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A NON–LINEAR GENERALISATION OF THE LOOMIS–WHITNEY INEQUALITY AND APPLICATIONS

Jonathan Bennett, Anthony Carbery and James Wright

Abstract. We establish a diffeomorphism–invariant generalisation of the classical Loomis–Whitney inequality in $\mathbb{R}^n$. As a consequence we obtain a sharp trilinear restriction theorem for the Fourier transform in three dimensions.

1. Introduction

The classical Loomis–Whitney inequality states that if $\pi_j : \mathbb{R}^n \to \mathbb{R}^{n-1}$ is given by $\pi_j(x) = (x_1, .., \widehat{x}_j, .., x_n)^1$, then

$$\int_{\mathbb{R}^n} f_1(\pi_1(x)) \cdots f_n(\pi_n(x))\, dx \leq \|f_1\|_{n-1} \cdots \|f_n\|_{n-1} \tag{1}$$

for all $f_j \in L^{n-1}(\mathbb{R}^{n-1})$. The Loomis–Whitney inequality can be viewed as an n-parameter isoperimetric inequality and in fact the classical isoperimetric inequality in $\mathbb{R}^n$ can be easily derived from it (albeit not with the sharp constant depending on n). This was the main reason that Loomis and Whitney originally considered inequalities of the form (1), see [8].

The main subject of this paper is the following non–linear generalisation of (1).

Theorem 1. *Suppose $\pi_1, \cdots, \pi_n : \mathbb{R}^n \to \mathbb{R}^{n-1}$ are smooth submersions in a neighbourhood of a point $x_0 \in \mathbb{R}^n$. If in addition, the linear span of the kernels of $d\pi_1(x_0), \cdots, d\pi_n(x_0)$ is $\mathbb{R}^n$, then for all cut–off functions a supported in a sufficiently small neighbourhood of x_0, there exists a constant C such that*

$$\int_{\mathbb{R}^n} f_1(\pi_1(x)) \cdots f_n(\pi_n(x))\, a(x)\, dx \leq C\|f_1\|_{n-1} \cdots \|f_n\|_{n-1} \tag{2}$$

for all $f_1, ..., f_n \in L^{n-1}(\mathbb{R}^{n-1})$.

Our applications to the restriction theory of the Fourier transform (the content of Section 4) require a slightly stronger, uniform version of the above theorem. In order to formulate this it will be convenient to associate to each submersion π_j a non–vanishing smooth vector field $X_j : \mathbb{R}^n \to \mathbb{R}^n$ which flows along the fibres of π_j. More specifically, for each x in a sufficiently small neighbourhood

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[1] here ^ denotes omission

of the origin in $\mathbb{R}^n$, $X_j(x)$ will be defined to be the wedge product of the rows of the $(n-1)\times n$ matrix $d\pi_j(x)$. We point out that $X_j(x)$ is a non–zero element of the kernel of $d\pi_j(x)$.

Theorem 2. *Let $A, \epsilon > 0$ be given. Suppose $\pi_1, \cdots, \pi_n : \mathbb{R}^n \to \mathbb{R}^{n-1}$ are smooth submersions in a neighbourhood of a point $x_0 \in \mathbb{R}^n$, and satisfy $\|\pi_j\|_{C^3} \le A$ for all j. If in addition the corresponding vector fields $X_1, ..., X_n$ are such that*

$$\det(X_1(x_0), ..., X_n(x_0)) > \epsilon, \tag{3}$$

then there exists a neighbourhood U of x_0, depending only on ϵ, A and n, such that for all cut–off functions a supported in U, there is a constant C depending only on n and a for which

$$\int_{\mathbb{R}^n} f_1(\pi_1(x)) \cdots f_n(\pi_n(x))\, a(x)\, dx \le C\epsilon^{-1/(n-1)} \|f_1\|_{n-1} \cdots \|f_n\|_{n-1} \tag{4}$$

for all $f_1, ..., f_n \in L^{n-1}(\mathbb{R}^{n-1})$.

Remark. The bound $\epsilon^{-1/(n-1)}$ appearing in the above theorem is the natural bound predicted by affine–invariance considerations.

When the mappings $\{\pi_j\}$ interact trivially in the sense that the corresponding vector fields $\{X_j\}$ commute, then one can choose coordinates reducing (4) to the classical Loomis–Whitney inequality (1). Therefore it is only of interest when the vectors fields $\{X_j\}$ do not commute and in this case, Theorems 1 and 2 can be viewed as the optimal result measured with respect to Lebesgue spaces under the nondegeneracy condition (3). It would also be of interest to establish optimal estimates when the vector fields fail to span. The bilinear situation where there are only two submersions π_1, $\pi_2 : \mathbb{R}^n \to \mathbb{R}^{n-1}$ is related to singular averaging operators along curves and has been well-studied by many authors; see [12] for a discussion of some aspects of this problem.

Theorems 1 and 2 are local results as they are stated above. One might ask whether there are global versions of these theorems, possibly even weighted versions to quantify how the spanning condition (3) may degenerate. Simple examples show that naive versions, for example, requiring only that the mappings $\{\pi_j\}$ be submersions at every point in $\mathbb{R}^n$, cannot possibly hold. For example, when $n = 2$ the determinant in (3) is simply the Jacobian of the mapping $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(x, y) = (\pi_1(x, y), \pi_2(x, y))$, and so the following formula holds:

$$\int_{\mathbb{R}^2} f(\pi_1(x,y))\, g(\pi_2(x,y))\ |\det(X_1(x,y), X_2(x,y))|\ dxdy$$

$$= \int_{\mathbb{R}^2} f(x)\, g(y)\ N(\pi_1, \pi_2; (x,y))\, dxdy$$

where $N(\pi_1, \pi_2; (x, y))$ counts the number of intersections of the fibres $\pi_1^{-1}\{x\}$ and $\pi_2^{-1}\{y\}$, see for example, [6]. Therefore if one can keep this number under control, a global version of (4) would then hold. Of course $N(\pi_1, \pi_2; (x, y))$

can be infinite almost everywhere, for example, consider $\pi_1(x,y) = e^x \cos y$ and $\pi_2(x,y) = e^x \sin y$.

Our proof of Theorem 2 (the content of Section 2) comes in two stages. The first uses a method of refinements due to Christ [5] (see also [12]), to establish (4) for arbitrary characteristic functions of sets. The second stage, which is based on an argument of the second author in [3], exploits a certain invariance under tensor products of our refinement argument, allowing us to pass from characteristic functions of sets to general functions.

The remaining parts of this paper are organised as follows. In Section 3 we use Theorems 1 and 2 in three dimensions to obtain L^2 estimates for a certain family of bilinear Radon–like transforms in the plane. In Section 4 we give applications of these estimates to the restriction theory of the Fourier transform in three dimensions. Finally, we collect together several observations of a technical nature in an appendix.

Notation. For $X, Y \in \mathbb{C}$, we write $X \lesssim Y$ ($X \gtrsim Y$) if $|X| \leq c|Y|$ ($|X| \geq c|Y|$) for some constant $c > 0$ which may depend only on the dimension n.

2. Proof of Theorem 2

Without loss of generality we may assume that $x_0 = 0$ in the statement of Theorem 2.

Before we begin it is convenient to make a number of observations about the vector fields $X_1, ..., X_n$, and identify the neighbourhood U appearing in the statement of the above theorem. In order to do this we associate to the X_j's the flow maps $e^{tX_j} : \mathbb{R}^n \to \mathbb{R}^n$. These will be defined in a neighbourhood of 0, for sufficiently small t, by the differential equations

$$\frac{d}{dt} e^{tX_j} x = X_j(e^{tX_j} x); \quad e^{0X_j} x = x.$$

The flow maps will be used to parametrise the fibres of the submersions π_j later in our argument. It will be implicit in our analysis that x and t are sufficiently localised so that the vector fields X_j never vanish, and for the corresponding flow maps to be well–defined. More specifically, by the hypotheses of Theorem 2 and the existence theory for ODE's, there exists a neighbourhood N of the origin in $\mathbb{R}^n$ and a time $t_0 > 0$, both depending only on ϵ, A and n, for which each of the vector fields X_j is bounded away from zero on N, and the flows $e^{tX_j} x$ exist for all $(t, x) \in [-t_0, t_0] \times N$ (see the proof of Proposition 3 in the Appendix).

Now the set U in the statement of Theorem 2 will be chosen sufficiently small so that it will satisfy several technical conditions needed at various stages of our argument. Throughout this article we will use $\gamma(t, x)$ to denote the iterated flow $e^{t_1 X_1} \cdots e^{t_n X_n} x$, where $t = (t_1, ..., t_n)$.

Proposition 3. *There exists a neighbourhood U of $0 \in \mathbb{R}^n$, and constants $C, r > 0$, depending only on ϵ, A and n, such that*

(i) $\det(X_1(x), ..., X_n(x)) \geq \epsilon/2$ *for all $x \in U$,*

(ii) $\pi_j^{-1}(y) \cap U$ *is a connected subset of* $\mathbb{R}^n$ *for all* $y \in \mathbb{R}^{n-1}$ *and* $1 \leq j \leq n$,
(iii) $e^{tX_j}(x) \cap U$ *is empty for all* $|t| > r$, $x \in U$, *and* $1 \leq j \leq n$, *and*
(iv) $\|\gamma\|_{C^2([-r,r]^n \times U)} \leq C$.

So as not to be distracted from the main line of our argument, a detailed proof of this proposition is left to an appendix.

Corollary 4. *There exists a neighbourhood* U *of* $0 \in \mathbb{R}^n$ *and an* $r > 0$, *depending only on* ϵ, A *and* n, *such that whenever* $x \in U$,

1. $t \mapsto \gamma(t,x)$ *is injective on* $[-r,r]^n$,
2. $|\det(D_1\gamma(t,x))| \gtrsim \epsilon$ *for every* $t \in [-r,r]^n$, *and*
3. $\pi_j^{-1}(\pi_j(x)) \cap U = \{e^{t_jX_j}(x) : t_j \in [-r,r]\} \cap U$ *for each* $1 \leq j \leq n$.

Proof. Part (1) is a direct consequence of parts (i) and (iv) of Proposition 3, the observation that $D_1\gamma(0,x) = (X_1(x), ..., X_n(x))$, and a quantitative versions of the inverse function theorem (see for example, [4], page 595). Part (2) is an elementary consequence of parts (i) and (iv) of Proposition 3 and the mean value theorem. Part (3) is an immediate consequence of parts (ii) and (iii) of Proposition 3. □

We now come to the main argument leading to Theorem 2. Both U and r will given by Corollary 4 throughout.

Initially we will focus on establishing (4) for characteristic functions of sets; i.e. that

$$\int_U \chi_{E_1}(\pi_1(x)) \cdots \chi_{E_n}(\pi_n(x))\, dx \lesssim \epsilon^{-1/(n-1)}|E_1|^{1/(n-1)} \cdots |E_n|^{1/(n-1)} \tag{5}$$

for all $E_1, ..., E_n \subset \mathbb{R}^{n-1}$. The extension to general functions will be dealt with at the end of the section.

Inequality (5) is easily seen to be equivalent to

$$|\Omega|^{n-1} \lesssim \epsilon^{-1}|\pi_1(\Omega)| \cdots |\pi_n(\Omega)|$$

for all $\Omega \subset U$. If we define

$$\delta_j = \frac{|\Omega|}{|\pi_j(\Omega)|}, \quad j = 1, ..., n,$$

then (5) may be restated as

$$|\Omega| \gtrsim \epsilon\delta_1 \cdots \delta_n \tag{6}$$

for all $\Omega \subset U$. In order to prove (6) we shall do some preliminary "refining" of our set Ω. We will say that a set $\Omega' \subset \mathbb{R}^n$ is a *refinement* of Ω, if $\Omega' \subset \Omega$ and the measure of Ω' is comparable to that of Ω.

By the coarea formula (see [6] for example),

$$\delta_1 = \frac{1}{|\pi_1(\Omega)|}\int_{\pi_1(\Omega)}\int_{\pi_1^{-1}(y)\cap\Omega} |X_1(x)|^{-1} d\mathcal{H}^1(x)dy, \tag{7}$$

where $d\mathcal{H}^1$ denotes 1–dimensional Hausdorff measure on $\mathbb{R}^n$.

We now define $E_1 \subset \pi_1(\Omega)$ by

$$E_1 = \left\{ y \in \pi_1(\Omega) : \int_{\pi_1^{-1}(y)\cap\Omega} |X_1(x)|^{-1} d\mathcal{H}^1(x) \geq \delta_1/10^n \right\},$$

and take $\Omega_1 = \Omega \cap \pi_1^{-1}(E_1)$. Clearly Ω_1 is a refinement of Ω since by the coarea formula,

$$|\Omega_1| = \int_{E_1} \int_{\pi_1^{-1}(y)\cap\Omega} |X_1(x)|^{-1} d\mathcal{H}^1(x) dy \geq |\Omega| - \frac{\delta_1}{10^n}|\pi_1(\Omega)| = \left(1 - \frac{1}{10^n}\right)|\Omega|.$$

Now let $E_2 \subset \pi_2(\Omega_1)$ be given by

$$E_2 = \left\{ y \in \pi_2(\Omega_1) : \int_{\pi_2^{-1}(y)\cap\Omega_1} |X_2(x)|^{-1} d\mathcal{H}^1(x) \geq \delta_2/10^n \right\},$$

and take $\Omega_2 = \Omega_1 \cap \pi_2^{-1}(E_2)$. Again, Ω_2 is a refinement of Ω, since

$$\begin{aligned} |\Omega_2| = \int_{E_2} \int_{\pi_2^{-1}(y)\cap\Omega_1} & |X_2(x)|^{-1} d\mathcal{H}^1(x) dy \\ & \geq |\Omega_1| - \frac{\delta_2}{10^n}|\pi_2(\Omega_1)| \\ & \geq \left(1 - \frac{1}{10^n}\right)|\Omega| - \frac{\delta_2}{10^n}|\pi_2(\Omega)| \\ & \geq \left(1 - \frac{2}{10^n}\right)|\Omega|. \end{aligned}$$

Proceeding in this way we generate a nested sequence of refinements Ω_j of Ω $(0 \leq j \leq n)$ satisfying $\Omega_0 = \Omega$, and $\Omega_j = \Omega_{j-1} \cap \pi_j^{-1}(E_j)$, where

$$E_j = \left\{ y \in \pi_j(\Omega_{j-1}) : \int_{\pi_j^{-1}(y)\cap\Omega_{j-1}} |X_j(x)|^{-1} d\mathcal{H}^1(x) \geq \delta_j/10^n \right\}.$$

Since $|\Omega| \neq 0$ (without loss of generality), the last refinement Ω_n is non–empty. Let x_0 be any element of Ω_n, and

$$I_{x_0} := \{t_n \in [-r, r] : e^{t_n X_n} x_0 \in \Omega_{n-1}\}.$$

Using Corollary 4, it is easily verified that

$$|I_{x_0}| = \int_{\pi_n^{-1}(\pi_n(x_0))\cap\Omega_{n-1}} |X_n(x)|^{-1} d\mathcal{H}^1(x) \geq \delta_n/10^n.$$

Now for each $t_n \in I_{x_0}$, $e^{t_n X_n} x_0 \in \Omega_{n-1}$, and so

$$I_{x_0,t_n} := \{t_{n-1} \in [-r, r] : e^{t_{n-1} X_{n-1}} e^{t_n X_n} x_0 \in \Omega_{n-2}\}$$

satisfies $|I_{x_0,t_n}| \geq \delta_{n-1}/10^n$.

Continuing in this way we generate a sequence of families of subsets of $\mathbb{R}$ given by

$$I_{x_0,t_n,\ldots,t_{\ell+1}} := \{t_\ell \in [-r, r] : e^{t_\ell X_\ell} \cdots e^{t_n X_n} x_0 \in \Omega_{\ell-1}\}; \quad 1 \leq \ell \leq n-1,$$

which satisfy the uniform bounds $|I_{x_0,t_n,\dots,t_{\ell+1}}| \geq \delta_\ell/10^n$.

Now define

$$\begin{aligned}\mathbf{P} = \{t = (t_1, \dots, t_n) \in [-r, r]^n : \; & t_n \in I_{x_0}, \\ & t_{n-1} \in I_{x_0,t_n}, \\ & t_{n-2} \in I_{x_0,t_n,t_{n-1}}, \\ & \vdots \\ & t_1 \in I_{x_0,t_n,t_{n-1},\dots,t_2}\}.\end{aligned}$$

Clearly,

$$|\mathbf{P}| \geq \delta_1 \cdots \delta_n/10^{n^2},$$

and $\gamma(\mathbf{P}, x_0) \subset \Omega$, and so by Corollary 4

$$|\Omega| \geq |\gamma(\mathbf{P}, x_0)| = \int_{\mathbb{P}} |\det (D_1\gamma(t, x_0))| \, dt \gtrsim \epsilon \, \delta_1 \cdots \delta_n.$$

This completes the proof of (6) and hence that of Theorem 2 in the case where the functions f_1,...,f_n are characteristic functions of sets.

In order to pass to general functions we will need to exploit a certain invariance under tensor products of the above proof.

Let $\pi_1, \dots, \pi_n : \mathbb{R}^n \to \mathbb{R}^{n-1}$ be as in the statement of Theorem 2. For each $1 \leq j \leq n$ and $k \in \mathbb{N}$ we define $\pi_j^{(k)} : \mathbb{R}^{kn} \to \mathbb{R}^{k(n-1)}$ by

$$\pi_j^{(k)}(x) = \left(\pi_j(x^{(1)}), \dots, \pi_j(x^{(k)})\right),$$

where $x := \left(x^{(1)}, \dots, x^{(k)}\right)$, and each $x^{(\ell)}$ is an element of $\mathbb{R}^n$.

Claim 5. *There exists a neighbourhood U of $0 \in \mathbb{R}^n$, depending only on ϵ, A and n, such that for all cut–off functions a supported in U^k, there is a constant $C > 0$ depending only on a and n, for which*

$$\begin{aligned}\int_{\mathbb{R}^{kn}} \chi_{E_1}(\pi_1^{(k)}(x)) \cdots \chi_{E_n}(\pi_n^{(k)}(x)) \, a(x) \, dx \\ \leq C^k \epsilon^{-k/(n-1)} |E_1|^{1/(n-1)} \cdots |E_n|^{1/(n-1)}\end{aligned}$$

for all $E_1, \dots, E_n \subset \mathbb{R}^{k(n-1)}$.

Before we prove the claim, let us see how it can be used to finish the proof of Theorem 2. We follow an argument of the second author in [3]. By an elementary density argument it is enough to prove (4) where f_1,...,f_n are simple functions. For each $1 \leq j \leq n$ let $F_j : \mathbb{R}^{k(n-1)} \to \mathbb{R}$ be given by

$$F_j(y) = f_j(y^{(1)}) \cdots f_j(y^{(k)}),$$

where $y := (y^{(1)}, ..., y^{(k)})$ and each $y^{(\ell)}$ is an element of $\mathbb{R}^{n-1}$. Now, there exists an integer N independent of k, sets $E_{j,\ell} \subset \mathbb{R}^{k(n-1)}$, and dyadic numbers $\alpha_{j,\ell}$ with $1 \le \ell \le kN$, such that for each $1 \le j \le n$,

$$F_j \le \sum_{\ell=1}^{kN} \alpha_{j,\ell} \chi_{E_{j,\ell}},$$

and

$$\|f_j\|_{n-1}^k = \|F_j\|_{n-1} \sim \left(\sum_{\ell=1}^{kN} \alpha_{j,\ell}^{n-1} |E_{j,\ell}| \right)^{1/(n-1)}.$$

Hence

$$\begin{aligned}
&\left(\int_U f_1(\pi_1(x)) \cdots f_n(\pi_n(x)) \, dx \right)^k \\
&\qquad = \int_{U^k} F_1(\pi_1^{(k)}(x)) \cdots F_n(\pi_n^{(k)}(x)) \, dx \\
&\qquad \le \sum_{\ell_1, \ldots, \ell_n = 1}^{kN} \alpha_{1,\ell_1} \cdots \alpha_{n,\ell_n} \int_{U^k} \chi_{E_{1,\ell_1}}(\pi_1^{(k)}(x)) \cdots \chi_{E_{n,\ell_n}}(\pi_n^{(k)}(x)) \, dx \\
&\qquad \le C^k \epsilon^{-k/(n-1)} \sum_{\ell_1, \ldots, \ell_n = 1}^{kN} \alpha_{1,\ell_1} \cdots \alpha_{n,\ell_n} |E_{1,\ell_1}|^{1/(n-1)} \cdots |E_{n,\ell_n}|^{1/(n-1)} \\
&\qquad \lesssim C^k \epsilon^{-k/(n-1)} (kN)^n \|F_1\|_{n-1} \cdots \|F_n\|_{n-1} \\
&\qquad = C^k \epsilon^{-k/(n-1)} (kN)^n \|f_1\|_{n-1}^k \cdots \|f_n\|_{n-1}^k.
\end{aligned}$$

On taking kth roots, letting k tend to infinity and using the fact that $(kN)^{n/k} \to 1$, we obtain

$$\int_U f_1(\pi_1(x)) \cdots f_n(\pi_n(x)) \, dx \le C \epsilon^{-1/(n-1)} \|f_1\|_{n-1} \cdots \|f_n\|_{n-1}.$$

Since the f_j's were arbitrary step functions, Theorem 2 now follows.

The proof of the claim is essentially a straightforward reprise of the above proof of (6). It will be enough to merely indicate the main steps in the argument. As before, we initially observe that we may reduce matters to proving that

$$|\Omega| \ge C^k \epsilon^k \delta_1 \cdots \delta_n,$$

for all $\Omega \subset U^k$, where now

$$\delta_j := \frac{|\Omega|}{|\pi_j^{(k)}(\Omega)|}.$$

Again, throughout this argument U and r will be given by Corollary 4.

As before we use the coarea formula to write

$$\delta_1 = \frac{1}{|\pi_1^{(k)}(\Omega)|} \int_{\pi_1^{(k)}(\Omega)} \int_{\pi_1^{(k)^{-1}}(y) \cap \Omega} |X_1(x^{(1)})|^{-1} \cdots |X_1(x^{(k)})|^{-1} d\mathcal{H}^k(x) \, dy,$$

where $d\mathcal{H}^k$ denotes k–dimensional Hausdorff measure on $\mathbb{R}^{kn}$. If we now define $\Omega_1 = \Omega \cap {\pi_1^{(k)}}^{-1}(E_1)$, where

$$E_1 = \left\{ y \in \pi_1^{(k)}(\Omega) : \int_{{\pi_1^{(k)}}^{-1}(y)\cap\Omega} |X_1(x^{(1)})|^{-1} \cdots |X_1(x^{(k)})|^{-1} d\mathcal{H}^k(x) \geq \frac{\delta_1}{10^n} \right\},$$

then, just as before, Ω_1 is a refinement of Ω. Continuing this process in the now familiar way we obtain a sequence of refinements Ω_j of Ω $(0 \leq j \leq n)$ satisfying $\Omega_0 = \Omega$ and $\Omega_j = \Omega_{j-1} \cap {\pi_j^{(k)}}^{-1}(E_j)$, where

$$E_j = \left\{ y \in \pi_j^{(k)}(\Omega_{j-1}) : \int_{{\pi_j^{(k)}}^{-1}(y)\cap\Omega_{j-1}} |X_j(x^{(1)})|^{-1} \cdots |X_j(x^{(k)})|^{-1} d\mathcal{H}^k \geq \frac{\delta_j}{10^n} \right\}.$$

Since $|\Omega| \neq 0$ (without loss of generality), the last refinement Ω_n is non–empty. Let $x_0 \in \Omega_n$, and

$$I_{x_0} := \left\{ t^{(n)} = (t_1^{(n)}, ..., t_k^{(n)}) \in [-r,r]^k : \left(e^{t_1^{(n)} X_n} x_0^{(1)}, \ \ldots, e^{t_k^{(n)} X_n} x_0^{(k)} \right) \in \Omega_{n-1} \right\},$$

then it is again easily verified that $|I_{x_0}| \geq \delta_n/10^n$. Continuing as before we may generate a sequence of families of subsets $I_{x_0,t^{(n)},\ldots,t^{(\ell+1)}}$ of $\mathbb{R}^k$ given by

$$\left\{ t^{(\ell)} \in [-r,r]^k : \left(e^{t_1^{(\ell)} X_\ell} \cdots e^{t_1^{(n)} X_n} x_0^{(1)}, \ \ldots, e^{t_k^{(\ell)} X_\ell} \cdots e^{t_k^{(n)} X_n} x_0^{(k)} \right) \in \Omega_{\ell-1} \right\},$$

$(1 \leq \ell \leq n-1)$, which satisfy the uniform bounds $|I_{x_0,t^{(n)},\ldots,t^{(\ell+1)}}| \geq \delta_\ell/10^n$.

Now define

$$\begin{aligned} \mathbf{P} = \{ t = (t^{(1)}, ..., t^{(n)}) \in [-r,r]^{kn} : \ & t^{(n)} \in I_{x_0}, \\ & t^{(n-1)} \in I_{x_0,t^{(n)}}, \\ & t^{(n-2)} \in I_{x_0,t^{(n)},t^{(n-1)}}, \\ & \qquad \vdots \\ & t^{(1)} \in I_{x_0,t^{(n)},t^{(n-1)},\ldots,t^{(2)}} \}. \end{aligned}$$

Clearly,

$$|\mathbf{P}| \geq \delta_1 \cdots \delta_n / 10^{n^2},$$

and $\gamma^{(k)}(\mathbf{P}, x_0) \subset \Omega$, where $\gamma^{(k)} : \mathbb{R}^{kn} \times \mathbb{R}^{kn} \to \mathbb{R}^{kn}$ is given by

$$\gamma^{(k)}(t,x) = \left(e^{t_1^{(1)} X_1} \cdots e^{t_1^{(n)} X_n} x^{(1)}, \ \ldots, e^{t_k^{(1)} X_1} \cdots e^{t_k^{(n)} X_n} x^{(k)} \right).$$

Now by Corollary 4, $t \mapsto \gamma^{(k)}(t, x_0)$ is injective on $[-r,r]^{kn}$, and since $D_1\gamma^{(k)}(t,x_0)$ is block–diagonal,

$$\left| \det \left(D_1 \gamma^{(k)}(t, x_0) \right) \right| \gtrsim \epsilon^k$$

for all $t \in [-r,r]^{kn}$. This completes the proof of the claim, and hence that of Theorem 2.

3. Bilinear Radon–like transforms in the plane

In this short section we use Theorems 1 and 2 to obtain $L^2 \times L^2 \to L^2$ estimates for certain bilinear Radon–like transforms in the plane. These transforms play a key role in the three–dimensional restriction theory that we develop in the following section.

Proposition 6. *If $F : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ is smooth, $F(0) = 0$, and*

$$\det(\nabla_x F(0), \nabla_y F(0)) \neq 0,$$

then there exists a neighbourhood V of the origin in $\mathbb{R}^2 \times \mathbb{R}^2$, and a constant C such that

$$\int_V f(x)g(y)h(x+y)\ \delta(F(x,y))\ dxdy \leq C\|f\|_2\|g\|_2\|h\|_2$$

for all $f, g, h \in L^2(\mathbb{R}^2)$.

Since our uniformity requirements in the next section are not addressed by this proposition, a stronger, more quantitative version is needed.

Proposition 7. *Let $A, \epsilon > 0$ be given. If $F : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ is smooth, such that $\|F\|_{C^3} \leq A$, $F(0) = 0$, and*

$$\det(\nabla_x F(0), \nabla_y F(0)) > \epsilon,$$

then there exists a neighbourhood V of the origin in $\mathbb{R}^2 \times \mathbb{R}^2$, depending only on A and ϵ, such that

$$\int_V f(x)g(y)h(x+y)\ \delta(F(x,y))\ dxdy \lesssim \epsilon^{-1/2}\|f\|_2\|g\|_2\|h\|_2$$

for all $f, g, h \in L^2(\mathbb{R}^2)$.

It is instructive to observe that if F is a linear mapping, the above propositions can be obtained from the classical 3–dimensional Loomis–Whitney inequality (1). It is therefore natural to expect the case of general F to be a consequence of a certain perturbed version of (1). As may be expected, in order to prove Proposition 6 we would need only appeal to Theorem 1, whereas the proof of (the stronger) Proposition 7 requires the more quantitative Theorem 2.

We now prove Proposition 7. We first remark that since the full gradient $\nabla F(0)$ is non–zero, the distribution $\delta(F)$ is well–defined in a sufficiently small neighbourhood of the origin in $\mathbb{R}^2 \times \mathbb{R}^2$. By straightforward considerations this neigbourhood may be taken to depend only on ϵ and A. Our next step is to find a local parametrisation of $\{(x,y) : F(x,y) = 0\}$. By symmetry, without loss of generality we may assume that $\frac{\partial F}{\partial y_2}(0) \gtrsim \epsilon^{1/2}$. Furthermore, by the mean value theorem, we may find a neighbourhood of the origin in $\mathbb{R}^2 \times \mathbb{R}^2$, which depends only on ϵ and A, upon which $\frac{\partial F}{\partial y_2}(x,y) \sim \frac{\partial F}{\partial y_2}(0)$. By the implicit function theorem, for each (x_1, x_2, y_1) in a sufficiently small neighbourhood W

of the origin in $\mathbb{R}^3$, there exists a smooth mapping $\eta : W \to \mathbb{R}$ such that for all $(x_1, x_2, y_1) \in W$,

$$F(x_1, x_2, y_1, \eta(x_1, x_2, y_1)) = 0. \tag{8}$$

Furthermore

$$\nabla\eta(0) = -\left(\frac{\partial F}{\partial y_2}(0)\right)^{-1} \nabla_{x_1,x_2,y_1} F(0). \tag{9}$$

That W and the C^3 norm of η may be chosen to depend only on A and ϵ is a consequence of quantitative versions of the implicit function theorem.

Let

$$\pi_1(u_1, u_2, u_3) = (u_1, u_2),$$
$$\pi_2(u_1, u_2, u_3) = (u_3, \eta(u_1, u_2, u_3))$$

and $\pi_3 = \pi_1 + \pi_2$. Hence the C^3–norms of π_1, π_2 and π_3 are uniformly bounded, and that their associated vector fields X_1, X_2 and X_3 (as defined in the introduction), satisfy

$$\det(X_1(0), X_2(0), X_3(0)) = \left(\frac{\partial F}{\partial y_2}(0)\right)^{-2} \det(\nabla_x F(0), \nabla_y F(0)).$$

Hence by Theorem 2, there exists a neighbourhood U of the origin, with $U \subset W$, depending only on A and ϵ, such that

$$\int_U f(\pi_1(u))g(\pi_2(u))h(\pi_3(u))du \lesssim \epsilon^{-1/2}\frac{\partial F}{\partial y_2}(0)\ \|f\|_2\|g\|_2\|h\|_2.$$

In order to complete the proof of Proposition 7 we make the elementary observation that there exists a neighbourhood V of the origin in $\mathbb{R}^2 \times \mathbb{R}^2$ (again depending only on A and ϵ), such that

$$\int_V f(x)g(y)h(x+y)\ \delta(F(x,y))\ dxdy$$
$$\lesssim \left(\frac{\partial F}{\partial y_2}(0)\right)^{-1} \int_U f(\pi_1(u))g(\pi_2(u))h(\pi_3(u))du.$$

(It is here where we explicitly use the definition of the distribution $\delta(F)$ – see [10] for further discussion.)

4. A trilinear restriction inequality

Let $\Sigma : \mathbb{R}^2 \to \mathbb{R}^3$ parametrise a two dimensional smooth submanifold. For a function $f \in L^1(\mathbb{R}^2)$ and $\xi \in \mathbb{R}^3$, let

$$\mathcal{E}f(\xi) = \int_{\mathbb{R}^2} f(x)e^{i\Sigma(x)\cdot\xi}dx.$$

We refer to $\mathcal{E}$ as the extension operator associated to Σ. We point out that $\mathcal{E}f$ coincides with the 3–dimensional Fourier transform of a density carried on the submanifold $\{\Sigma(x) : x \in \mathbb{R}^2\} \subset \mathbb{R}^3$.

Suppose now that we have three such mappings Σ_1, Σ_2 and Σ_3, and associated extension operators $\mathcal{E}_1$, $\mathcal{E}_2$ and $\mathcal{E}_3$.

Theorem 8. *If the kernels of the mappings $(d\Sigma_1(0))^*$, $(d\Sigma_2(0))^*$ and $(d\Sigma_3(0))^*$ span $\mathbb{R}^3$, then there exists a constant C such that*

$$\|\mathcal{E}_1 f\ \mathcal{E}_2 g\ \mathcal{E}_3 h\|_{L^2(\mathbb{R}^3)} \leq C\|f\|_{4/3}\|g\|_{4/3}\|h\|_{4/3} \tag{10}$$

for all f, g and h supported in a sufficiently small neighbourhood of $0 \in \mathbb{R}^2$.

Remarks.

1. The condition that the kernels of the linear mappings $(d\Sigma_j(0))^*$ span, amounts to requiring that the normals to the submanifolds $\{\Sigma_j(x) : x \in \mathbb{R}^2\}$ span at $x = 0$.
2. We will actually prove a slightly stronger, "uniform" version of the above restriction theorem. In particular, given $A, \epsilon > 0$, our conclusions will be uniform over all triples $(\Sigma_1, \Sigma_2, \Sigma_3)$ satisfying
$$\|\Sigma_j\|_{C^3} \leq A, \quad \text{and} \quad \det(Y_1, Y_2, Y_3) > \epsilon,$$
where $Y_j \in \mathbb{R}^3$ is the wedge product of the columns of the matrix $d\Sigma_j(0)$. In particular, the neighbourhood of the origin may be chosen in such a way that the constant C may be taken to be an absolute constant multiple of $\epsilon^{-1/4}$.
3. The exponent 4/3 on the right of (10) is optimal given the exponent 2 on the left. It should be emphasised that this continues to be the case even if we make additional curvature assumptions on the Σ_j's.
4. If the kernels of the mappings $(d\Sigma_j(0))^*$ fail to span, but are merely distinct the exponent 4/3 must be replaced by 3/2. This inequality can be seen as a consequence of the elementary bilinear estimate
$$\|\mathcal{E}_1 f\ \mathcal{E}_2 g\|_{L^2(\mathbb{R}^3)} \leq C\|f\|_2\|g\|_2.$$
However, it is well–known that in this context one can do better than 3/2 by imposing appropriate curvature hypotheses on the Σ_j's (it turns out that one may go down as far as 18/13 – see [9] and [11]).
5. If we place no condition at all on the kernels of the $(d\Sigma_j(0))^*$'s, it is easy to see that in general, the inequality (10) fails to hold even with L^∞ norms on the right. Such an inequality may of course be salvaged under curvature assumptions by appealing to classical linear restriction inequalities (such as the Stein–Tomas theorem), although the exponent can never drop below 3/2.
6. It may be conjectured that under the conditions of Theorem 8, one may obtain the stronger conclusion
$$\|\mathcal{E}_1 f\ \mathcal{E}_2 g\ \mathcal{E}_3 h\|_{L^1(\mathbb{R}^3)} \leq C\|f\|_2\|g\|_2\|h\|_2.$$

We point out that if the Σ_j's are *linear* mappings then this is a true inequality, and moreover is equivalent to the classical 3–dimensional Loomis–Whitney inequality (1). For some further partial results in this direction see [2].

7. Theorem 8 was proved for sections of paraboloid in [1].

Proof. We begin by observing that inequality (10) is equivalent to

$$\langle \mathcal{E}_1 f_1\ \mathcal{E}_2 g_1\ \mathcal{E}_3 h_1, \mathcal{E}_1 f_2\ \mathcal{E}_2 g_2\ \mathcal{E}_3 h_2\rangle \\ \leq C\|f_1\|_{4/3}\|g_1\|_{4/3}\|h_1\|_{4/3}\|f_2\|_{4/3}\|g_2\|_{4/3}\|h_2\|_{4/3},$$

for all f_1, g_1, h_1, f_2, g_2 and h_2 supported in a sufficiently small neighbourhood of the origin. It therefore suffices to show that there exist neighbourhoods $\Omega_1, \Omega_2, \Omega_3$ of $0 \in \mathbb{R}^2$, such that

$$\int_{\Omega_1}\int_{\Omega_2}\int_{\Omega_3}\int_{\Omega_1}\int_{\Omega_2}\int_{\Omega_3} f_1(x)g_1(y)h_1(z)f_2(x')g_2(y')h_2(z') \\ \times \delta\left(\Sigma_1(x)+\Sigma_2(y)+\Sigma_3(z)-\Sigma_1(x')-\Sigma_2(y')-\Sigma_3(z')\right)dx\,dy\,dz\,dx'dy'dz' \\ \leq C\|f_1\|_{4/3}\|g_1\|_{4/3}\|h_1\|_{4/3}\|f_2\|_{4/3}\|g_2\|_{4/3}\|h_2\|_{4/3}$$

for all $f_1, g_1, h_1, f_2, g_2, h_2 \in L^{4/3}$. By symmetry and multilinear interpolation it suffices to prove that

$$\int_{\Omega_1}\int_{\Omega_2}\int_{\Omega_3}\int_{\Omega_1}\int_{\Omega_2}\int_{\Omega_3} f_1(x)g_1(y)h_1(z)f_2(x')g_2(y')h_2(z') \\ \times \delta\left(\Sigma_1(x)+\Sigma_2(y)+\Sigma_3(z)-\Sigma_1(x')-\Sigma_2(y')-\Sigma_3(z')\right)dx\,dy\,dz\,dx'dy'dz' \\ \leq C\begin{cases} \|f_1\|_2\|g_1\|_2\|h_1\|_2\|f_2\|_1\|g_2\|_1\|h_2\|_1 \\ \|f_1\|_2\|g_1\|_2\|h_1\|_1\|f_2\|_1\|g_2\|_1\|h_2\|_2. \end{cases}$$

By considering the mapping $\Sigma_3 \mapsto -\Sigma_3$, we quickly see that it will be enough to prove only the first of the above two inequalities. Since f_2, g_2 and h_2 are measured in L^1 it is enough to deal with the case where $f_2 = \delta_{x'}$, $g_2 = \delta_{y'}$ and $h_2 = \delta_{z'}$, where now we view (x', y', z') as an arbitrary element of $\Omega_1 \times \Omega_2 \times \Omega_3$. It hence suffices to show that

$$\int_{\Omega_1}\int_{\Omega_2}\int_{\Omega_3} f_1(x)g_1(y)h_1(z) \\ \times \delta\left(\Sigma_1(x)+\Sigma_2(y)+\Sigma_3(z)-\Sigma_1(x')-\Sigma_2(y')-\Sigma_3(z')\right)dx\,dy\,dz \\ \leq C\|f_1\|_2\|g_1\|_2\|h_1\|_2,$$

uniformly in $(x', y', z') \in \Omega_1 \times \Omega_2 \times \Omega_3$.

Now, by a rotation of $\mathbb{R}^3$, we may assume that each $\Sigma_j(x)$ is of the form $(x, \Phi_j(x))$ in a small neighbourhood of the origin, where Φ_j is smooth and real–valued. In this notation, the hypothesis that the kernels of the $(d\Sigma_j(0))^*$'s span becomes

$$\det \begin{pmatrix} 1 & 1 & 1 \\ \frac{\partial \Phi_1}{\partial x_1} & \frac{\partial \Phi_2}{\partial x_1} & \frac{\partial \Phi_3}{\partial x_1} \\ \frac{\partial \Phi_1}{\partial x_2} & \frac{\partial \Phi_2}{\partial x_2} & \frac{\partial \Phi_3}{\partial x_2} \end{pmatrix} \neq 0$$

at the origin.

It remains to show that under the above condition

$$\int_{\Omega_1} \int_{\Omega_2} f(x)g(y)h(x'+y'+z'-x-y)\delta\left(F(x-x', y-y')\right) dxdy \leq C\|f\|_2\|g\|_2\|h\|_2 \tag{11}$$

uniformly in $(x', y', z') \in \Omega_1 \times \Omega_2 \times \Omega_3$. Here we have set

$$F(x,y) = \Phi_1(x'+x) + \Phi_2(y'+y) + \Phi_3(z'-x-y) - \Phi_1(x') - \Phi_2(y') - \Phi_3(z')$$

for each (x', y', z'). Note that

$$\det(\nabla_x F(0), \nabla_y F(0)) = \det \begin{pmatrix} 1 & 1 & 1 \\ \frac{\partial \Phi_1}{\partial x_1}(x') & \frac{\partial \Phi_2}{\partial x_1}(y') & \frac{\partial \Phi_3}{\partial x_1}(z') \\ \frac{\partial \Phi_1}{\partial x_2}(x') & \frac{\partial \Phi_2}{\partial x_2}(y') & \frac{\partial \Phi_3}{\partial x_2}(z') \end{pmatrix},$$

which is non–zero uniformly in x', y' and z' in a sufficiently small neighbourhood of the origin. On observing that the C^3–norm of F is bounded uniformly in x', y' and z', (11) now follows from Proposition 7. □

Concluding remarks.

1. Although we have only worked in three dimensions here, similar questions may be posed for n–linear expressions in n dimensions for all $n \geq 2$. [2] Let $\mathcal{E}_1, ..., \mathcal{E}_n$ be the extension operators associated to the smooth mappings $\Sigma_1, ..., \Sigma_n : \mathbb{R}^{n-1} \to \mathbb{R}^n$

$$\mathcal{E}_j f(\xi) = \int_{\mathbb{R}^{n-1}} f(x) e^{i\xi \cdot \Sigma_j(x)} dx.$$

One may conjecture that if the kernels of the mappings $(d\Sigma_1(0))^*, ..., (d\Sigma_n(0))^*$ span $\mathbb{R}^n$, then there exists a constant C such that

$$\|\mathcal{E}_1 f_1 \ \cdots \ \mathcal{E}_n f_n\|_{2/(n-1)} \leq C\|f_1\|_2 \cdots \|f_n\|_2$$

for all $f_1, ..., f_n$ supported in a sufficiently small neighbourhood of 0.

Again, when the mappings Σ_j are linear, this inequality is true, and is equivalent to the classical Loomis–Whitney inequality (1). However, rather curiously, the natural extension of Theorem 8 to dimensions $n \geq 4$ requires something other than Theorem 2. More specifically, if one attempts to follow the argument of Theorem 8 in higher dimensions, one is required to prove that

$$\int_{(\mathbb{R}^{n-1})^n} f_1(u_1) \cdots f_n(u_n)\ \delta(\Sigma_1(u_1) + \cdots + \Sigma_n(u_n) - \xi)\ du \lesssim \|f_1\|_{(n-1)'} \cdots \|f_n\|_{(n-1)'}$$

[2]The two dimensional situation is of course well established.

for all $f_1, ..., f_n$ supported in sufficiently small neighbourhoods of 0, uniformly in ξ. In dimensions $n \geq 4$ this inequality appears to require non–linear versions of more general inequalities of Brascamp–Lieb type. Some recent progress on this by other methods can be found in [2].

2. Although the statements of Theorems 1 and 2 are diffeomorphism invariant, our subsequent trilinear restriction theorem (Theorem 8) is not. It seems likely that Theorem 8 may be generalised to a wider (diffeomorphism invariant) family of oscillatory integral operators satisfying a certain non–degeneracy condition in the spirit of Hörmander's conjecture. This line of investigation will be taken up in a subsequent paper.

Appendix: The proof of Proposition 3

It will suffice to deduce the existence of the required U, C and r for each of the parts (i)– (iv) individually.

Part (i) is an elementary consequence of the mean value theorem and the fact that for each j, $\|X_j\|_{C^1} \leq A'$ where A' is such that A'/A^{n-1} is a non–zero constant depending only on the dimension. We may henceforth assume that we are dealing only with x satisfying

$$\det(X_1(x), ..., X_n(x)) \geq \epsilon/2.$$

Parts (ii) and (ii) rely on the simple geometrical observation that for each $1 \leq j \leq n$, we have the lower bound $|X_j(x)| \geq \frac{\epsilon}{2(A')^{(n-1)}}$. Fix $1 \leq j \leq n$. By considering a rotation of $\mathbb{R}^n$ we may suppose that the nth component of X_j, denoted by $(X_j)_n$, satisfies $|(X_j)_n(0)| \gtrsim \epsilon/(A')^{n-1}$. By the mean value theorem there exists a neighbourhood U' of $0 \in \mathbb{R}^n$, depending only on ϵ, A and n, upon which $|(X_j)_n(x)| \gtrsim \epsilon/(A')^{n-1}$. We now define $G : U' \to \mathbb{R}^n$ by $G(x) = (\pi_j(x), x_n)$. It is now easily verified that

$$\det(DG(x)) \gtrsim \epsilon/(A')^{n-1}$$

on U', and so there exist neighbourhoods $U \subset U'$ and $V \subset \mathbb{R}^n$ of $0 \in \mathbb{R}^n$, such that $G : U \to V$ is invertible. That U and V may be chosen to depend only on ϵ, A and n is a consequence of quantitative versions of the inverse function theorem as in [4]. Furthermore, we may clearly choose V to be a convex set, such as an open ball. We now observe that $\pi_j \circ G^{-1} : V \to \mathbb{R}^{n-1}$ is the canonical submersion $(x', x_n) \mapsto x'$, and hence for any $y \in \mathbb{R}^{n-1}$, $(\pi_j \circ G^{-1})^{-1}(y) \cap V$ is connected. Since G is a homeomorphism,

$$G^{-1}\left((\pi_j \circ G^{-1})^{-1}(y)\right) \cap G^{-1}(V) = \pi_j^{-1}(y) \cap U$$

is connected. This completes the proof of part (ii).

In order to prove part (iii) we confine our attention, as we may, to $|t| \leq \frac{\epsilon}{2(A')^{n+1}}$. We first observe that our flows are close to linear in the sense that

$$\begin{aligned} |\gamma_j(t,x) - (x + tX_j(x))| &= |\gamma_j(t,x) - (\gamma_j(0,x) + tD_1\gamma_j(0,x))| \\ &= \left|\int_0^t \int_0^s D_1^2 \gamma_j(u,x)\, du\, ds\right| \leq \|X_j\|_{C^1}^2 t^2/2, \end{aligned} \tag{12}$$

where $\gamma_j(t,x) := e^{tX_j}(x)$. Consequently,

$$\begin{aligned} |\gamma_j(t,x)| &\geq |t|\frac{\epsilon}{2(A')^{n-1}}\left(1 - \frac{(A')^{n-1}}{\epsilon}\|X_j\|_{C^1}^2|t|\right) - |x| \\ &\geq |t|\frac{\epsilon}{4(A')^{n-1}} - |x|, \end{aligned} \tag{13}$$

and so if $0 < r < \frac{\epsilon}{2(A')^{n+1}}$, $|\gamma_j(t,x)| > \frac{r\epsilon}{4(A')^{n-1}}$ for all $|t| > r$ and $|x| \leq \frac{r\epsilon}{4(A')^{n-1}}$. On taking $U = B\left(0; \frac{r\epsilon}{4(A')^{n-1}}\right)$, part (iii) of the proposition is satisfied.

Part (iv) is a simple consequence of the chain rule and a quantitative version of the differential dependence on initial conditions of ODE's, see for example, [7].

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School of Mathematics, JCMB, Kings Buildings, Mayfield Road, Edinburgh, EH9 3JZ, Scotland.

E-mail address: J.Bennett@ed.ac.uk, A.Carbery@ed.ac.uk, J.R.Wright@ed.ac.uk