Mathematical Research Letters **12**, 335–339 (2005)

ON STABLE TORSION LENGTH OF A DEHN TWIST

Mustafa Korkmaz

Abstract. In this note we prove that the stable torsion length of a Dehn twist is positive. This, in particular, answers a question of T. E. Brendle and B. Farb in the negative. We also give upper bounds for this length.

1. Introduction

In a recent paper [1], Tara E. Brendle and Benson Farb showed that the mapping class group of a closed orientable surface of genus $g \geq 3$ is generated by three torsion elements, and also by six involutions. They asked whether there is a number $C = C(g)$ such that every element in the mapping class group can be written as a product of at most C torsion elements. In this note, we give a negative answer to this question. We first prove that the stable torsion lengths of Dehn twists are positive, implying the mentioned negative answer. We deduce our result from the finiteness of the conjugacy classes of torsion elements in the mapping class group and from the fact that the stable commutator length of a Dehn twist is positive. We then obtain an upper bound for the stable torsion length of a Dehn twist.

2. Preliminaries

Suppose that G is a group and that $f \in G$ can be written as a product of torsion elements in G. Let us define the *torsion length* $\tau(f)$ of f to be the minimum number of factors needed to express f as a product of torsion elements. Clearly, the sequence $\tau(f^n)$ is subadditive, that is, $\tau(f^{n+m}) \leq \tau(f^n) + \tau(f^m)$ for all positive integers n, m. Therefore, the limit

$$\lim_{n\to\infty} \frac{\tau(f^n)}{n}$$

exists. Let us denote this limit by $||f||_\tau$ and call it the *stable torsion length* of f. Similarly, one defines the *commutator length* $c(f)$ and the *stable commutator length* $||f||_c$ of $f \in G$ if f can be written as a product of commutators.

More generally, if $f^n \in G$ can be written as a product of torsion elements (respectively commutators) for some n, then one may define $||f||_\tau = \frac{1}{n}||f^n||_\tau$

Received July 9, 2004.

The author was supported in part by the Turkish Academy of Sciences under the Young Scientists Award Program (MK/TÜBA-GEBİP 2003-10).

(respectively $||f||_c = \frac{1}{n}||f^n||_c$). Clearly, this definition is independent of the choice of the exponent.

Lemma 1. *Let G be a group and let $q \geq 2$ be an integer. Suppose that an element $f \in G$ can be written as a product of $l \geq 2$ torsion elements such that at least two of these torsion elements have orders dividing q. Then $||f||_\tau \leq l - \frac{q}{q-1}$*

Proof. Suppose that $f = x_1 x_2 \cdots x_l$, where x_i are torsion elements and the orders of at least two x_i divide q. After conjugating with appropriate elements, we may assume that the order of x_l is q. Then we can write

$$\begin{aligned} f^q &= (x_1 x_2 \cdots x_l)^q \\ &= (x_1 x_2 \cdots x_{l-1})(x_1 x_2 \cdots x_{l-1})^{x_l} \cdots (x_1 x_2 \cdots x_{l-1})^{x_l^{q-1}} x_l^q. \end{aligned}$$

Since $x_l^q = 1$, f^q is a product of $ql - q$ torsion elements and at least q of these torsion elements are of order dividing q. By iterating this argument, it can be shown that f^{q^n} is a product of $q^n l - (q^n + \cdots + q^2 + q) = q^n l - q\frac{q^n - 1}{q-1}$ torsion elements for each positive integer n. That is, $\tau(f^{q^n}) \leq q^n l - \frac{q}{q-1}(q^n - 1)$. The conclusion of the lemma follows from this. □

Let S be a closed oriented surface of genus g and let Mod_g denote the mapping class group of S, the group of isotopy classes of orientation-preserving diffeomorphisms $S \to S$. For a simple closed curve a on S, we denote by t_a the (isotopy class of a) Dehn twist about a. We will denote a diffeomorphism and its isotopy class by the same symbol.

The next theorem is well known to the experts and a proof of it may be found in [6] or in [9].

Theorem 2. *The number of conjugacy classes of torsion elements in the mapping class group Mod_g is finite.*

Corollary 3. *Suppose that $g \geq 3$. There is a constant $T(g)$ such that every torsion element in the mapping class group Mod_g can be written as a product of at most $T(g)$ commutators.*

Proof. The mapping class group Mod_g is perfect for $g \geq 3$ (c.f. [8]), so that every mapping class is a product of commutators. Choose representatives from each conjugacy class of torsion elements and write them as products of commutators. Let $T(g)$ be the maximum number of such commutators. Since every conjugation of a commutator is again a commutator, it follows from Theorem 2 that every torsion element in Mod_g is a product of at most $T(g)$ commutators. □

Although the mapping class group Mod_g is perfect for $g \geq 3$, it was shown in [2] and [4] that it is not uniformly perfect. That is, there is no constant $C(g)$ such that every element in the mapping class group Mod_g can be written as a product of at most $C(g)$ commutators.

Theorem 4. [2, 4] *Suppose that a is a homotopically nontrivial simple closed curve on a closed oriented surface of genus $g \geq 2$. Then $||t_a||_c \geq \frac{1}{18g-6}$.*

3. The results

We first prove that the stable torsion length of a Dehn twist is positive. A corollary to this result answers a question of Brendle and Farb [1] in the negative. We then give an upper bound for the stable torsion lengths of Dehn twists about homotopically nontrivial simple closed curves on a closed oriented surface.

Theorem 5. *Let $g \geq 3$ and let a be a homotopically nontrivial simple closed curve on a closed oriented surface S. Then $||t_a||_\tau > 0$ in Mod_g.*

Proof. Let $T = T(g)$ denote the positive integer in Corollary 3. Any element $f \in \mathrm{Mod}_g$ is a product of $\tau(f)$ torsion elements. Thus f can be written as a product of at most $\tau(f)T$ commutators. Therefore $c(f) \leq \tau(f)T$. It follows that $\frac{1}{T}\,||f||_c \leq ||f||_\tau$. The theorem follows from this since $||t_a||_c > 0$. □

Corollary 6. *Suppose that $g \geq 3$. There is no constant $C(g)$ such that every element in the mapping class group Mod_g of a closed orientable surface of genus g can be written as a product of at most $C(g)$ torsion elements.*

Remark. In the first version of this paper, Theorem 5 was a conjecture and Corollary 6 was the main result. After the appearence of the first version on the arXiv, D. Kotschick [7] proved by a different method that $2||f||_c \leq ||f||_\tau$ for any element f in a group with finitely generated abelianization.

Lemma 7. *Suppose that a and b are two disjoint nonseparating simple closed curves on a closed oriented surface S. Then $t_a t_b^{-1}$ can be written as a product of two involutions.*

Proof. Suppose that f is an involution in the mapping class group with $f(a) = b$. Then $t_a t_b^{-1} = t_a t_{f(a)}^{-1} = (t_a f t_a^{-1}) f^{-1}$ is a product of two involutions. Therefore in order to prove the lemma, it suffices to show that there is an involution $f \in \mathrm{Mod}_g$ with $f(a) = b$.

We may assume that S is the surface in Figure 1 embedded in the 3-space. If a is not homologous to b, then there is a diffeomorphism $h : S \to S$ such that $h(a) = a_1$ and $h(b) = a_{2g+1}$. Since the rotation r by π about the z-axis maps a_1 to a_{2g+1}, the involution $h^{-1}rh$ maps a to b. If a is homologous to b, then there is a diffeomorphism $k : S \to S$ such that $k(a) = c$ and $k(b) = d$, where the curves c and d are the curves on S as in Figure 1 such that the genus of one of the components of the complement of $c \cup d$ is equal to that of $a \cup b$. Now the rotation s by π about the y-axis maps c to d and thus the involution $k^{-1}sk$ maps a to b. □

Theorem 8. *Let S be a closed oriented surface of genus $g \geq 1$ and a be a homotopically nontrivial simple closed curve on S.*

1. *If a is nonseparating, then $||t_a||_\tau \leq \frac{1}{2} - \frac{1}{2(4g+1)}$.*
2. *If a is separating and $g \geq 3$, then $||t_a||_\tau \leq 4$.*

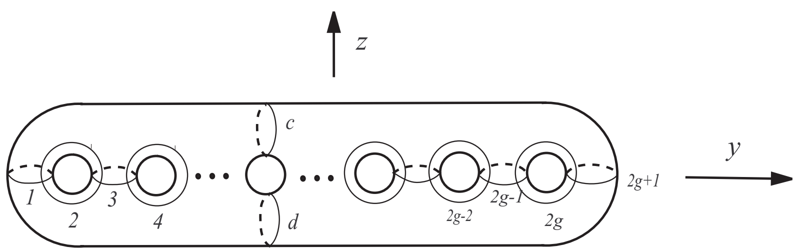

FIGURE 1. The curve labelled i is a_i.

3. *If a is separating and $g = 2$, then $||t_a||_\tau \leq \frac{15}{4}$.*

Proof. We may assume, once again, that S is the surface in Figure 1.

Suppose first that a is nonseparating. Since all Dehn twists about nonseparating simple closed curves form a single conjugacy class in Mod_g and a conjugate of a torsion element is again a torsion element, stable torsion lengths of all such Dehn twists are the same. Therefore, it suffices to prove (1) for some nonseparating simple closed curve.

Let us denote by t_i the Dehn twist about the simple closed curve a_i on S. It can easily be shown that the element $\sigma = t_{2g+1}t_{2g}\cdots t_2 t_1^2 t_2 \cdots t_{2g}t_{2g+1}$ is the isotopy class of the rotation by π about the y-axis and thus it is an involution. Also, it can easily be shown that the orders of the elements $x = (t_{2g+1}t_{2g}\cdots t_2)^{-1}$ and $y = (t_2 \cdots t_{2g}t_{2g+1})^{-1}$ are both $4g+2$. Moreover, it can easily be checked that σ commutes with x and y. Therefore, we can write $t_1^2 = xy\sigma$. Hence, we obtain $t_1^{2(4g+2)} = (xy)^{4g+2} = xx^y x^{y^2} \cdots x^{y^{4g+1}}$. That is, $t_1^{2(4g+2)}$ is a product of $4g+2$ elements of order $4g+2$. Now, (1) follows from Lemma 1.

Suppose next that a is separating and $g \geq 3$. By the use of the lantern relation, one can show as in [5] that there are six nonseparating simple closed curves $b_1, b_2, b_3, c_1, c_2, c_3$ on S such that b_i is disjoint from c_i and

$$t_a = (t_{b_1}t_{c_1}^{-1})(t_{b_2}t_{c_2}^{-1})(t_{b_3}t_{c_3}^{-1}).$$

By Lemma 7, t_a is a product of six involutions. We now apply Lemma 1 to get (2).

Suppose finally that a is separating and $g = 2$. In this case, let e be the boundary component of a regular neighborhood of $a_1 \cup a_2$. By the well-known one-holed torus relation in the mapping class group, $t_e = (t_1t_2)^6 = (t_2t_1)^6$. Let $x = (t_1t_2\cdots t_5)^2$ and $y = (t_2t_3t_4t_5)^{-2}$. It can be shown that $t_2t_1 = xy$ and that the orders of x and y are three and five respectively. Thus we get

$$\begin{aligned} t_e &= (xy)^6 \\ &= y^x y^{x^2} y^{yx} y^{yx^2} y^2. \end{aligned}$$

Hence, t_e is a product of five elements of order five. By applying Lemma 1 again, we get $||t_e||_\tau \leq \frac{15}{4}$. Since t_a is conjugate to t_e, we have finished the proof of (3), and hence the proof of theorem. □

References

[1] T. E. Brendle, B. Farb, *Every mapping class group is generated by* 6 *involutions*, J. Algebra **278** (2004), 187–198.

[2] H. Endo, D. Kotschick, *Bounded cohomology and non-uniform perfection of mapping class groups*, Invent. Math. **144** (2001), 169–175.

[3] B. Farb, A. Lubotzky, Y. N. Minsky, *Rank one phenomena of mapping class groups*, Duke Math Journal, **106**, (2001), 581–597.

[4] M. Korkmaz, *Stable commutator length of a Dehn twist*, Michigan Math. Journal, **52** (2004), 23–31.

[5] M. Korkmaz, B. Ozbagci, *Minimal number of singular fibers in a Lefschetz fibration*, Proc. Amer. Math. Soc. **129** (2001), 1545–1549.

[6] D. Kotschick, *On regularly fibered complex surfaces*, Proceedings of Kirbyfest (Berkeley, CA, 1998), 291–298, Geom. Topol. Publ., Coventry, 1999.

[7] ———, *Quasi-homomorphisms and stable lengths in mapping class groups*, Proc. Amer. Math. Soc. **132** (2004), 3167–3175

[8] J. Powell, *Two theorems on the mapping class group of a surface*, Proc. Amer. Math. Soc. **68** (1978), 347–350.

[9] K. Yokoyama, *Classification of periodic maps on compact surfaces: I*, Tokyo J. Math. **6** (1983), 75–94.

Department of Mathematics, Middle East Technical University, 06531 Ankara, Turkey

E-mail address: korkmaz@arf.math.metu.edu.tr