

EXISTENCE AND UNIQUENESS OF THE SOLUTION TO THE MODIFIED SCHRÖDINGER MAP

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ABSTRACT. We prove the local existence of the solution to a certain system of nonlinear Schrödinger equations arising from Schrödinger maps for the initial data in $H^s(\mathbf{R}^2)$ with $s > 1/2$. The uniqueness of the solution is also proved when the data belong to $H^1(\mathbf{R}^2)$.

1. Introduction

We consider the initial value problem for the system of the nonlinear Schrödinger equations in two space dimensions

$$(\text{MSM}) \begin{cases} i \partial_t u_1 + \Delta u_1 = -2i A \cdot \nabla u_1 + B u_1 + |A|^2 u_1 \\ \qquad \qquad \qquad \pm i \operatorname{Im}(u_2 \bar{u}_1) u_2, \quad (t, x) \in (0, T) \times \mathbf{R}^2, \\ i \partial_t u_2 + \Delta u_2 = -2i A \cdot \nabla u_2 + B u_2 + |A|^2 u_2 \\ \qquad \qquad \qquad \pm i \operatorname{Im}(u_1 \bar{u}_2) u_1, \quad (t, x) \in (0, T) \times \mathbf{R}^2, \\ u_1(0, x) = u_0^1(x), \quad u_2(0, x) = u_0^2(x), \quad x \in \mathbf{R}^2, \end{cases}$$

where u_1, u_2 are complex valued functions (we set $u = (u_1, u_2)$ in the following), and $A = (A_1[u], A_2[u])$, $B = B[u]$ are defined by

$$(1.1) \quad A_j[u] = \pm 2 G_j * \operatorname{Im}(u_1 \bar{u}_2), \quad j = 1, 2,$$

$$(1.2) \quad G_1(x) = \frac{1}{2\pi} \frac{x_2}{|x|^2}, \quad G_2(x) = -\frac{1}{2\pi} \frac{x_1}{|x|^2},$$

$$(1.3) \quad B[u] = \mp \sum_{j,k=1}^2 2R_j R_k \operatorname{Re}(u_j \bar{u}_k) \mp 2|u|^2.$$

Here, for a complex number z , $\operatorname{Re} z$ and $\operatorname{Im} z$ denotes the real part of z and the imaginary part respectively, and $R_j = \partial_j (-\Delta)^{-1/2}$ denotes the Riesz transforms. We notice that

$$\operatorname{div} A = 0 \quad \text{and} \quad \operatorname{rot} A = \partial_1 A_2 - \partial_2 A_1 = \pm 2 \operatorname{Im}(u_1 \bar{u}_2)$$

hold from the definition of A . These properties are used effectively to construct the solution to (MSM) for the low regularity initial data.

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The system (MSM) above is called the modified Schrödinger map which is derived by Nahmod-Stefanov-Uhlenbeck [11] from Schrödinger maps from $\mathbf{R} \times \mathbf{R}^2$ to the unit sphere S^2 or to the hyperbolic space \mathbf{H}^2 by using appropriate gauge change. The sign \pm corresponds to the each case respectively. As for the modified Schrödinger map, Nahmod-Stefanov-Uhlenbeck [12] showed the existence and uniqueness of the solution for the data $u_0 \in H^s(\mathbf{R}^2)$ with $s > 1$ by using the energy method. In this paper, we show the improvement of their result.

Theorem 1.1. *Let $u_0 \in H^s(\mathbf{R}^2)$ with $s > 1/2$. Then, there exists $T > 0$ satisfying*

$$\min\{1, C/((1 + \|u_0\|_{L^2}^q)\|u_0\|_{H^s}^q)\} \leq T \leq 1,$$

and at least one solution $u \in L^\infty(0, T; H^s) \cap C([0, T]; L^2)$ to (MSM) such that

$$(1.4) \quad J^\delta u \in L^p(0, T; L^q),$$

where $J^\delta = (I - \Delta)^{\delta/2}$, $s - 1/2 > \delta > 2/q > 0$, and $1/p = 1/2 - 1/q$.

Remark 1.2. (1) The modified Schrödinger map is invariant with respect to the scale transformation

$$u(t, x) \mapsto \lambda u(\lambda^2 t, \lambda x), \quad \lambda > 0.$$

Then, the scaling argument suggests that the critical space for the local well-posedness of the Cauchy problem (MSM) is $L^2(\mathbf{R}^2)$. We also notice that (MSM) conserves the L^2 -norm (see Proposition 2.5 below).

(2) Since the modified Schrödinger map is derived as the first order derivatives of the original Schrödinger map (see [11, Theorem 2.1]), the local well posedness of (MSM) in H^s corresponds to the local well-posedness of the original Schrödinger map in H^{s+1} .

(3) As is pointed out in [11, §3], it is not possible to go back directly from the solution of the (MSM) to the original Schrödinger map. However, a priori estimate and the estimate on the time of existence on the smooth solution to (MSM) are made use of in order to construct the low regularity solution to the Schrödinger map. See [11, §3] for details.

(4) After this work was completed, the author was informed that Kenig-Nahmod obtained the similar result for the Ishimori system ([7]). For the Ishimori system, see [9] and references therein. It is known that the Ishimori system is a generalization of the Heisenberg model for a ferromagnetic spin system, which is equivalent to the Schrödinger map in the case where the target manifold is S^2 (see [4, §1]).

In Theorem 1.1, the uniqueness of the solution is not obtained. However, for the uniqueness of the solution to (MSM) we obtain the following result by using the Vladimirov's argument [17] (see also [14]).

Theorem 1.3. *Let $u_0 \in L^2(\mathbf{R}^2)$. We assume that u and v are solutions to (MSM) on $(0, T) \times \mathbf{R}^2$ in the distribution sense with the same data u_0 satisfying*

$$\begin{aligned} u, v &\in C([0, T]; L^2(\mathbf{R}^2)), \\ \|u\|_{L_T^\infty H_x^1} &\leq M, \quad \|v\|_{L_T^\infty H_x^1} \leq M, \end{aligned}$$

for some constant $M > 0$. Then, we have $u(t) = v(t)$ in $L^2(\mathbf{R}^2)$ for $0 \leq t \leq T$.

Corollary 1.4. *If we assume $u_0 \in H^1(\mathbf{R}^2)$, then the solution in the class of Theorem 1.1 is uniquely determined.*

For the proof of Theorem 1.1 we use the compactness argument. Because the local well-posedness for smooth data is already known, our task is to show a priori estimate for the solution to (MSM). To recover the loss of the derivatives caused by the nonlinearity, the following type of estimate

$$(1.5) \quad \|J^s w\|_{L_T^p L_x^q} \lesssim \|w\|_{L_T^\infty H_x^{s+1/2+\varepsilon'}} + \|F\|_{L_T^2 H_x^{s-1/2}}$$

for the solution to $i\partial_t w + \Delta w = F$ is crucial in our argument, where p, q are the admissible exponent for Strichartz estimates (see Proposition 2.6 below for the precise statement). Compared with the usual Strichartz estimate

$$\|J^s w\|_{L_T^p L_x^q} \lesssim \|w(0)\|_{H^s} + \|F\|_{L_T^1 H_x^s},$$

estimate (1.5) says that we have a gain of the regularity $1/2$ on the inhomogeneous term at the cost of a loss of the regularity $1/2 + \varepsilon'$ on the homogeneous term. This type of estimate is first appeared in Koch-Tzvetkov [10] in the context of the Benjamin-Ono equation.

Throughout this paper we use the following notation. We denote by \widehat{f} or $\mathcal{F}f$ the Fourier transform of f . We denote $J^s = (I - \Delta)^{s/2}$ and $D^s = (-\Delta)^{s/2}$. H^s is the Sobolev space whose norm is defined by $\|f\|_{H^s} = \|J^s f\|_{L^2}$, and \dot{H}^s is the homogeneous Sobolev space whose semi-norm is defined by $\|f\|_{\dot{H}^s} = \|D^s f\|_{L^2}$. The function space $L^p(0, T; L^q(\mathbf{R}^2))$ is simply described by $L_T^p L_x^q$, and $L^\infty(0, T; H^s(\mathbf{R}^2))$ is also $L_T^\infty H_x^s$. We also denote $U(t) = e^{it\Delta}$.

This paper is organized as follows. In section 2, we prepare the basic estimates. In section 3, we show a priori estimates of the solution and give the proof of Theorem 1.1. In section 4, we give the proof of Theorem 1.3.

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2. Linear estimates and the energy estimate

In this section, we prepare the linear estimates and the energy estimate which we use for the proof of Theorems 1.1, 1.3. We first state the Strichartz estimates. For the proof, see [2] for example.

Lemma 2.1. *Let $n = 2$. We assume $2 < p \leq \infty$, $2 \leq q < \infty$, and $1/p = 1/2 - 1/q$. Then, the following estimates hold.*

$$(2.1) \quad \|U(t)f\|_{L_T^p L_x^q} \lesssim \|f\|_{L^2},$$

$$(2.2) \quad \left\| \int_0^t U(t-t')F(t')dt' \right\|_{L_T^p L_x^q} \lesssim \|F\|_{L_T^1 L_x^2}.$$

Lemma 2.2. *For $s > 0$, $1 < p < \infty$, we have*

$$\begin{aligned} \|D^s(fg)\|_{L^p} &\lesssim \|f\|_{L^{p_1}} \|D^s g\|_{L^{p_2}} + \|g\|_{L^{r_1}} \|D^s f\|_{L^{r_2}}, \\ \|J^s(fg)\|_{L^p} &\lesssim \|f\|_{L^{p_1}} \|J^s g\|_{L^{p_2}} + \|g\|_{L^{r_1}} \|J^s f\|_{L^{r_2}}, \end{aligned}$$

where $1/p = 1/p_1 + 1/p_2 = 1/r_1 + 1/r_2$ with $p < p_1, r_1 \leq \infty$.

Proof. See [3, Proposition 1.2], for example. Note that $\|f\|_{\dot{F}_{p,2}^s} \simeq \|D^s f\|_{L^p}$ and $\|f\|_{F_{p,2}^s} \simeq \|J^s f\|_{L^p}$ for $1 < p < \infty$. \square

The Hardy-Littlewood-Sobolev inequality below is used to estimate the potential term of the equation. For the proof, see [15, V. §1] for example.

Lemma 2.3. *Let $n = 2$. For $0 < \alpha < 2$, we define the operator I_α by*

$$I_\alpha f(x) = \int_{\mathbf{R}^2} \frac{f(y)}{|x-y|^{2-\alpha}} dy.$$

If $1/q = 1/p - \alpha/2$, $1 < p < 2/\alpha$, then we have

$$\|I_\alpha f\|_{L^q(\mathbf{R}^2)} \lesssim \|f\|_{L^p(\mathbf{R}^2)}.$$

Below we collect the estimates on the potential term A defined by (1.1).

Lemma 2.4. *We assume $s \geq 0$, $1 > \delta > 2/q > 0$, and $2 \leq p < \infty$ if $s > 0$; $2 < p < \infty$ if $s = 0$. Then, we have*

$$(2.3) \quad \|\nabla A\|_{\dot{H}^s} \lesssim \|u\|_{L^\infty} \|u\|_{\dot{H}^s},$$

$$(2.4) \quad \|\nabla A\|_{L^\infty} \lesssim \|u\|_{L^\infty} \|J^\delta u\|_{L^q},$$

$$(2.5) \quad \|A\|_{L^\infty} \lesssim \|u\|_{L^2} \|J^\delta u\|_{L^q},$$

$$(2.6) \quad \|D^s A\|_{L^p} \lesssim \|u\|_{L^p} \|u\|_{\dot{H}^s}.$$

Proof. We first notice that

$$(2.7) \quad \partial_k A_j = CR_k R_{j'} \text{Im}(u_1 \bar{u}_2),$$

where $j' = 2$ if $j = 1$, and $j' = 1$ if $j = 2$. This is a consequence of the fact that

$$\mathcal{F}[\partial_k A_j] = C \xi_k \widehat{G_j}(\xi) \mathcal{F}[\text{Im}(u_1 \bar{u}_2)],$$

and $\widehat{G_j}(\xi) = C \xi_{j'}/|\xi|^2$, since the Riesz transforms R_j is defined by $\widehat{R_j} f = -i \xi_j / |\xi| \widehat{f}$. For the Fourier transform of G_j , see [15, III. §3] for example.

Using (2.7) and Lemma 2.2 it is easy to see that

$$\begin{aligned} \|\nabla A\|_{\dot{H}^s} &\lesssim \sum_{i,j=1}^2 \|R_i R_j D^s \operatorname{Im}(u_1 \bar{u}_2)\|_{L^2} \\ &\lesssim \|D^s \operatorname{Im}(u_1 \bar{u}_2)\|_{L^2} \lesssim \|u\|_{L^\infty} \|u\|_{\dot{H}^s}, \end{aligned}$$

since the Riesz transforms are bounded on L^p for $1 < p < \infty$. Similarly, using (2.7) and the Sobolev embedding, we have

$$\begin{aligned} \|\nabla A\|_{L^\infty} &\lesssim \sum_{i,j=1}^2 \|J^\delta R_i R_j \operatorname{Im}(u_1 \bar{u}_2)\|_{L^q} \\ &\lesssim \|J^\delta \operatorname{Im}(u_1 \bar{u}_2)\|_{L^q} \lesssim \|u\|_{L^\infty} \|J^\delta u\|_{L^q}, \end{aligned}$$

where $\delta > 2/q$, and we applied Lemma 2.2 for the last inequality. For the proof of (2.5) we use Lemma 2.3. In fact, since $|G_j(x)| \lesssim 1/|x|$, we have

$$\|A\|_{L^\infty} \lesssim \|J^\delta A\|_{L^q} \lesssim \|I_1 |J^\delta \operatorname{Im}(u_1 \bar{u}_2)|\|_{L^q}.$$

Since $q > 2$, we are able to apply Lemma 2.3 for $1/q = 1/r - 1/2$ to obtain

$$\|I_1 |J^\delta \operatorname{Im}(u_1 \bar{u}_2)|\|_{L^q} \lesssim \|J^\delta \operatorname{Im}(u_1 \bar{u}_2)\|_{L^r} \lesssim \|u\|_{L^2} \|J^\delta u\|_{L^q}.$$

The proof of (2.6) are divided into three cases. The case $s = 0$ is similar to (2.5). In fact, we have

$$\|A\|_{L^p} \lesssim \|I_1 |u|^2\|_{L^p} \lesssim \| |u|^2 \|_{L^r} \lesssim \|u\|_{L^p} \|u\|_{L^2},$$

where r is determined by $1/p = 1/r - 1/2$ with $2 < p < \infty$. We next consider the case $0 < s < 1$. In this case, we first notice that

$$(2.8) \quad \mathcal{F}[D^s A_j] = C |\xi|^s \widehat{G}_j(\xi) \mathcal{F}[\operatorname{Im}(u_1 \bar{u}_2)] = C \frac{\xi_{j'}}{|\xi|^{2-s}} \mathcal{F}[\operatorname{Im}(u_1 \bar{u}_2)].$$

Since $\mathcal{F}^{-1}[\xi_{j'}/|\xi|^{2-s}] = C x_{j'}/|x|^{2+s}$ (see [15, III. §3] for example), we have

$$\|D^s A\|_{L^p} \lesssim \|I_{1-s} |u|^2\|_{L^p} \lesssim \| |u|^2 \|_{L^r} \lesssim \|u\|_{L^p} \|u\|_{L^{2/(1-s)}},$$

where r is determined by $1/p = 1/r - (1-s)/2$ with $2 \leq p < \infty$. Then, the Sobolev embedding $\dot{H}^s(\mathbf{R}^2) \hookrightarrow L^{2/(1-s)}(\mathbf{R}^2)$ (note that $u \in L^p$) enables us to obtain (2.6). In the case where $s \geq 1$, we observe that from (2.8) $\mathcal{F}[D^s A_j]$ is written as

$$\mathcal{F}[D^s A_j] = C \frac{\xi_{j'}}{|\xi|^{2-\varepsilon}} |\xi|^{s-\varepsilon} \mathcal{F}[\operatorname{Im}(u_1 \bar{u}_2)]$$

for $0 < \varepsilon < 1$. Thus, similarly as above, we obtain

$$\begin{aligned} \|D^s A\|_{L^p} &\lesssim \|I_{1-\varepsilon} |D^{s-\varepsilon} \operatorname{Im}(u_1 \bar{u}_2)|\|_{L^p} \\ &\lesssim \|D^{s-\varepsilon} \operatorname{Im}(u_1 \bar{u}_2)\|_{L^r} \\ &\lesssim \|u\|_{L^p} \|D^{s-\varepsilon} u\|_{L^{2/(1-\varepsilon)}} \lesssim \|u\|_{L^p} \|D^s u\|_{L^2}, \end{aligned}$$

where r is determined by $1/p = 1/r - (1-\varepsilon)/2$ with $2 \leq p < \infty$. \square

We next prove the energy estimate for the solution to the modified Schrödinger map.

Proposition 2.5. *Let $s \geq 0$ and let u be a solution to (MSM) on $(0, T) \times \mathbf{R}^2$. Then, for $T > 0$, we have*

$$(2.9) \quad \|u\|_{L_T^\infty L_x^2} = \|u_0\|_{L^2},$$

$$(2.10) \quad \|u\|_{L_T^\infty H_x^s} \leq \|u_0\|_{H^s} \exp\{C(1 + \|u_0\|_{L^2}^2) T^{2/q} \|J^\delta u\|_{L_T^p L_x^q}^2\},$$

where $\delta > 2/q > 0$, $1/p = 1/2 - 1/q$.

Proof. The conservation of the L^2 -norm (2.9) is easily obtained from the equation (MSM). In fact, multiplying the first equation in (MSM) by \bar{u}_1 and integrating over \mathbf{R}^n , and then taking the imaginary part, we obtain

$$\partial_t \|u_1(t)\|_{L^2}^2 = 2 \int \operatorname{Im}(u_2 \bar{u}_1) \operatorname{Re}(u_2 \bar{u}_1) dx,$$

since $\operatorname{div} A = 0$, and B is real valued. Similarly, we obtain

$$\begin{aligned} \partial_t \|u_2(t)\|_{L^2}^2 &= 2 \int \operatorname{Im}(u_1 \bar{u}_2) \operatorname{Re}(u_1 \bar{u}_2) dx \\ &= -2 \int \operatorname{Im}(u_2 \bar{u}_1) \operatorname{Re}(u_2 \bar{u}_1) dx. \end{aligned}$$

Thus, we obtain $\partial_t \|u(t)\|_{L^2}^2 = \partial_t \|u_1(t)\|_{L^2}^2 + \partial_t \|u_2(t)\|_{L^2}^2 = 0$ which implies (2.9).

For the proof of the energy estimate (2.10), we employ [12, Propositions 2] which states

$$(2.11) \quad \begin{aligned} \partial_t \|u(t)\|_{\dot{H}^s}^2 &\lesssim (\|\nabla A(t)\|_{\dot{H}^s} \|u(t)\|_{L^\infty} + \|\nabla A(t)\|_{L^\infty} \|u(t)\|_{\dot{H}^s} + \|B(t)u(t)\|_{\dot{H}^s} \\ &\quad + \| |A(t)|^2 u(t) \|_{\dot{H}^s} + \| \operatorname{Im}(u_1(t) \bar{u}_2(t)) u(t) \|_{\dot{H}^s}) \|u(t)\|_{\dot{H}^s}. \end{aligned}$$

In what follows, we estimate each term on the right hand side of (2.11) to obtain (2.10) assuming $s > 0$.

From (2.3) we have

$$(2.12) \quad \|\nabla A(t)\|_{\dot{H}^s} \|u(t)\|_{L^\infty} \lesssim \|u(t)\|_{L^\infty}^2 \|u(t)\|_{\dot{H}^s} \lesssim \|J^\delta u(t)\|_{L^q}^2 \|u(t)\|_{\dot{H}^s}$$

by the Sobolev embedding, where $\delta > 2/q$. Similarly, from (2.4) we have

$$(2.13) \quad \begin{aligned} \|\nabla A(t)\|_{L^\infty} \|u(t)\|_{\dot{H}^s} &\lesssim \|u(t)\|_{L^\infty} \|J^\delta u(t)\|_{L^q} \|u(t)\|_{\dot{H}^s} \\ &\lesssim \|J^\delta u(t)\|_{L^q}^2 \|u(t)\|_{\dot{H}^s}. \end{aligned}$$

From the definition (1.3) we have

$$(2.14) \quad \|B(t)u(t)\|_{\dot{H}^s} \lesssim \sum_{j,k=1}^2 \| (R_j R_k \operatorname{Re}(u_j(t) \bar{u}_k(t)) u(t) \|_{\dot{H}^s} + \| |u(t)|^2 u(t) \|_{\dot{H}^s}.$$

As for the first term on the right hand side of (2.14), we apply Lemma 2.2 to get

$$\begin{aligned}
& \left\| D^s (R_j R_k \operatorname{Re}(u_j \bar{u}_k) u) \right\|_{L^2} \\
& \lesssim \|u\|_{L^\infty} \|D^s R_j R_k \operatorname{Re}(u_j \bar{u}_k)\|_{L^2} + \|R_j R_k \operatorname{Re}(u_j \bar{u}_k)\|_{L^\infty} \|D^s u\|_{L^2} \\
& \lesssim \|u\|_{L^\infty} \|D^s \operatorname{Re}(u_j \bar{u}_k)\|_{L^2} + \|J^\delta R_j R_k \operatorname{Re}(u_j \bar{u}_k)\|_{L^q} \|D^s u\|_{L^2} \\
& \lesssim \|u\|_{L^\infty}^2 \|D^s u\|_{L^2} + \|J^\delta \operatorname{Re}(u_j \bar{u}_k)\|_{L^q} \|D^s u\|_{L^2} \\
& \lesssim \|u\|_{L^\infty}^2 \|D^s u\|_{L^2} + \|u\|_{L^\infty} \|J^\delta u\|_{L^q} \|D^s u\|_{L^2}.
\end{aligned}$$

Meanwhile, the following estimate of the second term on the right hand side of (2.14)

$$(2.15) \quad \left\| D^s (|u|^2 u) \right\|_{L^2} \lesssim \|u\|_{L^\infty}^2 \|D^s u\|_{L^2}$$

is also derived by applying Lemma 2.2 repeatedly. Thus, combining estimates above we obtain

$$(2.16) \quad \|B(t)u(t)\|_{\dot{H}^s} \lesssim \|J^\delta u(t)\|_{L^q}^2 \|u(t)\|_{\dot{H}^s}.$$

Here, we notice that the following estimate of the fifth term on the right hand side of (2.11)

$$(2.17) \quad \left\| \operatorname{Im}(u_1(t) \bar{u}_2(t)) u(t) \right\|_{\dot{H}^s} \lesssim \|u(t)\|_{L^\infty}^2 \|D^s u(t)\|_{L^2}$$

is similarly obtained as (2.15). Finally we estimate the fourth term on the right hand side of (2.11). Applying (2.5) and (2.6) we have

$$\begin{aligned}
(2.18) \quad & \left\| D^s (|A|^2 u) \right\|_{L^2} \lesssim \|u\|_{L^\infty} \|D^s |A|^2\|_{L^2} + \|A\|_{L^\infty}^2 \|D^s u\|_{L^2} \\
& \lesssim \|u\|_{L^\infty} \|A\|_{L^\infty} \|D^s A\|_{L^2} + \|A\|_{L^\infty}^2 \|D^s u\|_{L^2} \\
& \lesssim \|u\|_{L^2}^2 \|u\|_{L^\infty} \|J^\delta u\|_{L^q} \|u\|_{\dot{H}^s} + \|u\|_{L^2}^2 \|J^\delta u\|_{L^q}^2 \|u\|_{\dot{H}^s} \\
& \lesssim \|u_0\|_{L^2}^2 \|J^\delta u\|_{L^q}^2 \|u\|_{\dot{H}^s}
\end{aligned}$$

Combining (2.11), (2.12), (2.13) (2.16), (2.17), and (2.18), we obtain

$$\partial_t \|u(t)\|_{\dot{H}^s}^2 \leq C(1 + \|u_0\|_{L^2}^2) \|J^\delta u(t)\|_{L^q}^2 \|u(t)\|_{\dot{H}^s}^2.$$

Thus, applying the Gronwall inequality we have

$$\|u(t)\|_{\dot{H}^s} \leq \|u_0\|_{\dot{H}^s} \exp\left(C(1 + \|u_0\|_{L^2}^2) \int_0^T \|J^\delta u(t')\|_{L^q}^2 dt'\right).$$

Therefore, (2.9) and the Hölder inequality with respect to the time integral enables us to obtain (2.10). \square

We finally state the crucial estimate for the proof of Theorem 1.1. This type of estimate was first given by Koch-Tzvetkov [10] and refined by Kenig-Koenig [8] in the context of the Benjamin-Ono equation.

Proposition 2.6. *Let $T \leq 1$. We assume that w is a solution to the equation*

$$(2.19) \quad i\partial_t w + \Delta w = F_1 + F_2, \quad (t, x) \in (0, T) \times \mathbf{R}^2.$$

Then, for $s \in \mathbf{R}$, $\varepsilon' > 0$, we have

$$(2.20) \quad \|J^s w\|_{L_T^p L_x^q} \lesssim \|w\|_{L_T^\infty H^{s+1/2+\varepsilon'}} + \|F_1\|_{L_T^2 H_x^{s-1/2}} + \|F_2\|_{L_T^1 H_x^s},$$

where $1/p = 1/2 - 1/q$, $2 \leq q < \infty$.

Proof. The proof is essentially due to [8, Proposition 2.8]. We first introduce the Littlewood-Paley decomposition. Let $\varphi \in C_0^\infty(\mathbf{R}^2)$ satisfy $\varphi(\xi) = 1$ for $|\xi| \leq 1/2$, $\varphi(\xi) = 0$ for $|\xi| \geq 1$. And we set $\eta(\xi) = \varphi(\xi/2) - \varphi(\xi)$ and set $\eta_k(\xi) = \eta(\xi/2^k)$ for $k \geq 0$ so that $\text{supp } \eta_k \subset \{2^{k-1} \leq |\xi| \leq 2^{k+1}\}$ and $1 = \varphi(\xi) + \sum_{k=0}^\infty \eta_k(\xi)$. Then, we define Δ_k by $\widehat{\Delta_k f} = \eta_k \widehat{f}$ and S_0 by $\widehat{S_0 f} = \varphi \widehat{f}$, where \widehat{g} denotes the Fourier transform of g . Using the notation above, it is known that

$$\|f\|_{L^r(\mathbf{R}^n)} \simeq \left\| \left(|S_0 f|^2 + \sum_{k=0}^\infty |\Delta_k f|^2 \right)^{1/2} \right\|_{L^r(\mathbf{R}^n)}$$

holds for $1 < r < \infty$.

We next prepare the disjoint decomposition of the time interval $[0, T] = \cup_{j=1}^m I_j$, where $I_j = [a_j, a_{j+1})$ satisfy $|I_j| = 2^{-j}$ for $1 \leq j \leq m-1$ and $2^{-j} \leq |I_m| \leq 2^{-j+1}$. Then, we have $m \leq 2^j$, since $(m-1)2^{-j} + |I_m| = T$ implies $2^{-j}m \leq T - 2^{-j} + 2^{-j} = T \leq 1$.

Since $2 \leq q < \infty$, using the Littlewood-Paley decomposition of $J^s w$ we have

$$\begin{aligned} \|J^s w\|_{L_T^p L_x^q} &\lesssim \left\| \left(|S_0 J^s w|^2 + \sum_{k=0}^\infty |\Delta_k J^s w|^2 \right)^{1/2} \right\|_{L_T^p L_x^q} \\ &\lesssim \left(\|J^s S_0 w\|_{L_T^p L_x^q}^2 + \sum_{k=0}^\infty \|J^s \Delta_k w\|_{L_T^p L_x^q}^2 \right)^{1/2}. \end{aligned}$$

For the last inequality above we used Minkowski's integral inequality, since $p, q \geq 2$. Before applying Strichartz estimates, we use the decomposition $[0, T] = \cup_{j=1}^m I_j$ to estimate $J^s \Delta_k w$ as follows.

$$\begin{aligned} \|J^s \Delta_k w\|_{L_T^p L_x^q} &\lesssim 2^{sk} \|\Delta_k w\|_{L_T^p L_x^q} \\ &= 2^{sk} \left(\sum_{j=1}^m \|\Delta_k w\|_{L_{I_j}^p L_x^q} \right)^{1/p} \\ &\lesssim 2^{sk} \left(\sum_{j=1}^m \|\Delta_k w\|_{L_{I_j}^p L_x^q}^2 \right)^{1/2}, \end{aligned}$$

since $l^2 \hookrightarrow l^p$ for $p > 2$. Since $\Delta_k w$ satisfies the following integral equation

$$(2.21) \quad \Delta_k w(t) = U(t) \Delta_k w(a_j) - i \int_{a_j}^t U(t-t') \Delta_k (F_1(t') + F_2(t')) dt'$$

for $t \in I_j$, applying Lemma 2.1 we have

$$\|\Delta_k w\|_{L_{I_j}^p L_x^q} \lesssim \|\Delta_k w(a_j)\|_{L^2}^2 + \|\Delta_k F_1\|_{L_{I_j}^1 L_x^2} + \|\Delta_k F_2\|_{L_{I_j}^1 L_x^2}.$$

Thus, we obtain

$$\begin{aligned} & 2^{sk} \left(\sum_{j=1}^m \|\Delta_k w\|_{L_{I_j}^p L_x^q}^2 \right)^{1/2} \\ & \lesssim 2^{sk} \left\{ \sum_{j=1}^m (\|\Delta_k w(a_j)\|_{L^2}^2 + \|\Delta_k F_1\|_{L_{I_j}^1 L_x^2}^2 + \|\Delta_k F_2\|_{L_{I_j}^1 L_x^2}^2) \right\}^{1/2} \\ & \lesssim 2^{sk} \left\{ m \|\Delta_k w\|_{L_T^\infty L_x^2}^2 + \sum_{j=1}^m |I_j| \|\Delta_k F_1\|_{L_{I_j}^2 L_x^2}^2 \right\}^{1/2} + 2^{sk} \|\Delta_k F_2\|_{L_T^1 L_x^2} \\ & \lesssim 2^{(s+1/2)k} \|\Delta_k w\|_{L_T^\infty L_x^2} + 2^{(s-1/2)k} \|\Delta_k F_1\|_{L_T^2 L_x^2} + 2^{sk} \|\Delta_k F_2\|_{L_T^1 L_x^2} \\ & \lesssim 2^{-\varepsilon'k} \|w\|_{L_T^\infty H_x^{s+1/2+\varepsilon'}} + \|\Delta_k J^{s-1/2} F_1\|_{L_T^2 L_x^2} + \|\Delta_k J^s F_2\|_{L_T^1 L_x^2}. \end{aligned}$$

For the last term in the second inequality above, we used the triangle inequality in l^2 and then applied $l^1 \hookrightarrow l^2$. Combining the estimates above we obtain

$$\begin{aligned} & \|J^s \Delta_k w\|_{L_T^p L_x^q} \\ & \lesssim 2^{-\varepsilon'k} \|w\|_{L_T^\infty H_x^{s+1/2+\varepsilon'}} + \|\Delta_k J^{s-1/2} F_1\|_{L_T^2 L_x^2} + \|\Delta_k J^s F_2\|_{L_T^1 L_x^2}. \end{aligned}$$

Meanwhile, applying Lemma 2.1 it is easy to see that

$$\begin{aligned} \|J^s S_0 w\|_{L_T^p L_x^q} & \lesssim \|S_0 w(0)\|_{L^2} + \|S_0 F_1\|_{L_T^1 L_x^2} + \|S_0 F_2\|_{L_T^1 L_x^2} \\ & \lesssim \|w\|_{L_T^\infty L_x^2} + \|S_0 J^{s-1/2} F_1\|_{L_T^2 L_x^2} + \|S_0 J^s F_2\|_{L_T^1 L_x^2}. \end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
& \left(\|J^s S_0 w\|_{L_T^p L_x^q}^2 + \sum_{k=0}^{\infty} \|J^s \Delta_k w\|_{L_T^p L_x^q}^2 \right)^{1/2} \\
& \lesssim \left(\sum_{k=0}^{\infty} 2^{-2\varepsilon' k} \|w\|_{L_T^\infty H_x^{s+1/2+\varepsilon'}}^2 \right)^{1/2} \\
& \quad + \left(\|S_0 J^{s-1/2} F_1\|_{L_T^2 L_x^2}^2 + \sum_{k=0}^{\infty} \|\Delta_k J^{s-1/2} F_1\|_{L_T^2 L_x^2}^2 \right)^{1/2} \\
& \quad + \left(\|S_0 J^s F_2\|_{L_T^1 L_x^2}^2 + \sum_{k=0}^{\infty} \|\Delta_k J^s F_2\|_{L_T^1 L_x^2}^2 \right)^{1/2} \\
& \lesssim \left(\sum_{k=0}^{\infty} 2^{-2\varepsilon' k} \right)^{1/2} \|w\|_{L_T^\infty H_x^{s+1/2+\varepsilon'}} \\
& \quad + \left\| \left(\|S_0 J^{s-1/2} F_1\|^2 + \sum_{k=0}^{\infty} \|\Delta_k J^{s-1/2} F_1\|^2 \right)^{1/2} \right\|_{L_T^2 L_x^2} \\
& \quad + \left\| \left(\|S_0 J^s F_2\|^2 + \sum_{k=0}^{\infty} \|\Delta_k J^s F_2\|^2 \right)^{1/2} \right\|_{L_T^1 L_x^2} \\
& \lesssim \|w\|_{L_T^\infty H_x^{s+1/2+\varepsilon'}} + \|J^{s-1/2} F_1\|_{L_T^2 L_x^2} + \|J^s F_2\|_{L_T^1 L_x^2},
\end{aligned}$$

where we applied Minkowski's integral inequality again for the second inequality above. This completes the proof of Proposition 2.6. \square

3. A priori estimate

In this section, we prove a priori estimate for the solution to the modified Schrödinger map applying the estimates in the preceding section. Then, using the a priori estimate, we give the proof of Theorem 1.1.

Theorem 3.1. *Let $u_0 \in H^\infty(\mathbf{R}^2)$ and let u be a solution to (MSM). Then, for any $\varepsilon > 0$, there exists T satisfying*

$$\min\{1, C/((1 + \|u_0\|_{L^2}^q)\|u_0\|_{H^{1/2+\varepsilon}}^q)\} \leq T \leq 1$$

and $M > 0$ such that

$$(3.1) \quad \|J^\delta u\|_{L_T^p L_x^q} \leq M \|u_0\|_{H^{1/2+\varepsilon}},$$

where $\varepsilon > \delta > 2/q > 0$, $1/p = 1/2 - 1/q$.

Proof. Applying Proposition 2.6 for $s = \delta$, $F_1 = -2iA \cdot \nabla u_j$, and $F_2 = Bu_j + |A|^2 u_j \pm \text{Im}(u_2 \bar{u}_1) u_j$, we obtain

$$\begin{aligned}
(3.2) \quad & \|J^\delta u\|_{L_T^p L_x^q} \lesssim \|u\|_{L_T^\infty H_x^{1/2+\varepsilon}} + \|A \cdot \nabla u\|_{L_T^2 H_x^{-1/2+\delta}} \\
& \quad + \|Bu\|_{L_T^1 H_x^\delta} + \||A|^2 u\|_{L_T^1 H_x^\delta} + \|\text{Im}(u_2 \bar{u}_1) u\|_{L_T^1 H_x^\delta},
\end{aligned}$$

where we substituted $\delta + \varepsilon' = \varepsilon$. In what follows we estimate each term on the right hand side of (3.2) to obtain (3.1).

The first term is easily estimated by applying energy estimate (2.10) directly,

$$\|u\|_{L_T^\infty H_x^{1/2+\varepsilon}} \leq \|u_0\|_{H^{1/2+\varepsilon}} \exp\{C(1 + \|u_0\|_{L^2}^2) T^{2/q} \|J^\delta u\|_{L_T^p L_x^q}^2\}.$$

As for the second term we notice that $A \cdot \nabla u = \operatorname{div}(Au)$, since $\operatorname{div} A = 0$. Using this, we have

$$\begin{aligned} \|A \cdot \nabla u\|_{L_T^2 H_x^{-1/2+\delta}} &\lesssim \|J^{-1/2+\delta} \nabla(Au)\|_{L_T^2 L_x^2} \\ &\lesssim \|D^{1/2+\delta}(Au)\|_{L_T^2 L_x^2} + \|Au\|_{L_T^2 L_x^2} \\ &\lesssim \|u\|_{L_T^2 L_x^\infty} \|D^{1/2+\delta} A\|_{L_T^\infty L_x^2} \\ &\quad + \|A\|_{L_T^2 L_x^\infty} (\|D^{1/2+\delta} u\|_{L_T^\infty L_x^2} + \|u\|_{L_T^\infty L_x^2}). \end{aligned}$$

Then, applying (2.5), (2.6), we have

$$\begin{aligned} \|D^{1/2+\delta} A\|_{L_T^\infty L_x^2} &\lesssim \|u\|_{L_T^\infty L_x^2} \|D^{1/2+\delta} u\|_{L_T^\infty L_x^2}, \\ \|A\|_{L_T^2 L_x^\infty} &\lesssim \|u\|_{L_T^\infty L_x^2} \|J^\delta u\|_{L_T^2 L_x^q}. \end{aligned}$$

Thus, combining (2.9) and the energy estimate (2.10), we obtain

$$\begin{aligned} \|A \cdot \nabla u\|_{L_T^2 H_x^{-1/2+\delta}} &\lesssim \|u\|_{L_T^\infty L_x^2} \|J^\delta u\|_{L_T^2 L_x^q} \|u\|_{L_T^\infty H^{1/2+\delta}} \\ &\lesssim (1 + \|u_0\|_{L^2}^2 \|J^\delta u\|_{L_T^2 L_x^q}^2) \|u\|_{L_T^\infty H^{1/2+\delta}} \\ &\lesssim \|u_0\|_{H^{1/2+\delta}} \exp\{C(1 + \|u_0\|_{L^2}^2) T^{2/q} \|J^\delta u\|_{L_T^p L_x^q}^2\}. \end{aligned}$$

As for the third and the fifth term, we are able to use (2.16), (2.17) to obtain

$$\begin{aligned} \|Bu\|_{L_T^1 H_x^\delta} + \|\operatorname{Im}(u_2 \bar{u}_1)u\|_{L_T^1 H_x^\delta} \\ &\lesssim \|J^\delta u\|_{L_T^2 L_x^q}^2 \|u\|_{L_T^\infty H^\delta} \\ &\lesssim \|u_0\|_{H^\delta} \exp\{C(1 + \|u_0\|_{L^2}^2) T^{2/q} \|J^\delta u\|_{L_T^p L_x^q}^2\}. \end{aligned}$$

Similarly, we are able to use (2.18) to obtain

$$\begin{aligned} \| |A|^2 u \|_{L_T^1 H_x^\delta} &\lesssim \|u_0\|_{L^2}^2 \|J^\delta u\|_{L_T^2 L_x^q}^2 \|u\|_{L_T^\infty H^\delta} \\ &\lesssim \|u_0\|_{H^\delta} \exp\{C(1 + \|u_0\|_{L^2}^2) T^{2/q} \|J^\delta u\|_{L_T^p L_x^q}^2\}. \end{aligned}$$

Combining estimates above, we obtain

$$(3.3) \quad \|J^\delta u\|_{L_T^p L_x^q} \leq C_0 \|u_0\|_{H^{1/2+\varepsilon}} \exp\{C_1(1 + \|u_0\|_{L^2}^2) T^{2/q} \|J^\delta u\|_{L_T^p L_x^q}^2\}.$$

Now we set $K(T) = \|J^\delta u\|_{L_T^p L_x^q}^2$. Then, $K(T)$ is a continuous function with respect to T since $2 \leq p < \infty$, and (3.3) implies

$$(3.4) \quad K(T) \leq C_0^2 \|u_0\|_{H^{1/2+\varepsilon}}^2 \exp\{2C_1(1 + \|u_0\|_{L^2}^2) T^{2/q} K(T)\}.$$

If $K(T) \leq C_0^2 e \|u_0\|_{H^{1/2+\varepsilon}}^2$ holds for $0 \leq T \leq 1$, then the conclusion of Theorem 3.1 follows. On the other hand, in the case of there exists $T_1 \in (0, 1)$ such that $K(T_1) > C_0^2 e \|u_0\|_{H^{1/2+\varepsilon}}^2$, we set

$$T_0 = \inf\{T > 0; K(T) > C_0^2 e \|u_0\|_{H^{1/2+\varepsilon}}^2\}.$$

Then, $T_0 > 0$ and we have $K(T_0) = C_0^2 e \|u_0\|_{H^{1/2+\varepsilon}}^2$. Thus, (3.4) with $T = T_0$ implies that

$$e \leq \exp\{2C_1(1 + \|u_0\|_{L^2}^2) T_0^{2/q} C_0^2 e \|u_0\|_{H^{1/2+\varepsilon}}^2\}.$$

Therefore, we obtain the lower bound of T_0 ,

$$T_0 \geq \frac{1}{(2C_0^2 C_1 e)^{q/2} (1 + \|u_0\|_{L^2}^q) \|u_0\|_{H^{1/2+\varepsilon}}^q},$$

and for $0 \leq T \leq T_0$, $K(T) \leq K(T_0) = C_0^2 e \|u_0\|_{H^{1/2+\varepsilon}}^2$ holds. This completes the proof of Theorem 3.1. \square

Using Theorem 3.1, we are able to prove Theorem 1.1 as follows.

Proof of Theorem 1.1. Let $\rho \in C_0^\infty(\mathbf{R}^2)$ satisfy $\rho(x) = 1$ if $|x| \leq 1$, $\rho(x) = 0$ if $|x| \geq 2$. Then, for $u_0 \in H^s(\mathbf{R}^2)$ with $s > 1/2$ we define $\rho_\lambda(x) = 1/\lambda^2 \rho(x/\lambda)$ and set $u_0^\lambda = \rho_\lambda * u_0$ and the corresponding solution u^λ . The existence of such solution is assured by [12, Theorem 4] (see also [1, Theorem 2.1]).

From Theorem 3.1, u^λ has a priori uniform bound

$$(3.5) \quad \|J^\delta u^\lambda\|_{L_T^p L_x^q} \leq M \|u_0^\lambda\|_{H^{1/2+\varepsilon}} \leq CM \|u_0\|_{H^{1/2+\varepsilon}},$$

for $\varepsilon > \delta > 2/q > 0$, $1/p = 1/2 - 1/q$. Thus, by Proposition 2.5 we obtain

$$(3.6) \quad \|u^\lambda\|_{L_T^\infty H_x^s} \leq C \|u_0\|_{H^s} \exp(CM^2(1 + \|u_0\|_{L^2}^2) \|u_0\|_{H^{1/2+\varepsilon}}^2),$$

where we notice that ε can be chosen so that $s \geq 1/2 + \varepsilon$ with $\varepsilon > \delta$. Note that we may consider the solution u^λ exists on $[0, T]$ satisfying (3.5), (3.6), where the lower bound of T is determined independently of λ satisfying

$$\min\{1, C/((1 + \|u_0\|_{L^2}^q) \|u_0\|_{H^{1/2+\varepsilon}}^q)\} \leq T.$$

Using (3.5), (3.6), and the equation (MSM), we also obtain

$$(3.7) \quad \|\partial_t u^\lambda\|_{L_T^{p/2} H_x^{-2}} \leq M'.$$

Form uniform estimates on u^λ above, we observe that there exist subsequence $\{u^{\lambda'}\}$ and $u \in L_T^\infty H_x^s \cap L_T^p(J^{-\delta} L^q)_x$ such that

$$(3.8) \quad \begin{aligned} u^{\lambda'} &\rightarrow u \quad * \text{-weakly in } L_T^\infty H_x^s, \\ u^{\lambda'} &\rightarrow u \quad \text{weakly in } L_T^p(J^{-\delta} L^q)_x, \\ u^{\lambda'} &\rightarrow u \quad \text{in } L_T^p L_{\text{loc}, x}^2, \end{aligned}$$

as $\lambda' \rightarrow 0$. As for the convergence (3.8), we refer to [16, Ch. III, Theorem 2.1]. Using convergence above, we are able to observe that the limit u satisfy (MSM) in the distribution sense. In fact, we observe that u_j^λ satisfy

$$(3.9) \quad \begin{aligned} \frac{d}{dt} \langle u_j^\lambda, \varphi \rangle + \langle u_j^\lambda, \Delta \varphi \rangle &= -2i \langle A[u^\lambda] u_j^\lambda, \nabla \varphi \rangle \\ &\quad + \langle B[u^\lambda] u_j^\lambda, \varphi \rangle + \langle |A[u^\lambda]|^2 u_j^\lambda, \varphi \rangle \pm i \langle \text{Im}(u_1^\lambda \bar{u}_2^\lambda) u_j^\lambda, \varphi \rangle \end{aligned}$$

for $\varphi \in C_0^\infty(\mathbf{R}^2)$, $j = 1, 2$, where $j' = 2$ if $j = 1$, and $j' = 1$ if $j = 2$. As for the convergence of the first term on the right hand side of (3.9), we consider

$$\begin{aligned} &\langle A[u^\lambda] u_j^\lambda, \nabla \varphi \rangle - \langle A[u] u_j, \nabla \varphi \rangle \\ &= \langle A[u^\lambda] (u_j^\lambda - u_j), \nabla \varphi \rangle + \langle (A[u^\lambda] - A[u]) u_j^\lambda, \nabla \varphi \rangle. \end{aligned}$$

Now we set $\text{supp } \varphi \subset B_R \equiv \{x \in \mathbf{R}^n; |x| \leq R\}$. Then, applying (2.5) we obtain

$$|\langle A[u^\lambda] (u_j^\lambda - u_j), \nabla \varphi \rangle| \lesssim \|u^\lambda\|_{L^2} \|J^\delta u^\lambda\|_{L^q} \|u^\lambda - u\|_{L^2(B_R)} \|\nabla \varphi\|_{L^2}.$$

Similarly, we have

$$|\langle (A[u^\lambda] - A[u]) u_j^\lambda, \nabla \varphi \rangle| \lesssim \|A[u^\lambda] - A[u]\|_{L^q(B_R)} \|u\|_{L^2} \|\nabla \varphi\|_{L^p},$$

where $1/q + 1/p = 1/2$. To show the convergence, we set $L \geq 2R$ to estimate

$$\begin{aligned} &\|A[u^\lambda] - A[u]\|_{L^q(B_R)} \\ &\lesssim \|I_1\{(|u^\lambda| + |u|)|u^\lambda - u|\chi_{B_L}\}\|_{L^q} + \|I_1\{(|u^\lambda|^2 + |u|^2)\chi_{B_L^c}\}\|_{L^q(B_R)} \\ &\lesssim \|(|u^\lambda| + |u|)|u^\lambda - u|\chi_{B_L}\|_{L^r} + R^{n/q} \|I_1\{(|u^\lambda|^2 + |u|^2)\chi_{B_L^c}\}\|_{L^\infty(B_R)} \\ &\lesssim (\|u^\lambda\|_{L^q} + \|u\|_{L^q}) \|u^\lambda - u\|_{L^2(B_L)} + R^{n/q} (\|u^\lambda\|_{L^2}^2 + \|u\|_{L^2}^2) / L, \end{aligned}$$

where $B_L^c \equiv \{x \in \mathbf{R}^n; |x| > L\}$, r is determined by $1/q = 1/r - 1/2$, and χ_E denotes the characteristic function of a set E . For the last inequality above, we used the estimate

$$\begin{aligned} I_1\{(|u^\lambda|^2 + |u|^2)\chi_{B_L^c}\}(x) &= C \int_{|y|>L} \frac{1}{|x-y|} (|u^\lambda(y)|^2 + |u(y)|^2) dy \\ &\leq \frac{C}{L} \int (|u^\lambda(y)|^2 + |u(y)|^2) dy, \end{aligned}$$

for $|x| \leq R \leq L/2$, since $|x-y| \geq |y| - |x| \geq L/2$. Thus, using the uniform bound (3.5), (3.6), we obtain

$$\begin{aligned} &\left| \int_0^T \langle A[u^{\lambda'}] u_j^{\lambda'}, \nabla \varphi \rangle \psi(t) dt - \int_0^T \langle A[u] u_j, \nabla \varphi \rangle \psi(t) dt \right| \\ &\lesssim \|u^{\lambda'} - u\|_{L_T^2 L^2(B_L)} + 1/L, \end{aligned}$$

which converges to zero as $\lambda' \rightarrow 0$, $L \rightarrow \infty$. The convergence of other terms of (3.9) is also proved in the same manner. \square

Remark 3.2. It is worth noticing that the solution u also belongs to $C([0, T]; H^{s'})$ for $s' < s$, since $\partial_t u \in L_T^{p/2} H_x^{-2}$.

4. Uniqueness of H^1 solution

In this section, we give the proof of Theorem 1.3. For the proof, the following Gagliardo-Nirenberg inequality with the specific constant, which we refer to [14, Lemma 2], is essentially used.

Lemma 4.1. *For $2 \leq q < \infty$, we have*

$$\|u\|_{L^q(\mathbf{R}^2)} \leq (4\pi)^{(2-q)/2q} (q/2)^{1/2} \|u\|_{L^2(\mathbf{R}^2)}^{2/q} \|\nabla u\|_{L^2(\mathbf{R}^2)}^{1-2/q}.$$

Proof of Theorem 1.3. If we set $w_j = u_j - v_j$, then w_j satisfies

$$(4.1) \quad \begin{aligned} (i\partial_t + \Delta)w_j &= -2iA[u] \cdot \nabla w_j + B[u]w_j + |A[u]|^2 w_j \pm i\text{Im}(u_1 \bar{u}_2) w_{j'} \\ &\quad - 2i(A[u] - A[v]) \cdot \nabla v_j + (B[u] - B[v])v_j \\ &\quad + (|A[u]|^2 - |A[v]|^2)v_j \pm i(\text{Im}(u_1 \bar{u}_2) - \text{Im}(v_1 \bar{v}_2))v_{j'}, \end{aligned}$$

for $j = 1, 2$, where j' denotes 2 if $j = 1$, 1 if $j = 2$. Multiplying \bar{w}_j by the equation (4.1), integrating over \mathbf{R}^n , and then taking the imaginary part, we obtain

$$(4.2) \quad \partial_t \|w(t)\|_{L^2}^2 \lesssim qM^{2+4/q}(1+M^2)\|w(t)\|_{L^2}^{2(1-2/q)}$$

for $q > 2$. Note that as we shall see below we finally take the limit $q \rightarrow \infty$, so it is important to investigate the dependence on q of the constant appeared in the estimate. Once we obtain (4.2), then substituting $\varepsilon = 2/q$ we have

$$\frac{1}{\varepsilon} \partial_t \|w(t)\|_{L^2}^{2\varepsilon} \lesssim \frac{1}{\varepsilon} M^{2+2\varepsilon} (1+M^2).$$

Since $\|w(0)\|_{L^2} = 0$, we obtain

$$\|w(t)\|_{L^2} \leq (CM^{2+2\varepsilon}(1+M^2)T)^{1/2\varepsilon}$$

for $0 \leq t \leq T$. Thus, if we take T' satisfying $CM^{2+2\varepsilon}(1+M^2)T' < 1$, then letting $\varepsilon \rightarrow 0$ we conclude that $\|w(t)\|_{L^2} = 0$ for $0 \leq t \leq T'$. Using this argument repeatedly, the uniqueness holds for $0 \leq t \leq T$.

In what follows, we devote to derive (4.2) by estimating the right hand side of (4.1). As for the first term, integrating by parts it is easy to see that

$$\text{Im} \left\{ i \int A[u] \cdot (\nabla w_j) \bar{w}_j dx \right\} = \frac{1}{2} \int (\text{div} A[u]) |w_j|^2 dx = 0,$$

since $\text{div} A[u] = 0$. The second and the third terms are also 0, since $B[u]$ and $|A[u]|^2$ are real valued. To estimate other terms, we set p, q satisfying

$$1/2 = 1/p + 1/q, \quad 2 < p \leq 4, \quad 4 \leq q < \infty.$$

Then, the fourth term is estimated by applying Lemma 4.1 as follows.

$$\begin{aligned} \left| \int \text{Im}(u_1 \bar{u}_2) w_{j'} \bar{w}_j dx \right| &\leq \|u\|_{L^q}^2 \|w\|_{L^p}^2 \\ &\lesssim q \|u\|_{L^2}^{4/q} \|\nabla u\|_{L^2}^{2-4/q} \|w\|_{L^2}^{4/p} \|\nabla w\|_{L^2}^{2-4/p} \\ &\lesssim qM^{2+4/q} \|w\|_{L^2}^{2-4/q}, \end{aligned}$$

since $4/p = 2 - 4/q$, where M is the upper bound of H^1 norm in the assumption.

As for the fifth term, we have

$$(4.3) \quad \left| \int (A[u] - A[v]) \nabla v_j \bar{w}_j dx \right| \lesssim \|A[u] - A[v]\|_{L^q} \|\nabla v\|_{L^2} \|w\|_{L^p}.$$

Since $|\operatorname{Im}(u_2 \bar{u}_1) - \operatorname{Im}(v_2 \bar{v}_1)| \leq (|u| + |v|)|w|$, we have

$$|A[u] - A[v]| \lesssim I_1((|u| + |v|)|w|).$$

Then, applying Lemma 2.3 we obtain

$$(4.4) \quad \begin{aligned} \|A[u] - A[v]\|_{L^q} &\lesssim \|(|u| + |v|)|w|\|_{L^r} \\ &\lesssim (\|u\|_{L^q} + \|v\|_{L^q}) \|w\|_{L^2}, \end{aligned}$$

where r is determined by $1/q = 1/r - 1/2$. Thus, applying Lemma 4.1 again, the right hand side of (4.3) is estimated by a constant multiple of

$$q^{1/2} M^{2+2/q} \|w\|_{L^2}^{2-2/q} \leq q^{1/2} M^{2+4/q} \|w\|_{L^2}^{2-4/q}.$$

Similarly, since $|\operatorname{Re}(u_j \bar{u}_k) - \operatorname{Re}(v_j \bar{v}_k)| \leq (|u| + |v|)|w|$, we have

$$\|B[u] - B[v]\|_{L^2} \lesssim (\|u\|_{L^q} + \|v\|_{L^q}) \|w\|_{L^p}.$$

Thus, we obtain

$$\begin{aligned} \left| \int (B[u] - B[v]) v_j \bar{w}_j dx \right| &\leq \|B[u] - B[v]\|_{L^2} \|v\|_{L^q} \|w\|_{L^p} \\ &\lesssim (\|u\|_{L^q} + \|v\|_{L^q}) \|v\|_{L^q} \|w\|_{L^p}^2. \end{aligned}$$

Then, we are able to estimate in the same way as the fourth term. We note that the eighth term is estimated in the same manner.

Finally, as for the seventh term, we use (4.4) to estimate

$$\begin{aligned} &\left| \int (|A[u]|^2 - |A[v]|^2) v_j \bar{w}_j dx \right| \\ &\lesssim (\|A[u]\|_{L^q} + \|A[v]\|_{L^q}) \|A[u] - A[v]\|_{L^q} \|v\|_{L^p} \|w\|_{L^p} \\ &\lesssim M (\|u\|_{L^q}^2 + \|v\|_{L^q}^2) \|v\|_{L^p} \|w\|_{L^p} \|w\|_{L^2} \\ &\lesssim q M^{4+2/q} \|w\|_{L^2}^{2-2/q} \\ &\lesssim q M^{4+4/q} \|w\|_{L^2}^{2-4/q}. \end{aligned}$$

Therefore, we obtain (4.2). \square

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