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WELL-POSEDNESS AND LOCAL SMOOTHING OF SOLUTIONS OF SCHRÖDINGER EQUATIONS

Alexandru D. Ionescu and Carlos E. Kenig

1. Introduction

In $\mathbb{R}^n \times \mathbb{R}$ consider the initial value problem for the Schrödinger equation with potential V

$$(1)\qquad \begin{cases} (i\partial_t + \Delta_x)u = Vu; \\ u(.,0) = u_0. \end{cases}$$

In the case $V \equiv 0$, it was established by P. Constantin and J. C. Saut [1], P. Sjölin [7], and L. Vega [9] that the solution of the initial value problem (1) gains 1/2 derivative (locally) over the initial data at almost every time. This type of gain is referred to as *local smoothing*. A. Ruiz and L. Vega [6] proved well-posedness and local smoothing in the case of potentials $V \in L^{n/2}_x L^\infty_t + L^r_t L^\infty_x$, $r > 1$, $n \geq 3$, with small $L^{n/2}_x L^\infty_t$ part. For more references on local smoothing estimates for linear dispersive equations, as well as their applications to nonlinear problems, see the introduction of [6].

In this note we prove well-posedness and local smoothing for potentials V in the larger space $L^1_t L^\infty_x + L^\infty_t L^{n/2}_x$, $n \geq 3$, with an additional smallness assumption on the $L^\infty_t L^{n/2}_x$ part. We also show that our space of potentials is optimal for well-posedness, in the scale of Strichartz spaces $L^p_t L^q_x$, and that the smallness assumption on the $L^\infty_t L^{n/2}_x$ part is necessary.

To state our theorems, we define the set $\mathcal{A}$ of *acceptable* Strichartz exponents (p, q) by the conditions $2/p + n/q = (n + 4)/2$ and $p, q \in [1, 2]$. In dimension $n = 2$, we require, in addition, that $(p, q) \neq (2, 1)$. For any $(p, q) \in \mathcal{A}$ let (p', q') denote the dual exponent, i.e. $1/p + 1/p' = 1/q + 1/q' = 1$. Clearly $2/p' + n/q' = n/2$, $p', q' \in [2, \infty]$, and $(p', q') \neq (2, \infty)$ in dimension $n = 2$; let $\mathcal{A}'$ denote the set of such exponents (p', q').

We define three Banach spaces of functions X, X', and Y on $\mathbb{R}^n \times \mathbb{R}$: if $n \geq 3$ then $X = L^1_t L^2_x + L^2_t L^{2n/(n+2)}_x$, $X' = L^\infty_t L^2_x \cap L^2_t L^{2n/(n-2)}_x$, and $Y =$

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$L^1_tL^\infty_x + L^\infty_tL^{n/2}_x$, i.e.,

$$\begin{cases} \|f\|_X = \inf_{f_1+f_2=f}[\|f_1\|_{L^1_tL^2_x(\mathbb{R}^n\times\mathbb{R})} + \|f_2\|_{L^2_tL^{2n/(n+2)}_x(\mathbb{R}^n\times\mathbb{R})}],\\ \|u\|_{X'} = \sup[\|u\|_{L^\infty_tL^2_x(\mathbb{R}^n\times\mathbb{R})}, \|u\|_{L^2_tL^{2n/(n-2)}_x(\mathbb{R}^n\times\mathbb{R})}],\\ \|V\|_Y = \inf_{V_1+V_2=V}[\|V_1\|_{L^1_tL^\infty_x(\mathbb{R}^n\times\mathbb{R})} + \|V_2\|_{L^\infty_tL^{n/2}_x(\mathbb{R}^n\times\mathbb{R})}]. \end{cases}$$

If $n = 1$ we define $X = L^1_tL^2_x + L^{4/3}_tL^1_x$, $X' = L^\infty_tL^2_x \cap L^4_tL^\infty_x$ and $Y = L^1_tL^\infty_x + L^2_tL^1_x$. In dimension $n = 2$ we have to exclude the endpoint spaces $L^2_tL^1_x$ and $L^2_tL^\infty_x$ for which the Strichartz estimates fail (cf. [5]). For this purpose we fix an acceptable pair $(p_0, q_0) \in \mathcal{A}$, $1 \le p_0 < 2$, and define $X = X_{p_0} = L^1_tL^2_x + L^{p_0}_tL^{q_0}_x$, $X' = X'_{p_0} = L^\infty_tL^2_x \cap L^{p'_0}_tL^{q'_0}_x$, and $Y = Y_{p_0} = L^1_tL^\infty_x + L^{p_0/(2-p_0)}_tL^{q_0/(2-q_0)}_x$. For any $a < b \in \mathbb{R}$, let $X'([a, b])$ denote the Banach space of measurable functions $u : \mathbb{R}^n\times[a, b] \to \mathbb{C}$ with $\|u\|_{X'([a,b])} = \|E(u)\|_{X'} < \infty$, where $E(u)(x, t) = u(x, t)$ if $t \in [a, b]$ and $E(u) = 0$ if $t \notin [a, b]$. The Banach spaces $X([a, b])$ and $Y([a, b])$ are defined in a similar way. These spaces were used in recent work by the authors [3].

For any measurable functions V and u we have

$$\|Vu\|_X \le \|V\|_Y\|u\|_{X'}. \tag{2}$$

With our notation, the Strichartz estimate of M. Keel and T. Tao [4] is equivalent to

$$\|u\|_{X'} \le C\|(i\partial_t + \Delta_x)u\|_X$$

for any $u \in C^\infty_0(\mathbb{R}^n \times \mathbb{R})$. Here, and in the rest of the paper, we use the letter C to denote constants that may depend only on the dimension n if $n \ne 2$, and on the exponent p_0 if $n = 2$. For the classical Strichartz estimates, see [8].

For any unit vector $w_0 \in \mathbb{R}^n$ let $D^{1/2}_{w_0} : L^2(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ denote the operator defined by the Fourier multiplier $\xi \to |\xi \cdot w_0|^{1/2}$. For any set S let $\mathbf{1}_S$ denote its characteristic function. Our first main theorem is the following:

Theorem 1. *Assume that $V : \mathbb{R}^n \times [0, T] \to \mathbb{C}$ has the property that*

$$\exists\delta_0 > 0 \text{ such that } \|V\mathbf{1}_{[a,a+\delta_0]}(t)\|_Y \le 1/(2\overline{C}) \text{ for any } a \in [0, T - \delta_0], \tag{3}$$

where $\overline{C}$ is the constant in (7). Then the initial value problem (1), with $u_0 \in L^2(\mathbb{R}^n)$, admits a unique solution $u \in C([0, T] : L^2(\mathbb{R}^n)) \cap X'([0, T])$ with the property that the linear map $u_0 \to u$ is continuous from $L^2(\mathbb{R}^n)$ to $C([0, T] : L^2(\mathbb{R}^n)) \cap X'([0, T])$. In addition, we have the local smoothing estimate

$$\sup_{R>0,a\in\mathbb{R}} R^{-1/2}\|\mathbf{1}_{[a-R,a+R]}(x \cdot w_0)D^{1/2}_{w_0}u\|_{L^2_{x,t}(\mathbb{R}^n\times[0,T])} \le C_V\|u_0\|_{L^2}, \tag{4}$$

for any unit vector $w_0 \in \mathbb{R}^n$. The constant C_V may depend only on T, δ_0, the dimension n if $n \ne 2$, and the exponent p_0 if $n = 2$.

Our local smoothing estimate (4) is more precise than the local smoothing estimate of A. Ruiz and L. Vega [6]. Let $D^{1/2} : L^2(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ denote the operator defined by the Fourier multiplier $\xi \to |\xi|^{1/2}$. Then (4) can be localized to show that

$$\sup_{R>0, x_0 \in \mathbb{R}^n} R^{-1/2} ||\mathbf{1}_{[0,R]}(|x-x_0|) D^{1/2} u||_{L^2_{x,t}(\mathbb{R}^n \times [0,T])} \le C_V ||u_0||_{L^2},$$

which is the standard form of the local smoothing estimate.

Let $Z([0,T])$ denote the class of potentials V that satisfy (3). Clearly,

$$Z([0,T]) \subseteq Y([0,T]).$$

In fact,

$$Z([0,T]) = Y([0,T]) \text{ if } n = 1, 2.$$

In dimensions $n \ge 3$, $Z([0,T])$ does not contain $L^\infty_t L^{n/2}_x(\mathbb{R}^n \times [0,T])$. However, we show in section 2 that

$$L^p_t L^q_x(\mathbb{R}^n \times [0,T]) \subseteq Z([0,T]) \text{ if } 2/p + n/q \le 2 \text{ and } q \in (n/2, \infty], \tag{5}$$

and

$$C([0,T] : L^{n/2}(\mathbb{R}^n)) \subseteq Z([0,T]). \tag{6}$$

Our second theorem shows that the assumption (3) is essentially optimal: ill-posedness (i.e. lack of uniqueness) may occur below the critical line $2/p+n/q = 2$ (compare with (5)), as well as for some potentials $V \in L^\infty_t L^{n/2}_x$ for which the smallness assumption (3) fails (compare with (6)).

Theorem 2. *For any $N \ge 0$ there is a (not identically 0) function $u \in C(\mathbb{R} : H^N(\mathbb{R}^n))$, and a measurable potential V with the following properties:*

(i) $\operatorname{supp} V \subseteq \mathbb{R}^n \times [0,1]$, *$V \in L^p_t L^q_x(\mathbb{R}^n \times [0,1])$ for any $p, q \in [1, \infty]$ with $2/p + n/q > 2$, and $V \in L^\infty_t L^{n/2}_x(\mathbb{R}^n \times [0,1])$, $n \ge 2$;*

(ii) $(i\partial_t + \Delta_x)u = Vu$ as distributions on $\mathbb{R}^n \times \mathbb{R}$;

(iii) $u \equiv 0$ in $\mathbb{R}^n \times (-\infty, 0]$.

Our construction is inspired from counterexamples in unique continuation (see, for example, [10]). Our counterexample is easier, however, since we do not require vanishing of infinite order at time $t = 0$.

The rest of the paper is organized as follows: in section 2 we prove Theorem 1 and the inclusions (5) and (6). In section 3 we prove Theorem 2.

2. Proof of Theorem 1

We prove first the inclusions (5) and (6). For (5), fix $V \in L^p_t L^q_x(\mathbb{R}^n \times [0,T])$, $\varepsilon > 0$. Let $G(t) = ||V(.,t)||_{L^q_x}$, $V_s = V \cdot \mathbf{1}_{\{(x,t):|V(x,t)| \le \lambda(t)\}}$, $V_l = V \cdot \mathbf{1}_{\{(x,t):|V(x,t)| > \lambda(t)\}}$, $\lambda(t) = G(t)^{2q/(2q-n)}/\varepsilon^{n/(2q-n)}$. It follows easily that $V_s \in L^1_t L^\infty_x(\mathbb{R}^n \times [0,T])$ and $||V_l||_{L^\infty_t L^{n/2}_x(\mathbb{R}^n \times [0,T])} \le \varepsilon$. Thus V satisfies (3) for δ_0 small enough.

To prove (6), fix $V \in C([0,T] : L^{n/2}(\mathbb{R}^n))$, $\varepsilon > 0$. Since $V \in C([0,T] : L^{n/2}(\mathbb{R}^n))$ we can find a finite sequence $0 = t_0 < t_1 < \ldots < t_m = T$ with the property that $||V-\widetilde{V}||_{L^\infty_t L^{n/2}_x(\mathbb{R}^n\times[0,T])}\leq\varepsilon/2$, where $\widetilde{V}(x,t)=V(x,t_j)$ if $t\in[t_j,t_{j+1})$. Then, we further decompose $\widetilde{V}=\widetilde{V}_s+\widetilde{V}_l$, where $\widetilde{V}_s=\widetilde{V}\cdot\mathbf{1}_{\{(x,t):|\bar{V}(x,t)|\leq\lambda\}}$, $\widetilde{V}_l=\widetilde{V}\cdot\mathbf{1}_{\{(x,t):|\bar{V}(x,t)|>\lambda\}}$. For λ large enough, $\widetilde{V}_s\in L^1_tL^\infty_x(\mathbb{R}^n\times[0,T])$ and $||\widetilde{V}_l||_{L^\infty_t L^{n/2}_x(\mathbb{R}^n\times[0,T])}\leq\varepsilon/2$. Thus V satisfies (3) for δ_0 small enough.

We turn now to the proof of Theorem 1. Let H denote the Schrödinger operator $i\partial_t+\Delta_x$. To prove well-posedness we use Strichartz estimates. This type of argument is standard (see, for example, [6, Theorem 1.1]). Our main tool is the Strichartz estimate of M. Keel and T. Tao [4, Theorem 1.2]. For any $g\in\mathcal{S}(\mathbb{R}^n)$ (the Schwartz space on $\mathbb{R}^n$) define $A(g)$ by the formula

$$\widetilde{A(g)}(\xi,t)=e^{-it|\xi|^2}\widehat{g}(\xi),$$

where $\widehat{g}$ denotes the Fourier transform of g, and $\widetilde{F}(\xi,t)$ denotes the partial Fourier transform of the function F in the variable x. For any $f\in\mathcal{S}(\mathbb{R}^n\times\mathbb{R})$ supported in $\mathbb{R}^n\times[0,\infty)$ define $B(f)$ by the formula

$$\widetilde{B(f)}(\xi,t)=(-i)\int_0^t e^{-i(t-s)|\xi|^2}\widetilde{f}(\xi,s)\,ds.$$

By [4, Theorem 1.2] (see also [4, p. 972]), A extends to a bounded operator from $L^2(\mathbb{R}^n)$ to $X'([0,T'])$, and B extends to a bounded operator from $X([0,T'])$ to $X'([0,T'])$, i.e.

$$\begin{cases}||A||_{L^2(\mathbb{R}^n)\to X'([0,T'])}\leq\overline{C}^{1/2};\\ ||B||_{X([0,T'])\to X'([0,T'])}\leq\overline{C},\end{cases}\tag{7}$$

uniformly in T'. For uniqueness, we use the fact that if $u\in C([0,T]:L^2(\mathbb{R}^n))\cap X'([0,T])$ solves (1) then $u=\mathbf{1}_{[0,T]}(t)Au_0+\mathbf{1}_{[0,T]}(t)B(\mathbf{1}_{[0,T]}(t)Vu)$. By (2), (3) with $a=0$, and (7) with $T'=\delta_0$, we have $\mathbf{1}_{[0,\delta_0]}(t)u\equiv0$ if $u_0\equiv0$. This proves uniqueness of solutions in $C([0,T]:L^2(\mathbb{R}^n))\cap X'([0,T])$.

To prove existence, consider the Banach space $X'([0,\delta_0])$ and the operator $R(v)=\mathbf{1}_{[0,\delta_0]}(t)B(\mathbf{1}_{[0,\delta_0]}(t)Vv)$. By (2), (3) with $a=0$, and (7) with $T'=\delta_0$, R is a bounded operator on $X'([0,\delta_0])$ with $||R||_{X'([0,\delta_0])\to X'([0,\delta_0])}\leq1/2$. Therefore the operator $I-R$ is invertible on $X'([0,\delta_0])$, which shows that there is $v_1\in X'([0,\delta_0])$ with the property that

$$v_1=\mathbf{1}_{[0,\delta_0]}(t)Au_0+\mathbf{1}_{[0,\delta_0]}(t)B(\mathbf{1}_{[0,\delta_0]}(t)Vv_1).$$

This formula, the definition of the operators A and B, and the bound (7) show that $v_1\in C([0,\delta_0]:L^2(\mathbb{R}^n))$. In addition, by taking X' norms,

$$||v_1||_{X'([0,\delta_0])}\leq C||u_0||_{L^2}.$$

We can now continue the recursive procedure and construct solutions $v_2,v_3,\dots$ in $X'([\delta_0,2\delta_0]),X'([2\delta_0,3\delta_0]),\dots$. The global solution u is obtained by adjoining these solutions.

For the local smoothing bound (4) we may assume $w_0=(1,0,\dots,0)$, using the rotation invariance. We will prove the following *a priori* estimate:

Lemma 3. *If $v \in C_0^\infty(\mathbb{R}^n \times \mathbb{R})$ then, with $w_0 = (1, 0, \dots, 0)$,*

$$\Big[\int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} |D_{w_0}^{1/2} v(x_1, x', t)|^2 \, dx' dt\Big]^{1/2} \le C||Hv||_X, \tag{8}$$

for any $x_1 \in \mathbb{R}$, where, as before, $H = i\partial_t + \Delta_x$.

To deduce (4) from Lemma 3, we apply the inequality (8) to the function

$$v(x,t) = (u * \varphi_\delta)(x,t)\widetilde{\eta}_r(x)\eta_\varepsilon(t),$$

where $\varphi_\delta(x,t) = \delta^{-n-1}\varphi(x/\delta, t/\delta)$, $\delta > 0$, is a smooth approximation of the identity supported in $\{(x,t) : |x|, |t| \le \delta\}$, the smooth cutoff functions $\eta_\varepsilon : [0,T] \to [0,1]$, $2\delta \le \varepsilon \le T/10$, are supported in $[\varepsilon, T-\varepsilon]$ and equal to 1 in $[2\varepsilon, T-2\varepsilon]$, and the smooth cutoff functions $\widetilde{\eta}_r$, $r \ge 1$, are supported in $\{x : |x| \le 2r\}$ and equal to 1 in $\{x : |x| \le r\}$. Then we let r tend to ∞, δ tend to 0, and ε tend to 0 (in this order). The bound (4) follows easily from (8) and the bound

$$||u||_{X'([0,T])} \le C_V ||u_0||_{L^2},$$

which was proved before. See [3, Section 3] for more details.

To summarize, it remains to prove Lemma 3. In dimensions $n \ge 3$ we need an interpolation lemma of M. Keel and T. Tao [4] (see pages 964–967 in [4] for the proof):

Lemma 4. (M. Keel, T. Tao [4]). *Assume $n \ge 3$ and for any $f \in C_0^\infty(\mathbb{R}^n \times \mathbb{R})$*

$$U(f)(x,t) = \int_{\mathbb{R}^n} \int_{\mathbb{R}} f(y,s) K(x,y,t,s) \, dyds$$

is an operator with a locally integrable kernel K. For $l \in \mathbb{Z}$ let

$$U_l(f)(x,t) = \int_{\mathbb{R}^n} \int_{|t-s| \in [2^l, 2^{l+1}]} f(y,s) K(x,y,t,s) \, dyds.$$

Let

$$\beta(a,b) = \frac{n}{2} - 1 - \frac{n}{2}\left(\frac{1}{a'} + \frac{1}{b'}\right)$$

and assume that the estimate

$$||U_l(f)||_{L_t^2 L_x^{b'}} \le 2^{-l\beta(a,b)} ||f||_{L_s^2 L_y^a} \tag{9}$$

holds for the exponents

(i) $a = b = 1$,
(ii) $2n/(n+1) \le a \le 2$ *and* $b = 2$,
(iii) $2n/(n+1) \le b \le 2$ *and* $a = 2$.

Then

$$||U(f)||_{L_t^2 L_x^{2n/(n-2)}} \le C ||f||_{L_s^2 L_y^{2n/(n+2)}}.$$

We prove now Lemma 3. Let $g = (i\partial_t + \Delta_x)v$. With the same notation as before

$$\widetilde{v}(\xi,t) = C\int_{\mathbb{R}} \mathbf{1}_+(t-s)e^{-i(t-s)|\xi|^2}\widetilde{g}(\xi,s)\,ds,$$

where $\mathbf{1}_+$ denotes the characteristic function of the interval $[0,\infty)$. Thus

$$\widetilde{D^{1/2}_{w_0}v}(\xi,t) = C|\xi_1|^{1/2}\int_{\mathbb{R}} \mathbf{1}_+(t-s)e^{-i(t-s)|\xi|^2}\widetilde{g}(\xi,s)\,ds.$$

For $\varepsilon > 0$ let Q_ε denote the operator defined by the Fourier multiplier $(\xi,\tau) \to e^{-\varepsilon^2|\xi|^2}$. By taking the inverse Fourier transform in x we have

$$Q_\varepsilon D^{1/2}_{w_0}v(x,t) = C\int_{\mathbb{R}^n}\int_{\mathbb{R}} dyds\, g(y,s) \int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi}e^{-i(t-s)|\xi|^2}e^{-\varepsilon^2|\xi|^2}|\xi_1|^{1/2}\mathbf{1}_+(t-s)\,d\xi.$$

It remains to prove that the operators

$$T_{\pm,x_1}(f)(x',t) = \int_{\mathbb{R}^n}\int_{\mathbb{R}} dyds\, f(y,s)K_\pm(x_1,x',y,t,s)$$
$$= \int_{\mathbb{R}^n}\int_{\mathbb{R}} dyds\, f(y,s)e^{-\varepsilon'^2(t-s)^2}\int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi}e^{-i(t-s)|\xi|^2}e^{-\varepsilon^2|\xi|^2}\mu_\pm(\xi_1,t,s)\,d\xi$$

are bounded from X to $L^2_{x',t}$, uniformly in $\varepsilon,\varepsilon' > 0$ and $x_1 \in \mathbb{R}$. The multipliers $\mu_\pm$ are given by

$$\mu_\pm(\xi_1,t,s) = \mathbf{1}_+(t-s)\mathbf{1}_\pm(\xi_1)|\xi_1|^{1/2}. \tag{10}$$

By symmetry and translation invariance it suffices to prove the boundedness of the operator $T_{+,0}$. To cover all dimensions fix (p,q) an acceptable pair, $p \le 4/3$ if $n=1$, $p \le p_0$ if $n=2$, and $p \le 2$ if $n \ge 3$. Clearly an operator is bounded from X to $L^2_{x',t}$ if it is bounded from $L^p_sL^q_y$ to $L^2_{x',t}$, uniformly in (p,q). It suffices to prove that the operator $S_{+,0} = T^*_{+,0}T_{+,0}$ is bounded from $L^p_sL^q_y$ to $L^{p'}_tL^{q'}_x$. Let $L_{+,0}$ denote the kernel of the operator $S_{+,0}$, i.e.,

$$L_{+,0}(x,y,t,s) = \int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}} K_+(0,z',y,r,s)\overline{K}_+(0,z',x,r,t)\,dz'dr. \tag{11}$$

We will prove that the kernel $L_{+,0}(.,.,t,s)$ defines a bounded operator on L^2, i.e.,

$$\left|\left|\int_{\mathbb{R}^n} h(y)L_{+,0}(x,y,t,s)\,dy\right|\right|_{L^2_x} \le C||h||_{L^2_y} \tag{12}$$

uniformly in t and s. In addition, we will prove the dispersive bound

$$|L_{+,0}(x,y,t,s)| \le C|t-s|^{-n/2} \tag{13}$$

uniformly in all the variables.

We show first how to use these bounds to complete the proof. Assume first that $p < 2$. By interpolating between (12) and (13) we have

$$\left|\left|\int_{\mathbb{R}^n} h(y)L_{+,0}(x,y,t,s)\,dy\right|\right|_{L_x^{q'}} \le C|t-s|^{-n(1/q-1/2)}||h||_{L_y^q}.$$

By the Minkowski inequality for integrals we have

$$\begin{aligned}||S_{+,0}(f)(.,t)||_{L_x^{q'}} &\le \int_{\mathbb{R}} ds \left|\left|\int_{\mathbb{R}^n} dy\, f(y,s)L_{+,0}(x,y,t,s)\right|\right|_{L_x^{q'}} \\ &\le C\int_{\mathbb{R}} ds||f(.,s)||_{L_y^q}|t-s|^{-n(1/q-1/2)}.\end{aligned}$$

Since $n(1/q-1/2) = 2/p'$, it follows by fractional integration that

$$||S_{+,0}(f)||_{L_t^{p'}L_x^{q'}} \le C_p||f||_{L_s^pL_y^q},$$

with a constant C_p that blows up as p approaches 2.

It remains to consider the case $n \ge 3$, $p = 2$. For this we use Lemma 4; it remains to verify the estimate (9) for the operator $S_{+,0}^l$ defined by the kernel $L_{+,0}^l(x,y,t,s) = L_{+,0}(x,y,t,s)\mathbf{1}_{[2^l,2^{l+1}]}(|t-s|)$. We may assume that the function f is supported in a time interval of length 2^{l+1}, say $\mathbb{R}^n \times [s_0-2^l, s_0+2^l]$. Then $S_{+,0}^l(f)$ is supported in $\mathbb{R}^n \times [s_0 - 3\cdot 2^l, s_0 + 3\cdot 2^l]$. To check the estimate (9) with $a = b = 1$ we use (13):

$$\begin{aligned}||S_{+,0}^l(f)||_{L_t^2L_x^\infty} &\le C2^{l/2}||S_{+,0}^l(f)||_{L_{x,t}^\infty} \le C2^{l/2}\sup|L_{+,0}^l(x,y,t,s)|\,||f||_{L_{y,s}^1} \\ &\le C2^{l/2}2^{-l\cdot n/2}2^{l/2}||f||_{L_s^2L_y^1} = C2^{-l\beta(1,1)}||f||_{L_s^2L_y^1},\end{aligned}$$

as desired. For the $L_s^2L_y^a \to L_t^2L_x^2$ bound, recall that $a \in [2n/(n+1), 2]$ and let $p(a) \in [1, 4/3]$ denote the exponent with the property that $(p(a), a) \in \mathcal{A}$. We have already proved that the operator $S_{+,0}$ is bounded from $L_s^pL_y^q$ to $L_t^{p'}L_x^{q'}$ if $(p,q) \in \mathcal{A}$ and $1 \le p \le 4/3$. Since $S_{+,0} = T_{+,0}^*T_{+,0}$, the operator $T_{+,0}$ is bounded from $L_s^pL_y^q$ to $L_{x',t}^2$ if $(p,q) \in \mathcal{A}$ and $1 \le p \le 4/3$, i.e.,

$$||T_{+,0}(f)||_{L_{x',t}^2} \le C||f||_{L_s^pL_y^q}.$$

We use this bound with $p = 1$ and $p = p(a)$. This type of argument may be found in [2]. With the notation $\mathbf{1}_l(t-s) = \mathbf{1}_{[2^l,2^{l+1}]}(|t-s|)$ we have for any f, g

supported in $\mathbb{R}^n \times [s_0 - 3\cdot 2^l, s_0 + 3\cdot 2^l]$

$$\begin{aligned}\langle S^l_{+,0}(f), g\rangle &= \int_{\mathbb{R}^n}\int_{\mathbb{R}}\int_{\mathbb{R}^n}\int_{\mathbb{R}} dydsdxdt\, f(y,s)L_{+,0}(x,y,t,s)\mathbf{1}_l(t-s)\overline{g}(x,t)\\ &= \int_{\mathbb{R}} dt \int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}} dz'dr\left(\int_{\mathbb{R}^n}\overline{g}(x,t)\overline{K}_+(0,z',x,r,t)\,dx\right)\\ &\qquad\qquad\left(\int_{\mathbb{R}^n}\int_{\mathbb{R}} f(y,s)\mathbf{1}_l(t-s)K_+(0,z',y,r,s)\,dyds\right)\\ &\le C\int_{\mathbb{R}} dt\,||\overline{g}(.,t)||_{L^2_x}||f||_{L^{p(a)}_s L^a_y} = C||f||_{L^{p(a)}_s L^a_y}||g||_{L^1_t L^2_x}\\ &\le C2^{l(1/p(a)-1/2)}||f||_{L^2_s L^a_y}2^{l/2}||g||_{L^2_t L^2_x} = C2^{-l\beta(a,2)}||f||_{L^2_s L^a_y}||g||_{L^2_t L^2_x},\end{aligned}$$

as desired. The $L^2_s L^2_y \to L^2_t L^{b'}_x$ bound is similar. By Lemma 4, this completes the proof of Lemma 3.

It remains to prove the bounds (12) and (13). For (12), it is more convenient to prove that the operator $T_{+,0}$ is bounded from L^2_y to $L^2_{x',t}$,

$$\left|\left|\int_{\mathbb{R}^n} h(y)K_+(0,x',y,t,s)\,dy\right|\right|_{L^2_{x',t}} \le C||h||_{L^2_y},$$

uniformly in s. We may ignore the factor $\mathbf{1}_+(t-s)$ and assume that $h\in\mathcal{S}(\mathbb{R}^n)$. It suffices to prove that

$$\left|\left|e^{-\varepsilon'^2t^2}\int_{\mathbb{R}^n}\widehat{h}(\xi)e^{is|\xi|^2}e^{ix'\cdot\xi'}e^{-it|\xi|^2}e^{-\varepsilon^2|\xi|^2}\mathbf{1}_+(\xi_1)|\xi_1|^{1/2}\,d\xi\right|\right|_{L^2_{x',t}} \le C||h||_{L^2}.$$

We may ignore the factor $e^{is|\xi|^2}$ and take the Fourier transform in (x,t). By Plancherel's theorem it suffices to prove that

$$\left|\left|\int_{\mathbb{R}} h(\xi_1,\eta')G_{\varepsilon'}(\tau+|\eta'|^2+\xi_1^2)e^{-\varepsilon^2\xi_1^2}\mathbf{1}_+(\xi_1)|\xi_1|^{1/2}\,d\xi_1\right|\right|_{L^2_{\eta',\tau}} \le C||h||_{L^2}$$

for any $h\in\mathcal{S}(\mathbb{R}^n)$, where $G_{\varepsilon'}$ is the Fourier transform of the function $t\to e^{-\varepsilon'^2t^2}$. The change of variables $\tau\to\mu+|\eta'|^2$ shows that we may replace the factor $G_{\varepsilon'}(\tau+|\eta'|^2+\xi_1^2)$ with $G_{\varepsilon'}(\mu+\xi_1^2)$. Let $H(\xi_1) = [\int_{\mathbb{R}^{n-1}}|h(\xi_1,\eta')|^2\,d\eta']^{1/2}$. We apply the Minkowski inequality for integrals for the variables η'. It remains to prove that

$$\left|\left|\int_{\mathbb{R}} H(\xi_1)G_{\varepsilon'}(\mu+\xi_1^2)e^{-\varepsilon^2\xi_1^2}\mathbf{1}_+(\xi_1)|\xi_1|^{1/2}\,d\xi_1\right|\right|_{L^2_\mu} \le C||H||_{L^2(\mathbb{R})}. \tag{14}$$

We make the change of variable $\xi_1 = (\eta_1-\mu)^{1/2}$, $\eta_1\in[\mu,\infty)$, and apply the Minkowski inequality for integrals for the variable μ. Since $||G_{\varepsilon'}||_{L^1(\mathbb{R})}\le C$, the bound (14) follows. This completes the proof of (12).

For the dispersive bound (13), we may assume that

$$0 = s < t. \tag{15}$$

Recall that

$$K_+(0, z', y, r, s) = e^{-\varepsilon'^2(r-s)^2} \int_{\mathbb{R}^n} e^{i(z-y)\cdot\xi} e^{-i(r-s)|\xi|^2} e^{-\varepsilon^2|\xi|^2} \mu_+(\xi_1, r, s)\, d\xi,$$

where $z = (0, z')$. We substitute this into the formula (11) defining the kernel $L_{+,0}$, use (15), and integrate the variable z' first. The result is

$$\begin{aligned} L_{+,0}(x,y,t,s) = C \int_{\mathbb{R}^{n-1}} & e^{i(x'-y')\cdot\xi'} e^{-it|\xi'|^2} e^{-2\varepsilon^2|\xi'|^2}\, d\xi' \times \\ \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} & e^{i(x_1\eta_1 - y_1\xi_1)} e^{i(r\eta_1^2 - r\xi_1^2 - t\eta_1^2)} e^{-\varepsilon^2(\eta_1^2+\xi_1^2)} \\ & e^{-\varepsilon'^2(r^2+(r-t)^2)} \mathbf{1}_+(r-t)|\xi_1|^{1/2} \mathbf{1}_+(\xi_1)|\eta_1|^{1/2} \mathbf{1}_+(\eta_1)\, dr d\eta_1 d\xi_1. \end{aligned} \tag{16}$$

Assume that $b \in \mathbb{R}$ and $m \in C^1(\mathbb{R} \setminus \{b\})$ is a function that satisfies the Hörmander–Michlin bound

$$\sup_{\eta\in\mathbb{R}\setminus\{b\}} |m(\eta)| + \sup_{\eta\in\mathbb{R}\setminus\{b\}} |(\eta - b)\cdot m'(\eta)| \le 1. \tag{17}$$

Then we claim that

$$\left| \int_{\mathbb{R}} e^{i\delta\mu^2} e^{ia\mu} e^{-\varepsilon^2(\mu-b')^2} m(\mu)\, d\mu \right| \le C|\delta|^{-1/2} \tag{18}$$

for any $\delta \in \mathbb{R} \setminus \{0\}$, $a, b' \in \mathbb{R}$ and $\varepsilon \in (0, \infty)$. To see this, by a linear change of variable, we can assume that $\delta = \pm 1$ and $a = 0$. Let $\widetilde{m}(\mu) = e^{-\varepsilon^2(\mu-b')^2} m(\mu)$ and B denote the set of numbers b, b'. Clearly $\widetilde{m} \in L^1(\mathbb{R})$ and

$$|\widetilde{m}(\mu)| + \operatorname{dist}(\mu, B) \cdot |\widetilde{m}'(\mu)| \le C$$

for any $\mu \in \mathbb{R} \setminus B$. By breaking up the integral in (18) into at most 4 integrals we see that it suffices to prove that

$$\left| \int_A^{\widetilde{A}} e^{i\delta\mu^2} m(\mu)\, d\mu \right| \le C$$

uniformly in $A, \widetilde{A} \in \mathbb{R}$, provided that $\delta = \pm 1$ and

$$|m(\mu)| + |(\mu - A) \cdot m'(\mu)| \le 1.$$

This follows by a routine integration by parts argument.

By using (18), we see that the ξ' integral in (16) is bounded uniformly by $C|t-s|^{-(n-1)/2}$. For the integral in r, η_1, ξ_1, we make the change of variables $r = t + \mu$ and define

$$F_{\varepsilon',t}(\nu) = \int_{\mathbb{R}} \mathbf{1}_+(\mu) e^{-\varepsilon'^2[\mu^2+(t+\mu)^2]} e^{i\mu\nu}\, d\mu.$$

As the inverse Fourier transform of a Hörmander–Michlin multiplier, the function $F_{\varepsilon',t}$ is a Calderón-Zygmund kernel, i.e.,

$$|\nu^{k+1}\partial_\nu^k F_{\varepsilon',t}(\nu)| \leq C_k, \tag{19}$$

for any integer $k \geq 0$ and $\nu \in \mathbb{R}$, and

$$\left| \int_{|\nu|\leq R} F_{\varepsilon',t}(\nu)\, d\nu \right| \leq C, \tag{20}$$

for any $R > 0$. The second integral in (16) becomes

$$\int_0^\infty e^{i(x_1-y_1)\xi_1} e^{-it\xi_1^2} e^{-\varepsilon^2\xi_1^2} m_0(\xi_1)\, d\xi_1, \tag{21}$$

where

$$m_0(\xi_1) = \mathbf{1}_+(\xi_1) \int_0^\infty e^{ix_1(\eta_1-\xi_1)} F_{\varepsilon',t}(\eta_1^2 - \xi_1^2) e^{-\varepsilon^2\eta_1^2} |\xi_1|^{1/2} |\eta_1|^{1/2}\, d\eta_1.$$

We make the change of variable $\eta_1 = \xi_1(1+\mu)^{1/2}$. The formula for m_0 becomes

$$m_0(\xi_1) = C\mathbf{1}_+(\xi_1) \int_{-1}^\infty e^{ix_1\xi_1((1+\mu)^{1/2}-1)} \xi_1^2 F_{\varepsilon',t}(\xi_1^2\mu) e^{-\varepsilon^2\xi_1^2(1+\mu)} (1+\mu)^{-1/4}\, d\mu.$$

We break the multiplier m_0 into three parts using a partition of unity $1 = \psi_1(\mu) + \psi_2(\mu) + \psi_3(\mu)$, where $\psi_1, \psi_2, \psi_3 : \mathbb{R} \to [0,1]$ are smooth functions, ψ_1 is supported in $(-\infty, -1/4]$ and equal to 1 in $(-\infty, -3/4]$, ψ_2 is supported in $[-3/4, 3/4]$ and equal to 1 in $[-1/4, 1/4]$, and ψ_3 is supported in $[1/4, \infty)$ and equal to 1 in $[3/4, \infty)$. Then

$$m_0(\xi_1) = e^{-ix_1\xi_1} m_0^1(\xi_1) + m_0^2(\xi_1) + m_0^3(\xi_1),$$

where

$$m_0^1(\xi_1) = C\mathbf{1}_+(\xi_1) \int_{-1}^\infty \psi_1(\mu) e^{ix_1\xi_1(1+\mu)^{1/2}} \xi_1^2 F_{\varepsilon',t}(\xi_1^2\mu) e^{-\varepsilon^2\xi_1^2(1+\mu)} (1+\mu)^{-1/4}\, d\mu,$$

$$m_0^2(\xi_1) = C\mathbf{1}_+(\xi_1) \int_{\mathbb{R}} \psi_2(\mu) e^{ix_1\xi_1((1+\mu)^{1/2}-1)} \xi_1^2 F_{\varepsilon',t}(\xi_1^2\mu) e^{-\varepsilon^2\xi_1^2(1+\mu)} (1+\mu)^{-1/4}\, d\mu,$$

$$m_0^3(\xi_1) = C\mathbf{1}_+(\xi_1) \int_{\mathbb{R}} \psi_3(\mu) e^{ix_1\xi_1((1+\mu)^{1/2}-1)} \xi_1^2 F_{\varepsilon',t}(\xi_1^2\mu) e^{-\varepsilon^2\xi_1^2(1+\mu)} (1+\mu)^{-1/4}\, d\mu.$$

Recall that we are looking to prove that the integral in (21) is bounded by $Ct^{-1/2}$. By (18), it suffices to verify that the functions m_0^j, $j = 1, 2, 3$, satisfy the Hörmander–Michlin estimates (17) with $b = 0$.

For the multiplier m_0^1, we notice that $\mu \in [-1, -1/4]$, thus $|m_0^1(\xi_1)| \leq C$ by (19) with $k = 0$. To estimate $|\xi_1\partial_{\xi_1} m_0^1(\xi_1)|$, $\xi_1 > 0$, we use again (19) with $k = 0, 1$, and integrate by parts in μ when the ξ_1-derivative acts on the exponential term $e^{ix_1\xi_1(1+\mu)^{1/2}}$.

For the multiplier m_0^3, we notice that $\mu \in [1/4, \infty)$. By (19) with $k = 0$, $|m_0^3(\xi_1)| \leq C$. To estimate $|\xi_1 \partial_{\xi_1} m_0^3(\xi_1)|$, $\xi_1 > 0$, we use again (19) with $k = 0, 1$, and integrate by parts in μ when the ξ_1-derivative acts on the exponential term $e^{ix_1\xi_1((1+\mu)^{1/2}-1)}$.

For the multiplier m_0^2, we notice that $\mu \in [-3/4, 3/4]$. To estimate $|m_0^2(\xi_1)|$, $\xi_1 > 0$, we decompose the integral into two parts, corresponding to $|\mu| \leq (1 + |x_1\xi_1|)^{-1}$ and $|\mu| \geq (1 + |x_1\xi_1|)^{-1}$. For the first part we use the cancellation property (20); for the second part we integrate by parts and use (19) with $k = 0, 1$. To estimate $|\xi_1 \partial_{\xi_1} m_0^2(\xi_1)|$, $\xi_1 > 0$, notice that the function $\nu\partial_\nu F_{\varepsilon',t}(\nu)$ satisfies the same Calderón–Zygmund bounds (19) and (20) as the function $F_{\varepsilon',t}(\nu)$. When the ξ_1-derivative acts on the exponential term $e^{ix_1\xi_1((1+\mu)^{1/2}-1)}$, we decompose the resulting integral into two parts, and integrate by parts twice when $|\mu| \geq (1 + |x_1\xi_1|)^{-1}$.

3. Proof of Theorem 2

In this section we prove Theorem 2. Let $\varphi : \mathbb{R}^n \to [0, 1]$, $\varphi(0) = 1$, denote a smooth function supported in the unit ball $\{x : |x| \leq 1\}$. Let

$$f_0(x, t) = i\mathbf{1}_{[0,\varepsilon_0]}(t)\varphi(x),$$

where $\varepsilon_0 \leq 1/2$ is a small constant (depending on N) to be fixed later. Let v_0 denote the solution of the initial value problem $Hv_0 = f_0$, $v_0(.,0) \equiv 0$, and define $v_0 \equiv 0$ in $\mathbb{R}^n \times (-\infty, 0]$. Then $v_0 \in C(\mathbb{R} : H^N(\mathbb{R}^n))$ and

$$\widetilde{v}_0(\xi, t) = \int_0^t e^{-i(t-s)|\xi|^2} \widehat{\varphi}(\xi)\mathbf{1}_{[0,\varepsilon_0]}(s)\, ds. \tag{22}$$

We define the function u by the formula

$$u(x, t) = \sum_{j=1}^{\infty} \alpha_j v_0(2^j x, 2^{2j}t - 1), \tag{23}$$

for some coefficients $\alpha_j \in \mathbb{C}$, $|\alpha_j| \leq 2^{-\beta j}$. Clearly, $u \in C(\mathbb{R} : H^N(\mathbb{R}^n))$ if $\beta \geq N + 1$. Also $u \equiv 0$ in $\mathbb{R}^n \times (-\infty, 0]$. Let $f = Hu$, i.e.,

$$f(x, t) = \sum_{j=1}^{\infty} 2^{2j}\alpha_j f_0(2^j x, 2^{2j}t - 1).$$

By the definition of f_0, the function f is supported in $\bigcup_{j\geq 1} E_j$, where $E_j = \{(x, t) : |x| \leq 2^{-j}, t \in [2^{-2j}, (1 + \varepsilon_0)2^{-2j}]\}$.

We will show that we can choose the constants $\varepsilon_0 \ll 1$, $\beta \gg 1$, and $\alpha_j \in \mathbb{C}$, $|\alpha_j| \approx 2^{-\beta j}$, in such a way that

$$|u(x, t)| \geq C^{-1}|\alpha_k| \text{ in } E_k. \tag{24}$$

Assuming this, we would define $V = f/u$ in $\bigcup_{j\geq 1} E_j$ and $V \equiv 0$ outside $\bigcup_{j\geq 1} E_j$. By (24) we would have $|V(x, t)| \leq C2^{2k}$ in E_k, which would show easily that $V \in L_t^\infty L_x^{n/2}$ if $n \geq 2$, and $V \in L_t^p L_x^q$ for any $p, q \in [1, \infty]$ with $2/p + n/q > 2$.

It remains to prove (24) for a suitable choice of the constants ε_0, β, and α_j. For $j = 1, 2, \dots$ let $u_j(x,t) = \alpha_j v_0(2^j x, 2^{2j}t - 1)$ and notice that $\sum_{j=1}^{k-1} u_j \equiv 0$ in E_k. We will choose the parameters in such a way that the main contribution to the function u in E_k comes from the term u_{k+1}. By (22),

$$v_0(x,t) = C_1^{-1} \int_{\mathbb{R}^n} \int_0^{\varepsilon_0} \varphi(y)(t-s)^{-n/2} e^{i|x-y|^2/(4(t-s))}\, dyds, \tag{25}$$

for any $t \geq 1$, where $C_1 \in \mathbb{C} \setminus \{0\}$ is a constant. Let

$$\alpha_j = C_1^j 2^{-\beta' j},$$

where C_1 is the constant in (25) and β' is a large constant to be fixed. Then, if $j \geq k+1$ and $(x,t) \in E_k$,

$$\begin{aligned} u_j(x,t) &= 2^{-\beta' j} C_1^{j-1} \int_{\mathbb{R}^n} \int_0^{\varepsilon_0} \varphi(y)(2^{2j}t - 1 - s)^{-n/2} e^{i|2^j x - y|^2/(4(2^{2j}t-1-s))}\, dyds, \\ &= 2^{-\beta' j} C_1^{j-1} 2^{-n(j-k)} \varepsilon_0 (a_{k,j}(x,t) + i b_{k,j}(x,t)), \end{aligned}$$

where $a_{k,j}(x,t) \in [C_2^{-1}, C_2]$ and $|b_{k,j}(x,t)| \leq C_2$ if $(x,t) \in E_k$. It is important that the real part $a_{k,j}$ is positive and bounded away from 0; this is due to the fact that the function φ is nonnegative and $|2^j x - y|^2 \leq 4(2^{2j}t - 1 - s)$ when $(x,t) \in E_k$, $s \in [0, \varepsilon_0]$ and $|y| \leq 1$. We can now fix β' large enough (depending on C_1, C_2, and N) such that $\beta = \beta' - \log |C_1| \geq N + 1$ and

$$\sum_{j=k+1}^{\infty} u_j(x,t) = 2^{-\beta'(k+1)} C_1^k \varepsilon_0 (\widetilde{a}_k(x,t) + i\widetilde{b}_k(x,t)) \tag{26}$$

where $\widetilde{a}_k(x,t) \in [C_3^{-1}, C_3]$ and $|\widetilde{b}_k(x,t)| \leq C_3$ if $(x,t) \in E_k$. We estimate now the contribution of u_k on E_k. By (22), if $t \in [0, \varepsilon_0]$ then

$$\widetilde{v_0}(\xi, t) = \int_0^t e^{-i(t-s)|\xi|^2} \widehat{\varphi}(\xi)\, ds = \int_0^t 1 \cdot \widehat{\varphi}(\xi)\, ds + \int_0^t (e^{-i(t-s)|\xi|^2} - 1)\widehat{\varphi}(\xi)\, ds,$$

which shows that

$$v_0(x,t) = t\varphi(x) + \varepsilon_0^2 E(x,t),$$

where $|E(x,t)| \leq C$ if $t \in [0, \varepsilon_0]$. Then

$$u_k(x,t) = 2^{-\beta' k} C_1^k [(2^{2k}t - 1)\varphi(2^k x) + \varepsilon_0^2 E(2^k x, 2^{2k}t - 1)].$$

if $(x,t) \in E_k$. Recall that the function φ is nonnegative. By (26) and the fact that $\widetilde{a}_k(x,t) \geq C_3^{-1}$, we may fix ε_0 small enough in such a way that

$$\left| \sum_{j=k}^{\infty} u_j(x,t) \right| \geq C^{-1} 2^{-\beta' k} |C_1|^k,\ (x,t) \in E_k,$$

which proves (24).

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University of Wisconsin–Madison
E-mail address: `ionescu@math.wisc.edu`

University of Chicago and Institute for Advanced Study
E-mail address: `cek@math.uchicago.edu and cek@ias.edu`