and (ii*). Let E be the Koszul complex on a minimal generating set of I. It satisfies $\partial(E) \subseteq IE$ and is a free resolution of S, as φ is c.i. We get

$$\operatorname{Tor}^R(S,S) \cong \operatorname{H}(E \otimes_R S) = E \otimes_R S$$

so the S-module $\operatorname{Tor}_1^R(S,S)$ is free and λ^S is bijective. Thus, $\operatorname{Tor}_n^R(S,S)$ is finite free and vanishes for $n < 0$ or $n > \nu_S(I)$, so the Künneth homomorphisms

$$\operatorname{Tor}^R(S,S) \otimes_S \operatorname{H}(M) \longrightarrow \operatorname{Tor}^R(S,M)$$
$$\operatorname{Ext}_R(S,M) \longrightarrow \operatorname{Hom}_S(\operatorname{Tor}^R(S,S), \operatorname{H}(M))$$

are bijective. The definitions of λ^M and λ_M show that they are bijective as well.

(ii$_*$) $\implies$ (iii$_*$), and (ii*) $\implies$ (iii*). These implications are clear because the S-module I/I^2 is finite and $\operatorname{H}(M)$ is bounded.

(iii$_*$) or (iii*) $\implies$ (i). As φ is closed by (1.3.3), Theorem (2.1), respectively, Theorem (3.1), shows that condition (iii$_*$), respectively, (iii*), implies φ is c.i. □

5. Hochschild (co)homology

Finally, we return to the subject in the title of this article. First we recall a classical interpretation of the functors in question; see e.g. [25, (9.1.5)].

5.1. Let $\eta\colon K \to S$ be a homomorphism of rings and let $\varphi^S\colon S \otimes_K S \to S$ be the homomorphism of rings given by $\varphi^S(s \otimes_K s') = ss'$.

If the K-module S is flat, then for each $n \in \mathbb{Z}$ one has

$$\operatorname{HH}_n(S|K;M) = \operatorname{Tor}_n^{S\otimes_K S}(S,M)$$

Note that I/I^2, where $I = \operatorname{Ker}(\varphi^S)$, is a standard realization of the module of differentials $\Omega_{S|K}$. The map λ^M from (4.1) yields S-linear maps

$$\lambda_n^M\colon\ \wedge_S^n \Omega_{S|K} \otimes_S M \longrightarrow \operatorname{HH}_n(S|K;M)$$

If S is projective as a K-module, then also

$$\operatorname{HH}^n(S|K;M) = \operatorname{Ext}^n_{S\otimes_K S}(S,M)$$

so in this context the homomorphism λ_M from (4.1) reads

$$\lambda^n_M\colon\ \operatorname{HH}^n(S|K;M) \longrightarrow \operatorname{Hom}_S(\wedge^n_S \Omega_{S|K}, M)$$

The maps above are the homomorphisms that appear in the introduction. For the proof of the theorem stated there we need a characterization of smoothness proved by André [1, Proposition C], using André-Quillen homology. A short version of his argument may be found in [4, (1.1)].

5.2. A flat algebra S essentially of finite type over a noetherian ring K is smooth if and only if the homomorphism $(\varphi^S)_{\mathfrak{p}}$ is c.i., for each $\mathfrak{p} \in \operatorname{Spec} S$.

Proof of the Main Theorem. Let $\varphi^S\colon S\otimes_K S\to S$ be the product map. We claim that, for a given $\mathfrak{q}\in\operatorname{Spec} S$, condition (i): the K-algebra $S_\mathfrak{q}$ is smooth, is equivalent to: (i′) the homomorphism $(\varphi^S)_\mathfrak{q}\colon (S\otimes_K S)_{(\varphi^S)^{-1}(\mathfrak{q})}\to S_\mathfrak{q}$ is c.i.

Indeed, $(\varphi^S)_\mathfrak{q}$ is surjective, so it is c.i. if and only if $(\varphi^S)_\mathfrak{p}$ is c.i. for each $\mathfrak{p}\subseteq\mathfrak{q}$. However, the local homomorphisms $(\varphi^S)_\mathfrak{p}$ and $(\varphi^{S_\mathfrak{q}})_\mathfrak{p}$ coincide, and the latter is c.i. for each $\mathfrak{p}$ precisely when the K-algebra $S_\mathfrak{q}$ is smooth, by (5.2).

Given this translation and the identifications in (5.1), the desired result is contained in Theorem (4.2), for $s\mapsto 1\otimes s$ gives a section $S\to S\otimes_K S$ of φ^S. □

The example below shows that condition (ii$_*$) in the Main Theorem cannot be weakened in general. We do not know whether the conclusion of the theorem still holds if the vanishing intervals in condition (iii*) are shortened.

Example 5.3. Let $S=\mathbb{Z}[\sqrt{2}]$. The Hochschild homology of S over $\mathbb{Z}$ is

$$\operatorname{HH}_n(S|\mathbb{Z};S)=\begin{cases} S & \text{for } n=0\\ S/(2\sqrt{2}) & \text{for odd } n\geq 1\\ 0 & \text{for even } n\geq 2\end{cases}$$

while the Hochschild cohomology of S over $\mathbb{Z}$ is given by

$$\operatorname{HH}^n(S|\mathbb{Z};S)=\begin{cases} S & \text{for } n=0\\ 0 & \text{for odd } n\geq 1\\ S/(2\sqrt{2}) & \text{for even } n\geq 2\end{cases}$$

Indeed, $\operatorname{Ker}(S\otimes_\mathbb{Z} S\to S)$ is generated by $\sqrt{2}\otimes 1-1\otimes\sqrt{2}$. A free resolution of S as a module over $S\otimes_\mathbb{Z} S$ is given by the complex F below:

$$\cdots\to S\otimes_\mathbb{Z} S\xrightarrow{\sqrt{2}\otimes 1-1\otimes\sqrt{2}} S\otimes_\mathbb{Z} S\xrightarrow{\sqrt{2}\otimes 1+1\otimes\sqrt{2}} S\otimes_\mathbb{Z} S\xrightarrow{\sqrt{2}\otimes 1-1\otimes\sqrt{2}} S\otimes_\mathbb{Z} S\to 0$$

As S is finite free as a $\mathbb{Z}$-module, $\operatorname{HH}_*(S|\mathbb{Z};S)$ is the homology of the complex

$$F\otimes_{S\otimes_\mathbb{Z} S} S=\quad\cdots\longrightarrow S\xrightarrow{2\sqrt{2}} S\xrightarrow{0} S\xrightarrow{2\sqrt{2}} S\xrightarrow{0} S\longrightarrow 0$$

and $\operatorname{HH}^*(S|\mathbb{Z};S)$ is the homology of the complex

$$\operatorname{Hom}_{S\otimes_\mathbb{Z} S}(F,S)=\quad 0\longrightarrow S\xrightarrow{0} S\xrightarrow{2\sqrt{2}} S\xrightarrow{0} S\xrightarrow{2\sqrt{2}} S\longrightarrow\cdots$$

see (5.1). The desired expressions follow.

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