EXOTIC SMOOTH STRUCTURES ON $\mathbb{CP}^2\#5\overline{\mathbb{CP}^2}$

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Abstract. Motivated by a construction of Fintushel and Stern, we show that the topological 4–manifold $\mathbb{CP}^2\#5\overline{\mathbb{CP}^2}$ supports infinitely many distinct smooth structures.

1. Introduction

It is a basic problem in 4–dimensional topology to find exotic smooth structures on rational surfaces. The first such structures were found by Donaldson [4]; these examples were homeomorphic to $\mathbb{CP}^2\#9\overline{\mathbb{CP}^2}$. While in this homeomorphism type many exotic examples were constructed [6, 9, 18], the cases of $\mathbb{CP}^2\#k\overline{\mathbb{CP}^2}$ with $k < 9$ were more elusive. The Barlow surface [1] provided the first exotic structure on $\mathbb{CP}^2\#8\overline{\mathbb{CP}^2}$, see [13]. More recently, an exotic smooth structure on $\mathbb{CP}^2\#7\overline{\mathbb{CP}^2}$ has been constructed [15]. After this example many new exotic 4–manifolds with small Euler characteristic have been found. In [16] symplectic 4–manifolds homeomorphic but not diffeomorphic to $\mathbb{CP}^2\#6\overline{\mathbb{CP}^2}$ were constructed, implying the existence of an exotic smooth structure on $\mathbb{CP}^2\#6\overline{\mathbb{CP}^2}$. In a beautiful recent paper [7] Fintushel and Stern showed the existence of infinitely many distinct smooth structures on $\mathbb{CP}^2\#k\overline{\mathbb{CP}^2}$ with $k = 6, 7, 8$. Combining their technique of knot surgery in a double node neighborhood with a particular form of generalized rational blow–down, in this note we prove

Theorem 1.1. *There exist infinitely many pairwise nondiffeomorphic 4–manifolds all homeomorphic to $\mathbb{CP}^2\#5\overline{\mathbb{CP}^2}$.*

In Section 2 various constructions of 4–manifolds homeomorphic to $\mathbb{CP}^2\#5\overline{\mathbb{CP}^2}$ are described. In Section 3 we use Seiberg–Witten theory to show that many of these examples are mutually nondiffeomorphic, leading us to the proof of Theorem 1.1.

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2. The constructions

We construct our examples using knot surgery (in a double node neighborhood, as in [7]) when applied to particular elliptic fibrations. The special properties of the chosen elliptic fibration allow us to find a configuration in the result of the knot surgery such that after rationally blowing it down we arrive to a 4–manifold homeomorphic, but not diffeomorphic to $\mathbb{CP}^2 \# 5\overline{\mathbb{CP}^2}$. By using a suitable infinite set of knots (the twist knots already encountered in [7], cf. also [6, 18]), we get an infinite family of 4–manifolds all homeomorphic to $\mathbb{CP}^2 \# 5\overline{\mathbb{CP}^2}$.

2.1. Elliptic fibrations. Singular fibers of holomorphic elliptic fibrations have been classified [12] (cf. also [11]). In this note we will consider fibrations containing only singular fibers of type I_n $(n \geq 1)$. Recall that the singular fiber I_1 (also known as the *fishtail* fiber) is an immersed 2–sphere with one positive double point, and it is created from a regular torus fiber by collapsing a homologically essential simple closed curve (the *vanishing cycle* of the singular fiber). The I_n–fiber $(n \geq 2)$ is a collection of n 2–spheres of self–intersection (-2), intersecting each other in a circular pattern, see [11, page 35]. An elliptic fibration with singular fibers only of type I_n are *Lefschetz fibrations* in the sense of [10, Chapter 8]. The only subtlety we have to keep in mind is that here we allow a singular fiber to contain more than one singular points as well.

Lefschetz fibrations can be conveniently described by the monodromy factorization induced by the singular fibers of the fibration, that is, by a word involving right–handed Dehn twists which is equal to 1 in the mapping class group of the regular fiber. The mapping class group Γ_1 of the 2–torus T^2 can be presented as

$$\Gamma_1 = \{a, b \mid aba = bab, (ab)^6 = 1\},$$

where $a, b \in \Gamma_1$ denote the right–handed Dehn twists along the two standard simple closed curves A, B in T^2 intersecting each other transversally in a unique point. This group can identified with $\mathrm{SL}(2; \mathbb{Z})$ by mapping a to $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and b to $\left(\begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix}\right)$. For example, the standard elliptic fibration we get by blowing up nine base points of a generic elliptic pencil in $\mathbb{CP}^2$ results the monodromy factorization $(ab)^6$. Using the braid relation $aba = bab$ it can be shown that $(a^3b)^3$ also defines an elliptic fibration on $\mathbb{CP}^2 \# 9\overline{\mathbb{CP}^2}$. Furthermore, it is easy to see that for any expression $x \in \Gamma_1$ the mapping class $a^x = xax^{-1}$ can be identified with the right–handed Dehn twist along the image of a under a map giving x. Note, for example, that the braid relation implies that $b = a^{ab}$.

The monodromy of a fishtail fiber can be shown to be equal to the right–handed Dehn twist along the vanishing cycle corresponding to the given singular

fiber. An I_n–fiber can be created by collapsing n parallel (homologically essential) simple closed curves, therefore the monodromy of such a fiber is equal to the nth power of the right–handed Dehn twist along one of the parallel curves.

In our constructions we will need the existence of a section, which can also be read off from the monodromy factorization. In general, a Lefschetz fibration admits a section if the monodromy factorization induced by it can be lifted from the mapping class group of its generic fiber to the mapping class group of the fiber with one marked point. In the case of a genus–1 Lefschetz fibration, however, the forgetful map $f\colon \Gamma_1^1 \to \Gamma_1$ mapping from the mapping class group Γ_1^1 of T^2 with one marked point to Γ_1 is an isomorphism, implying in particular

Lemma 2.1. *Any genus–1 Lefschetz fibration over S^2 admits a section.* □

2.2. The definition of the 4–manifold X_n. Our first construction of exotic 4–manifolds relies on the following existence result. (For a schematic picture of the fibration see Figure 1.)
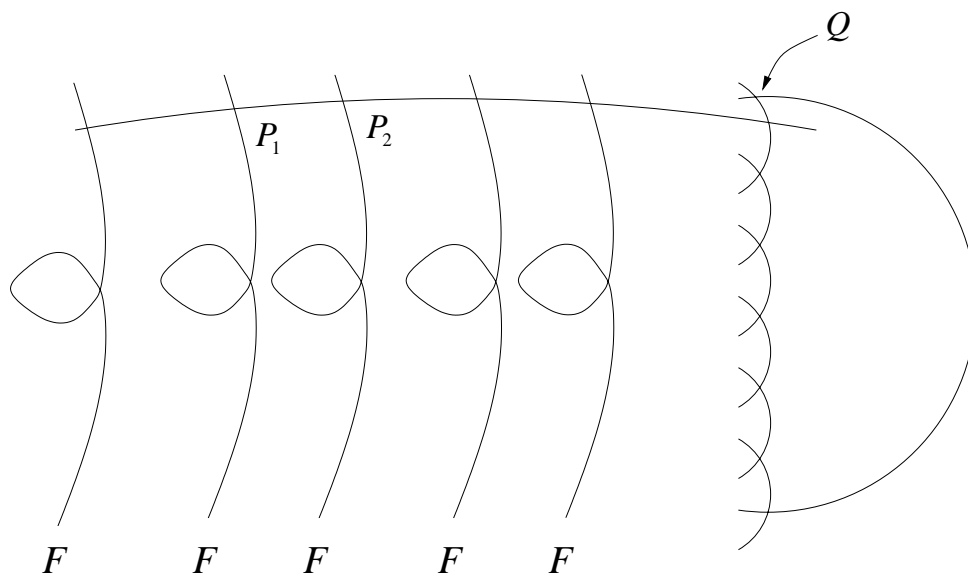

FIGURE 1.   The schematic diagram of the fibration with an I_7 fiber

Proposition 2.2. *There exists an elliptic fibration $\mathbb{CP}^2 \# 9\overline{\mathbb{CP}^2} \to \mathbb{CP}^1$ with five fishtail fibers, an I_7–fiber and a section. Furthermore, we can assume that two of the five fishtail fibers have isotopic vanishing cycles.*

Proof. We will show the existence of such fibration by finding an appropriate factorization of 1 in the mapping class group Γ_1 of the torus. Start with the fibration on $\mathbb{CP}^2 \# 9\overline{\mathbb{CP}^2}$ defined by the factorization

$$(a^3 b)^3$$

of $1 \in \Gamma_1$. Notice that

$$(a^3 b)^3 = a^7 (a^{-1}(a^{-3}ba^3)a)(a^{-1}ba)a^2 b = a^7 b^{a^{-4}} b^{a^{-1}} a^2 b.$$

Since a^7 is the monodromy of an I_7–fiber, its existence in the above fibration is verified. The term a^2 gives rise to two fishtail fibers with isotopic vanishing cycles in the complement of the I_7–fiber. Finally, Lemma 2.1 shows the existence of a section in the fibration. $\qquad\square$

Suppose now that $p > q > 0$ are relatively prime integers. Let us define the 4–manifold $C_{p,q}$ as the result of the linear plumbing with weights specified by the continued fraction coefficients of $-\frac{p^2}{pq-1}$. It is known [2] (cf. also [5, 14, 16, 17]) that the boundary $\partial C_{p,q} = L(p^2, pq-1)$ is a lens space which bounds a rational ball $B_{p,q}$. The replacement of an embedded copy of $C_{p,q} \subset X$ with $B_{p,q}$ is called the *(generalized) rational blow–down* of X along $C_{p,q}$. This operation was introduced and successfully applied by Fintushel and Stern [5] in the case of $q = 1$ and studied in [14, 17] in the above generality.

Now we are ready to turn to the construction of the 4–manifolds homeomorphic but not diffeomorphic to $\mathbb{CP}^2 \# 5\overline{\mathbb{CP}^2}$. Let K_n denote the n–twist knot as it is depicted in [7]. Let F_3, F_4 of Figure 1 denote the fishtail fibers with isotopic vanishing cycles. Following the convention of [7] we denote the result of the knot surgery in a double node neighborhood containing F_3, F_4 and with knot K_n by Y_n. Fintushel and Stern [7] prove the existence of a "pseudo–section" $S \subset Y_n$ which is an immersed sphere with one positive double point, homological self–intersection -1, and which transversally intersects F_1, F_2 and one of the spheres in the I_7–fiber: The section of the fibration, punctured by the fiber along which the knot surgery is performed, can be glued to the genus–1 Seifert surface of the knot K_n. In this way an embedded torus T of self–intersection -1 is found in Y_n. Using the two thimbles of the isotopic vanishing cycles, Fintushel and Stern find a disk U attached to T with relative self–intersection -1. From T and U now it is an easy task to find the immersed sphere with a positive double point and which is homologous to T. For more details of the construction see [7].

Let us blow up Y_n in the double point of the pseudo–section, and in the double points of the fishtail fibers F_1 and F_2. After smoothing the intersections P_1, P_2, we get a sphere of self–intersection -9 intersecting the I_7–fiber transversally at one point. Now we apply eight infinitely close blow–ups at the point Q as it is shown by Figure 2. This construction results in a chain of 2–spheres, with a neighborhood diffeomorphic to the 4–manifold C we get by plumbing along a linear chain with weights

$$(-9, -10, -2, -2, -2, -2, -2, -3, -2, -2, -2, -2, -2, -2, -2)$$

in the eleven–fold blow–up of Y_n. Simple computation identifies C with $C_{71,8}$. Define X_n as the (generalized) rational blow–down of Y_n along C, that is,

$$X_n = (Y_n \# 11\overline{\mathbb{CP}^2} - \text{ int } C) \cup B_{71,8}.$$

Theorem 2.3. X_n *is homeomorphic to* $\mathbb{CP}^2 \# 5\overline{\mathbb{CP}^2}$.
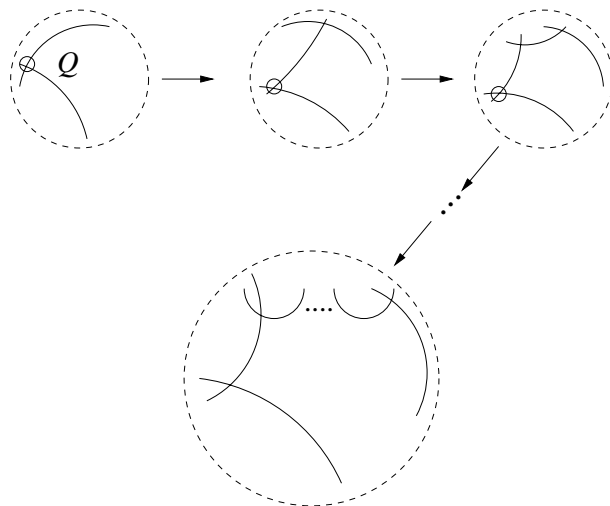
FIGURE 2.   Infinitely close blow–ups at Q

Proof. The 4–manifold Y_n has trivial fundamental group, since the fibration admits a section and two different vanishing cycles in the complement of the double node neighborhood. Simple connectivity of X_n follows from the fact that the complement of C in $Y_n \# 11\overline{\mathbb{CP}^2}$ is simply connected, since the generator of $\pi_1(\partial C)$ can be contracted along the fishtail fiber F_0 present in the fibration but not used in constructing the configuration C and from the surjectivity of the natural map $\pi_1(\partial B) \to \pi_1(B)$. Now simple Euler characteristic and signature computation together with Freedman's Theorem on the classification of topological 4–manifolds [8] imply the result. $\square$

2.3. Further constructions. Many similar constructions can be carried out using different elliptic fibrations or different sets of knots. Below we outline constructions relying on various types of elliptic fibrations.

2.3.1. *Another construction using the I_7–fiber.* A similar argument provides an embedding of $C_{212,55}$ into $Y_n \# 12\overline{\mathbb{CP}^2}$ by smoothing only at P_2 and keeping the transverse intersection P_1. In this case one further blow–up of a (-2)–sphere is necessary, leading to the chain

$$(-4, -7, -10, -2, -2, -2, -2, -2, -3, -2, -2, -2, -2, -3, -2, -2)$$

in $Y_n \# 12\overline{\mathbb{CP}^2}$. Blowing this configuration down we get a sequence of 4–manifolds with the same properties as X_n. (The hemisphere originated from the exceptional sphere of the last blow–up can be used to show that the resulting configuration of spheres in the twelve–fold blow–up of Y_n has simply connected complement.)

2.3.2. *Configurations using the I_8–fiber.* Many other examples can be given using the I_8–fiber. To see the existence of the required fibration, we need a result similar to Proposition 2.2.

Proposition 2.4. *There exists an elliptic fibration $\mathbb{CP}^2 \# 9\overline{\mathbb{CP}^2} \to \mathbb{CP}^1$ with four fishtail fibers, an I_8–fiber and a section. Furthermore, we can assume that two of the four fishtail fibers have isotopic vanishing cycles.*

Proof. Using the braid relation it is fairly easy to see that the expression

$$a^3 b a^2 b^2 a^2 b a$$

is equal to 1 in Γ_1^1, hence defines an elliptic fibration with a section. Since it can be written as

$$a^8 (b^{a^{-2}}) b^2 (b^{a^2}),$$

the resulting fibration can be chosen to have an I_8–fiber and two fishtails in its complement with isotopic vanishing cycles.　　　　□

Our further constructions rely on

Proposition 2.5. *Let Y_n be the 4–manifold defined above.*

(1) *The 4–manifold $C_{44,9}$ embeds into $Y_n \# 8\overline{\mathbb{CP}^2}$;*
(2) *$C_{79,44}$ admits an embedding into $Y_n \# 9\overline{\mathbb{CP}^2}$;*
(3) *$C_{89,9}$ embeds into $Y_n \# 13\overline{\mathbb{CP}^2}$;*
(4) *$C_{169,89}$ can be embedded into $Y_n \# 14\overline{\mathbb{CP}^2}$;*
(5) *$C_{301,62}$ admits an embedding into $Y_n \# 14\overline{\mathbb{CP}^2}$; and finally*
(6) *$C_{540,301}$ is a submanifold of $Y_n \# 15\overline{\mathbb{CP}^2}$.*

The complements of these configurations are simply connected.

Remark 2.6. Recall that the above 4–manifolds can be given by the linear plumbings as follows:

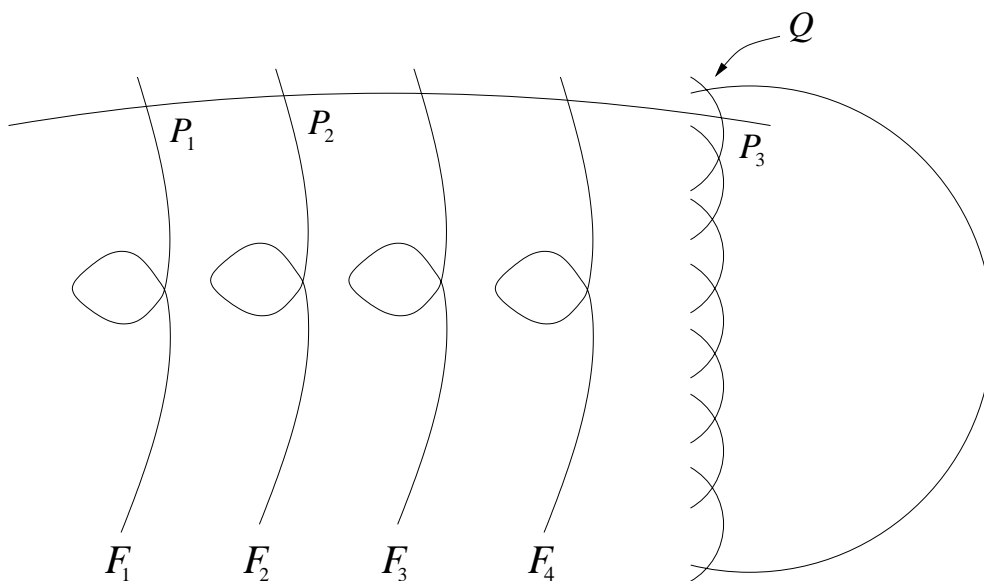$$C_{44,9} = (-5, -11, -2, -2, -2, -2, -2, -2, -3, -2, -2, -2),$$
$$C_{79,44} = (-2, -5, -11, -2, -2, -2, -2, -2, -2, -3, -2, -2, -3),$$

$$C_{89,9} = (-10, -11, -2, -2, -2, -2, -2, -2, -3, -2, -2, -2, -2, -2,$$
$$-2, -2, -2),$$

$$C_{169,89} = (-2, -10, -11, -2, -2, -2, -2, -2, -2, -3, -2, -2, -2, -2, -2,$$
$$-2, -2, -3),$$

$$C_{301,62} = (-5, -7, -11, -2, -2, -2, -2, -2, -2, -3, -2, -2, -2, -2, -3,$$
$$-2, -2, -2),$$

$$C_{540,301} = (-2, -5, -7, -11, -2, -2, -2, -2, -2, -2, -3, -2, -2, -2, -2, -3,$$
$$-2, -2, -3).$$

FIGURE 3.   A fibration with an I_8–fiber

Proof. We use the configuration of Figure 3 to indicate the embeddings given above. First of all, perform the knot surgery in the double node neighborhood of the fishtail fibers F_3, F_4 with isotopic vanishing cycles and blow up the two double points of the remaining two fishtail fibers F_1, F_2 together with the double point of the pseudo–section. To get the first embedding, smooth the transverse intersections P_2, P_3 and apply four infinitely close blow–ups at Q, resulting the configuration

$$(-4, -11, -2, -2, -2, -2, -2, -2, -3, -2, -2, -2).$$

One further blow–up of the (-4)–sphere provides the first embedding. If we blow up this sphere as instructed by Figure 4, a final blow–up of the last (-2)–sphere in the chain gives the second embedding.

    If we smooth the intersections P_1 and P_2 then eight infinitely close blow–ups at Q, together with a final blow–up on any of the former fishtail fibers F_1 or F_2 results the third embedding. Once again, the last blow–up can be performed as in Figure 4, in which case we need to blow up the other end of the chain, resulting the fourth embedding. Finally, resolving only P_2, eight infinitely close blow–ups at Q, one further blow–up on the appropriate (-2)–sphere in the I_8–fiber and one more on the fishtail passing through P_1 gives the fifth configuration. If this last blow–up is performed as in Figure 4, by blowing up the last (-2)–sphere of the configuration we get the last promised embedding. Since in any of the above constructions the last blow–up provides an exceptional
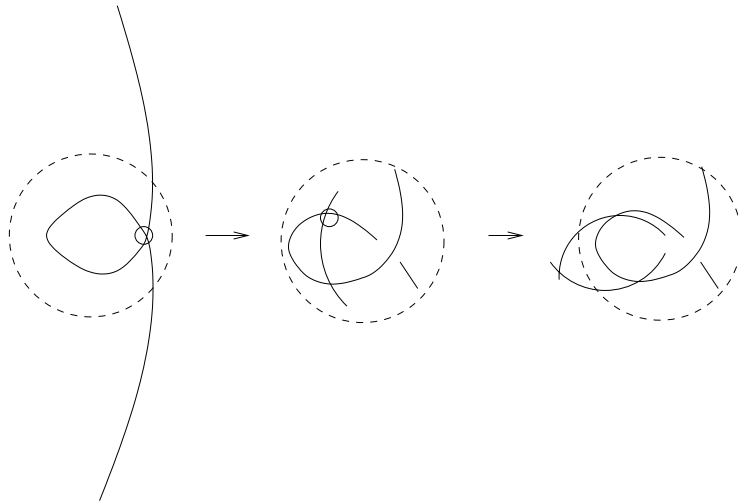
FIGURE 4.   Infinitely close blow–ups at the double point of
the fishtail fiber

divisor transversally intersecting the first or last sphere of the configuration, the complements of the configurations are obviously simply connected.   □

Simple Euler characteristic computation and Freedman's Theorem imply that after rationally blowing down any of the configurations presented in Proposition 2.5 we get further interesting examples of 4–manifolds homeomorphic to $\mathbb{CP}^2 \# 5\overline{\mathbb{CP}^2}$.

2.3.3. *A configuration using the I_6–fiber.* A slightly different procedure can be applied if we start with a fibration containing an I_6–fiber and two pairs of fishtail fibers with isotopic vanishing cycles. This example was also discovered independently by Fintushel and Stern [7].

Proposition 2.7. *There is an elliptic fibration $\mathbb{CP}^2 \# 9\overline{\mathbb{CP}^2} \to \mathbb{CP}^1$ with an I_6–fiber, six fishtail fibers $F_1, \ldots, F_6$ and a section. Furthermore we can assume that the vanishing cycles of F_1 and F_2 are isotopic, and the vanishing cycles of F_3 and F_4 are isotopic.*

Proof. Start again with the fibration given by the relation $(a^3b)^3$ and notice that it is equal to $a^6 b^{a^{-3}} b^2 (a^{b^{-1}})^3$. This expression shows the existence of the required fibration.   □

Consider the 4–manifold $V_{K_{n_1}, K_{n_2}}$ we get from $\mathbb{CP}^2 \# 9\overline{\mathbb{CP}^2}$ by doing two double node surgeries in the neighborhoods of F_1, F_2 and F_3, F_4 respectively, using the knot K_{n_1} for the first and K_{n_2} for the second surgery. By choosing K_{n_1}, K_{n_2} to be twist knots (as in [7]) we get a pseudo–section $S \subset V_{K_{n_1}, K_{n_2}}$ which is now a sphere with two positive double points and self-intersection -1. Blowing up

the two self–intersections we get a sphere of square -9 in $V_{K_{n_1},K_{n_2}} \# 2\overline{\mathbb{CP}^2}$. Using five (-2)–spheres of the I_6 fiber, we get a chain of spheres according to the linear plumbing

$$(-9, -2, -2, -2, -2, -2),$$

giving rise to an embedding of $C_{7,1}$ into $V_{K_{n_1},K_{n_2}} \# 2\overline{\mathbb{CP}^2}$. We define our 4–manifolds by rationally blowing down these copies of $C_{7,1}$. Simple connectivity of $V_{K_{n_1},K_{n_2}}$ follows from the presence of two different vanishing cycles and the pseudo–section, while the complement of $C_{7,1}$ in $V_{K_{n_1},K_{n_2}} \# 2\overline{\mathbb{CP}^2}$ is simply connected because there are two more fishtail fibers in the fibration which we did not use in the construction. Since $V_{K_{n_1},K_{n_2}}$ is homeomorphic to $\mathbb{CP}^2 \# 9\overline{\mathbb{CP}^2}$ and we used two blow–ups to find the above chain of spheres, after rationally blowing down we get a 4–manifold homeomorphic to $\mathbb{CP}^2 \# 5\overline{\mathbb{CP}^2}$. Recall that K_n denotes the n–twist knot (as depicted in [7]); let V_n denote V_{K_1,K_n}. The result of the rational blow–down of $C_{7,1} \subset V_n \# 2\overline{\mathbb{CP}^2}$ will be denoted by Q_n.

3. Seiberg–Witten computations

We will prove Theorem 1.1 by computing Seiberg–Witten invariants of the 4–manifolds constructed above. We will give details of the computation for the first construction, resulting the manifolds X_n, very similar ideas work for all the other manifolds. The argument sketched below is closely modeled on the argument encountered in [7]. We will finish this section by an explicit computation of the Seiberg–Witten invariants of the manifolds Q_n constructed in Subsection 2.3.3.

It is shown in [6, 18] that Y_n has two Seiberg–Witten basic classes $\pm K$, moreover $|SW_{Y_n}(\pm K)| = n$. Furthermore, we can choose the sign of K so that it evaluates on the pseudo–section S as -1. Consequently

$$(K - e_1 - \ldots - e_{11})(u_i) = u_i \cdot u_i + 2$$

for each sphere u_i appearing in the plumbing C. Let L be the extension of $K|_{Y_n - C}$ to X_n. Using the blow–up and the rational blow–down formula together with the wall–crossing formula we get

Proposition 3.1. *The Seiberg–Witten invariant $SW_{X_n}(L)$ is an element of the set $\{\pm n, \pm n \pm 1\}$. Therefore the 4–manifold X_n with $n \geq 2$ admits a Seiberg–Witten basic class.* $\square$

This computation leads us to

Corollary 3.2. *There exists an exotic smooth structure on $\mathbb{CP}^2 \# 5\overline{\mathbb{CP}^2}$.*

Proof. Since the Seiberg–Witten function is a diffeomorphism invariant for manifolds with $b_2^+ = 1$ and $b_2^- \leq 9$, and by the existence of a positive scalar curvature metric we have $SW_{\mathbb{CP}^2 \# 5\overline{\mathbb{CP}^2}} \equiv 0$, we get that X_n is not diffeomorphic to $\mathbb{CP}^2 \# 5\overline{\mathbb{CP}^2}$, hence the corollary follows. $\square$

Since Y_n has exactly two basic classes, the same computation as above actually shows

Lemma 3.3. *The Seiberg–Witten function of X_n takes its values in a subset of $\{0, \pm 1, \pm n, \pm n \pm 1\}$, and for $n \geq 3$ there are exactly two basic classes $\pm L$ with Seiberg–Witten values in $\{\pm n, \pm n \pm 1\}$.* $\square$

Proof of Theorem 1.1. Combining Proposition 3.1 with Lemma 3.3 it follows that X_n and X_{n+3k} are not diffeomorphic once $n \geq 2$ and $k > 0$. This observation proves the existence of infinitely many distinct smooth structures on $\mathbb{CP}^2 \# 5\overline{\mathbb{CP}^2}$. The blow–up formula and the fact that for $n \geq 3$ there are only two basic classes of X_n with Seiberg–Witten values in the set $\{\pm n, \pm n \pm 1\}$ show that for $n \geq 3$ the manifold X_n is actually minimal. $\square$

The argument above was sufficient for proving Theorem 1.1, but with some additional work the complete Seiberg–Witten invariants of the 4–manifolds encountered above can be determined. We demonstrate this for the 4–manifolds Q_n defined in Subsection 2.3.3 and prove

Theorem 3.4. *For $n \geq 1$ the 4–manifold Q_n admits exactly two basic classes $\pm L$ and $SW_{Q_n}(\pm L) = \pm n$. Consequently the manifolds Q_n are all minimal and pairwise nondiffeomorphic.*

The heart of the argument is to find a simple way to relate the Seiberg–Witten invariants of Q_n to those of V_n. As a stepping stone we will need the following construction.

Start with the fibration $\mathbb{CP}^2 \# 9\overline{\mathbb{CP}^2} \to S^2$ provided by Proposition 2.7. Instead of doing the double node surgery, blow up the 4–manifold $\mathbb{CP}^2 \# 9\overline{\mathbb{CP}^2}$ twice and in the two new $\overline{\mathbb{CP}^2}$'s choose embedded spheres representing twice the generator of $H_2(\overline{\mathbb{CP}^2}; \mathbb{Z})$. By tubing these two (-4)–spheres to a fixed section of $\mathbb{CP}^2 \# 9\overline{\mathbb{CP}^2} \to S^2$ we get a (-9)–sphere, which, together with five (-2)–spheres of the I_6–fiber gives rise to an embedded copy of $C_{7,1}$ in $(\mathbb{CP}^2 \# 9\overline{\mathbb{CP}^2}) \# 2\overline{\mathbb{CP}^2} = \mathbb{CP}^2 \# 11\overline{\mathbb{CP}^2}$. Let R denote the 4–manifold we get by rationally blowing down this copy of $C_{7,1}$.

Proposition 3.5. *The Seiberg–Witten invariant SW_R is identically zero.*

Proof. Note that $b_2^-(R) = 5$, hence the Seiberg–Witten function SW_R is well–defined. Let $D \subset \mathbb{CP}^2 \# 9\overline{\mathbb{CP}^2}$ denote the tubular neighborhood of the chosen section and the chain of five (-2)–spheres in the I_6–fiber. Notice that $\partial D = S^3$. By performing the blow–ups and the rational blow–down process in D (resulting in a negative definite 4–manifold W), we get a decomposition of R as $P \# W$. Since $P \# 6\overline{\mathbb{CP}^2} = \mathbb{CP}^2 \# 9\overline{\mathbb{CP}^2}$, the blow–up formula and $SW_{\mathbb{CP}^2 \# 9\overline{\mathbb{CP}^2}} \equiv 0$ imply that $SW_P \equiv 0$. Now the usual gluing formula along S^3 implies the result. $\square$

Notice that by the construction of V_n there is a natural bijection

$$\mathcal{F} \colon H_2(\mathbb{CP}^2 \# 9\overline{\mathbb{CP}^2}; \mathbb{Z}) \to H_2(V_n; \mathbb{Z})$$

mapping the chosen section of $\mathbb{CP}^2 \# 9\overline{\mathbb{CP}^2} \to S^2$ to the pseudo–section in V_n. The map $\mathcal{F}$ induces a natural extension to the double blow–ups

$$\mathcal{F}' \colon H_2(\mathbb{CP}^2 \# 11\overline{\mathbb{CP}^2}; \mathbb{Z}) \to H_2(V_n \# 2\overline{\mathbb{CP}^2}; \mathbb{Z}).$$

In these double blow–ups we have found copies of $C_{7,1}$; it follows from the constructions of these submanifolds that $\mathcal{F}'$ maps the homology classes of the chains of spheres into each other.

In addition, homology classes of R (resp. Q_n) can be naturally constructed from homology classes of $\mathbb{CP}^2\#11\overline{\mathbb{CP}^2}$ (resp. $V_n\#2\overline{\mathbb{CP}^2}$) by appropriately extending them to the rational blow–down. In particular, the map $\mathcal{F}'$ gives rise to a bijection

$$\mathcal{F}_1\colon H_2(R;\mathbb{Z}) \to H_2(Q_n;\mathbb{Z}).$$

Let $K \in H^2(V_n;\mathbb{Z})$ be a characteristic element. For odd integers a,b we get extensions $K + aE_1 + bE_2 \in H^2(V_n\#2\overline{\mathbb{CP}^2};\mathbb{Z})$, where E_i denote the Poincaré duals of the exceptional divisors of the blow–ups. Suppose that the restriction of $K + aE_1 + bE_2$ to $V_n\#2\overline{\mathbb{CP}^2} - C_{7,1}$ extends to a characteristic cohomology class to Q_n and denote this extension by $K(a,b)$. Suppose furthermore that the formal dimension $d(K + aE_1 + bE_2)$ of the Seiberg–Witten moduli space on $V_n\#2\overline{\mathbb{CP}^2}$ corresponding to $K + aE_1 + bE_2$ is nonnegative.

Lemma 3.6. *Let K, a, b be chosen as above. Then*

$$SW_{Q_n}(K(a,b)) - SW_{V_n}(K) = SW_R(f_1(K(a,b))) - SW_{\mathbb{CP}^2\#9\overline{\mathbb{CP}^2}}(f(K)),$$

where f and f_1 are duals of $\mathcal{F}$ and $\mathcal{F}_1$.

Proof. Since the blow–up, wall–crossing and rational blow–down formulae involve only homological computations, and $\mathcal{F}'$ identifies the two copies of $C_{7,1}$, the lemma follows. $\square$

Proof of Theorem 3.4. Let $L \in H^2(Q_n;\mathbb{Z})$ be a characteristic element with $SW_{Q_n}(L) \neq 0$. By the rational blow–down formula there is a class $K + aE_1 + bE_2 \in H^2(V_n\#2\overline{\mathbb{CP}^2};\mathbb{Z})$ with

$$SW_{Q_n}(L) = SW_{V_n\#2\overline{\mathbb{CP}^2}}(K + aE_1 + bE_2)$$

where the right–hand side is taken in the appropriate chamber. In particular, $L = K(a,b)$ for some $K \in H^2(V_n;\mathbb{Z})$ and $d(K+aE_1+bE_2) \geq 0$. Since $SW_R \equiv 0$ and $SW_{\mathbb{CP}^2\#9\overline{\mathbb{CP}^2}} \equiv 0$, Lemma 3.6 implies that $SW_{V_n}(K) \neq 0$. On the other hand, the Seiberg–Witten invariants of V_n are known [6], hence it follows that $K = \pm T, \pm 3T$ where T is the Poincaré dual of the fiber. Since $d(T) = d(3T) = 0$, it follows that $a = \pm 1$ and $b = \pm 1$. A simple homological computation shows that in the family $\{\pm T \pm E_1 \pm E_2,\ \pm 3T \pm E_1 \pm E_2\} \subset H^2(V_n\#2\overline{\mathbb{CP}^2};\mathbb{Z})$ there are only two cohomology classes — which are equal to $\pm(3T - E_1 - E_2)$ — admitting extensions to Q_n. Since $SW_{V_n}(\pm 3T) = \pm n$ the theorem follows from Lemma 3.6. $\square$

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