Mathematical Research Letters **11**, 883–904 (2004)

RICCI FLOW AND NONNEGATIVITY OF SECTIONAL CURVATURE

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Abstract. In this paper, we extend the general maximum principle in [NT3] to the time dependent Lichnerowicz heat equation on symmetric tensors coupled with the Ricci flow on complete Riemannian manifolds. As an application we exhibit complete Riemannian manifolds with bounded nonnegative sectional curvature of dimension greater than three such that the Ricci flow does not preserve the nonnegativity of the sectional curvature, even though the nonnegativity of the sectional curvature was proved to be preserved by Hamilton in dimension three. This fact is proved through a general splitting theorem on the complete family of metrics with nonnegative sectional curvature, deformed by the Ricci flow.

Introduction

The Ricci flow has been proved to be an effective tool in the study of the geometry and topology of manifolds. One of the good properties of the Ricci flow is that it preserves the 'nonnegativity' of various curvatures. In dimension three, Hamilton [H1] proves that on compact manifolds the Ricci flow preserves the nonnegativity of the Ricci curvature and the sectional curvature. Using this property and the quantified version, curvature pinching estimate, it was proved in [H1] that the normalized Ricci flow converges to a Einstein metric if the initial metric has positive Ricci curvature. In particular, it implies that a simply-connected compact three-manifold is diffeomorphic to the three sphere if it admits a metric with positive Ricci curvature. One can refer [Ch] for an updated survey and [P2] for some recent developement on the Ricci flow on three manifolds. Later in [H2] it was proved that the Ricci flow also preserves the nonnegativity of the curvature operator in all dimensions (on compact manifolds). In the Kähler case, Bando and Mok [B, M] proved that the flow also preserves the nonnegativity of the holomorphic bisectional curvature. The Ricci flow on complete manifolds was initiated in [Sh2]. In [Sh3] Shi generalized the above mentioned result of Bando and Mok to the complete Kähler manifolds with bounded curvature. Interesting applications were also obtained therein.

In this paper, we shall study the topological consequences of the assumption that Ricci flow preserves the nonnegativity of the sectional curvature on complete Riemannian manifolds. The basic method is to study the heat equation, time

Received May 3, 2003. Revised August 2004.
Research partially supported by NSF grant DMS-0328624, USA.

dependent, deformation of the Busemann function via the new tensor maximum principle proved in [NT3]. The maximum principle of this type was first proved by Hamilton for compact manifolds [H2]. The proof of [H2] can be generalized to the complete noncompact manifolds with bounded curvature with additional assumption that the tensor satisfying certain heat equation is *uniformly bounded* on the space-time. See for example [H3, Theorem 5.3], [NT2, Proposition 1.1] and [Ca]. However in the study of the deformation of a continuous function, the Busemann function (with respect to all rays from a fixed point) in our case, one in general can not expect uniform pointwise control of its Hessian since the Busemann function is not even differentiable in general. Therefore one needs a more general maximum principle than those developed by Hamilton in [H3]. This is the main technical difficulty. This difficulty was resolved in [NT3] and an (optimal in a certain sense) maximum principle was established there for the time-independent heat equation. The tensor maximum principle proved in Theorem 2.1 of this paper is a time-dependent analogue of the corresponding result, Theorem 2.1 in [NT3].

By studying the deformation of the Busemann function, we shall prove that on a simply-connected complete Riemannian manifolds with bounded nonnegative sectional curvature, if the Ricci flow preserves the nonnegativity of the sectional curvature, then the manifold splits as the product of a compact manifold with nonnegative sectional curvature with a complete manifold which is diffeomorphic to the Euclidean space. As a consequence of such splitting result, we give examples of complete Riemannian manifolds with *bounded* nonnegative sectional curvature of dimension ≥ 4 such that the Ricci flow does not preserve the nonnegativity of the sectional curvature. As far as we know, this is the first example of this kind even though this is believed to be the case by the experts. Noticing that in dimension three the Ricci flow does preserve the nonnegativity of the sectional curvature by [H1] on compact manifolds and complete manifolds with bounded curvature. Another application of our approach is a classification of complete manifolds with *bounded* nonnegative curvature operator, a result which has been previously established in [N] using different methods without assuming the boundedness of the curvature (see also [CC, CY, GaM, H2], but more importantly [MM, SiY] for the compact case). The use of the heat equation deformation of Busemann functions to study the structure of complete manifolds was initiated in [NT3]. Therefore this paper can be viewed as a continuation of the pervious work. The difference between current paper and [NT3] is that we have to consider the heat equation with respect to metrics evolved by the Ricci flow in order to show that the Hessian of the solution to the heat equation satisfies the Lichnerowicz heat equation. Therefore we have to derive the heat kernel estimate of Li-Yau type (cf. [LY]) for the time dependent heat equation. The estimate of this type was considered before in [Gr1, Sa] for a fixed complete Riemannian metric satisfying the volume doubling properties for the balls and the Neumann-Poincaré inequality. However, the heat equation considered here does not belong to the classes considered in the previous cases (see Remark

1.1 for more details). Therefore we devote the first section in establishing the heat kernel estimate as well as the Harnack inequality for the time dependent heat equation, following the approach of Grigoy'an in [Gr1]. The result itself maybe has its own interests. There exists also related (weaker) lower bound estimates on the fundamental solution of time-dependent heat equation in [Gu] by Guenther.

1. Time-dependent heat equation

Let $(M, g^0_{ij}(x))$ be a complete Riemannian manifold (of dimension n) with bounded curvature tensor. We denote k_0 to be the upper bound of $|R_{ijkl}|^2$, the curvature tensor of g^0. By [Sh2, Theorem 1.1, p. 224] we know that there exists a constant $T(n, k_0) > 0$ such that the Ricci flow

$$\frac{\partial}{\partial t} g_{ij}(x,t) = -2R_{ij}(x,t) \tag{1.1}$$

has solution on $M \times [0, T]$. Moreover, there exists $A'_m = A'_m(n, m, k_0)$ such that for all $(x,t) \in M \times [0,T]$,

$$\|\nabla^m R_{ijkl}\|^2(x,t) \le \frac{A'_m}{t^m}. \tag{1.2}$$

In particular,

$$\|R_{ijkl}\|(x,t) \le \sqrt{A_0}. \tag{1.3}$$

The argument of [H2, H3] (see also [NT2, Proposition 1.1]) can be adapted to show that $g_{ij}(x,t)$ has nonnegative curvature operator if the initial metric $g_{ij}(x,0)$ has the nonnegative curvature operator. We are going to study the initial value problem of the heat equation

$$\left(\frac{\partial}{\partial t} - \Delta\right) v(x,t) = 0 \tag{1.4}$$

with initial value $v(x,0) = u(x)$. Here $\Delta v = g^{ij}(x,t) v_{ij}$, where v_{ij} denotes the Hessian of v. Namely Δ is time-dependent. The following lemma is well-known to experts. For example, it was known and used in [CH] by Chow and Hamilton in their study of the linear trace differential Harnack or Li-Yau-Hamilton inequality for the Ricci flow.

Lemma 1.1. *Let $v(x,t)$ be a solution to* (1.4). *Then the complex Hessian $v_{ij}(x,t)$ satisfies*

$$\left(\frac{\partial}{\partial t} - \Delta\right) v_{ij} = 2R_{ipjq}v_{pq} - R_{ip}v_{pj} - R_{pj}v_{ip}. \tag{1.5}$$

Here we have used Einstein convention and a normal frame.

Proof. Direct calculation, using formulae on page 274 of [H1], one has that

$$(v_{ij})_t = (v_t)_{ij} + (\nabla_i R_{jk} + \nabla_j R_{ik} - \nabla_k R_{ij})\, v_k. \tag{1.6}$$

On the other hand, the commutator calculation shows that

$$v_{ijkk} = v_{kkij} + (-\nabla_s R_{ij} + \nabla_i R_{js} + \nabla_j R_{is})\, v_s + R_{is}v_{sj} + R_{js}v_{is} - 2R_{isjk}v_{sk}. \tag{1.7}$$

Now using (1.4) we have $(v_t)_{ij} = v_{kkij}$. Then lemma follows from (1.6) and (1.7).

Following [CH], the equation (1.5) is called Lichnerowicz heat equation for symmetric tensors.

Corollary 1.2. *Denote briefly by η the symmetric tensor v_{ij}. Denote by $\|\eta\|^2$ the norm of v_{ij} with respect to $g_{ij}(x,t)$. Then $\exp(-2\sqrt{A_0}t)\|\eta\|(x,t)$ is a subsolution of* (1.4).

Proof. Direct calculation shows that

$$\begin{aligned}\left(\Delta - \frac{\partial}{\partial t}\right)\|\eta\|^2 &\geq -4R_{ipjq}\eta_{pq}\eta_{ij} + 4R_{ip}\eta_{pk}\eta_{ik} + 2\|\nabla\eta\|^2 - 4R_{ip}\eta_{pk}\eta_{ik} \\ &\geq 2\|\nabla\eta\|^2 - 4\sqrt{A_0}\|\eta\|^2.\end{aligned}$$

Here we have used Lemma 1.1, namely the fact that η satisfies (1.5). The claim of the corollary follows easily from the above calculation.

In the following we collect some fundamental results on solution (subsolutions) of (1.4). Our basic assumption is (1.3). For the purpose of the later section we also assume $T \leq 1$ and $g_{ij}(x,0)$ has nonnegative Ricci curvature. By (1.1) and (1.3) we know that, if $g_{ij}(x,t)$ has nonnegative Ricci curvature,

$$C(n, A_0)g_{ij}(x,0) \leq g_{ij}(x,t) \leq g_{ij}(x,0). \tag{1.8}$$

Since $g_{ij}(x,0)$ has nonnegative Ricci curvature, by (1.8), for any $0 \leq t \leq T$, we still have the following Neumann type Poincaré inequality for $g_{ij}(x,t)$.

Lemma 1.2. *Let $(M, g_{ij}(x,t))$ be a solution to the Ricci flow such that the initial metric $g_{ij}(x,0)$ has nonnegative Ricci curvature. For any domain $\Omega \subset B_0(o,R)$ and any Lipschitz function φ on $\overline{\Omega}$, which vanishes on $\partial\Omega$*

$$\int_\Omega |\nabla\varphi|^2(y)\,dy \geq \frac{b}{R^2}\left(\frac{V_0(o,R)}{|\Omega|_0}\right)^\beta \int_\Omega \varphi^2(y)\,dy \tag{1.9}$$

for some positive constants β, b which only depends on n and A_0. Here $|\nabla\varphi|^2$ is calculated using $g_{ij}(x,t)$, while $|\Omega|_0$, the volume of Ω, and $V_0(o,R)$, the volume of $B_0(o,R)$, are calculated using $g_{ij}(x,0)$.

Proof. The lemma follows easily from Theorem 1.4 of [Gr1]. The point is that only the weak form Neumann-Poincaré inequality and the volume doubling property are needed in the proof of Theorem 1.4 of [Gr1]. Since $g_{ij}(x,0)$ has nonnegative Ricci curvature these two properties hold for $(M, g_{ij}(x,0))$. On the other hand, the metric $g_{ij}(x,t)$ is equivalent to $g_{ij}(x,0)$. Therefore these two sufficient properties preserve.

Remark 1.1. *Here we assume that $g_{ij}(x,0)$ has nonnegative Ricci curvature to make our presentation easier. In fact, one can replace it by assuming $(M, g_{ij}(x,0))$ satisfies* (1.9) *and a volume doubling property for balls as in* [Gr1]. *This applies to some other results in this section.*

The next result is a mean value inequality. The proof is just a modification of the one in [Gr1] for the time-independent heat equation. Note that it is known from [H1], applying the maximum principle for functions, that the scalar curvature $\mathcal{R}(x,t)$ of $g_{ij}(x,t)$ is nonnegative, under the assumption that $g_{ij}(x,0)$ has nonnegative Ricci/scalar curvature.

Theorem 1.1. *Let $(M, g(t))$ be as in Lemma 1.2. Let $w(x,t)$ be a smooth function satisfying*

$$\left(\Delta - \frac{\partial}{\partial t}\right) w(x,t) \geq 0 \tag{1.10}$$

on $\coprod_{\sqrt{t}}$ with $t \leq T$, where $\coprod_R = B_0(x,R) \times (0,R^2)$ and $B_\tau(x,\sqrt{t})$ is the ball of radius $\sqrt{t}$ with respect to $g_{ij}(x,\tau)$. Then

$$w_+^2(x,t) \leq \frac{C(n,A_0,T)}{V_0(x,\sqrt{t})t} \int_0^t \int_{B_0(x,\sqrt{t})} w_+^2(y,\tau)\, dy d\tau \tag{1.11}$$

Here $w_+ = \max\{0,w\}$, $V_0(x,r)$ denotes the volume of $B_0(x,r)$ with respect to $g_{ij}(x,0)$.

Proof. We essentially repeat the argument of the proof of Theorem 3.1 in [Gr1]. The key to the argument is the fact that $g_{ij}(x,t)$ satisfying the Neumann-Poincaré inequality (1.9) and the volume double property for balls. We have these two properties if we assume that the initial metric has nonnegative Ricci curvature. To make the iteration argument work using Lemma 1.2 we need also to prove that the Lemma 3.1 of [Gr1] still holds for our case. In fact, for any $R \leq \sqrt{t}$, let $\phi(x,t)$ be a cut-off function supported in $B_0(x,R)$ such that $\phi(x,0) = 0$. For $\theta > 0$, let $w_\theta = (w-\theta)_+$. Multiplying $w_\theta\phi^2$ on both sides of

(1.10) we have that

$$
\begin{aligned}
\int_{\{w\geq\theta\}} w_t w_\theta \phi^2\, dy &\leq \int_{\{w\geq\theta\}} (\Delta w) w_\theta \phi^2\, dy \\
&= -2\int_{\{w\geq\theta\}} \langle \nabla w_\theta, \nabla \phi\rangle w_\theta \phi\, dy - \int_{\{w\geq\theta\}} |\nabla w_\theta|^2 \phi^2\, dy \\
&= -\int_{\{w\geq\theta\}} |\nabla (w_\theta \phi)|^2\, dy + \int_M |\nabla \phi|^2 w_\theta^2\, dy.
\end{aligned} \tag{1.12}
$$

Integrating the time variable and noticing that $\phi \in C_0^\infty(B_0(x,R))$ we have that

$$\int_0^t \int_{B_0(x,R)} w_\theta (w_\theta)_\tau \phi^2\, dyd\tau \leq -\int_0^t \int_{B_0(x,R)} |\nabla (w_\theta\phi)|^2\, dyd\tau + \int_0^t \int_{B_0(x,R)} |\nabla\phi|^2 w_\theta^2\, dyd\tau.$$

The left hand side above equals to

$$
\begin{aligned}
\frac{1}{2}\int_0^t \int_{B_0(x,R)} (w_\theta^2)_\tau \phi^2\, dyd\tau &= \left(\frac{1}{2}\int_{B_0(x,R)} w_\theta^2\phi\, dy\right)(t) - \left(\frac{1}{2}\int_{B_0(x,R)} w_\theta^2\phi\, dy\right)(0) \\
&\quad + \int_0^t \int_{B_0(x,R)} w_\theta^2 \left(-\phi_\tau\phi + \frac{1}{2}\mathcal{R}(y,\tau)\phi^2\right) dyd\tau.
\end{aligned}
$$

Combining the above two inequalities and using the fact $\mathcal{R} \geq 0$ we have that

$$\left.\int_{B_0(x,R)} w_\theta^2\phi^2\, dy\right|_t + 2\int_0^t \int_{B_0(x,R)} |\nabla(w_\theta\phi)|^2 dyd\tau \leq 2\int_0^t \int_{B_0(x,R)} w_\theta^2 \left(|\nabla\phi|^2 + |\phi\phi_\tau|\right) dyd\tau. \tag{1.13}$$

Similarly, one can prove Lemma 3.2 of [Gr1], noticing that Lemma 1.2 holds for metric $g_{ij}(x,t)$. Then the iteration scheme in [Gr1] can be applied to complete the proof of the theorem.

Next is the Harnack inequality for positive solutions. Let v be a positive solution to (1.4) on $\coprod_{8R}$ where $\coprod_R = B_0(x,R)\times(0,R^2)$.

Theorem 1.2. *Let $(M, g_{ij}(x,t))$ be as in Lemma 1.2. Then there exists a constant $\gamma = \gamma(n, A_0) > 0$ such that*

$$v(x, 64R^2) \geq \gamma \sup_{B_0(x,R)\times(3R^3,4R^2)} v. \tag{1.14}$$

Proof. The proof follows similarly as the proof of Theorem 4.1 in [Gr1]. Since Lemma 4.2–4.4 in [Gr1] are robust enough to be adapted to the current situation we only need to establish the following result corresponding to Lemma 4.1 of [Gr1].

Lemma 1.3. *Let $v(x,t)$ be a positive solution to* (1.4) *in* $\coprod_{2R}$ *and set*

$$H = \{(x,t) \in \coprod_R : v(x,t) > 1\}, \qquad \widetilde{\coprod}_R = B_0(x,R) \times (3R^2, 4R^2).$$

Then for any $\delta > 0$ there exists $\epsilon = \epsilon(\delta, A_0, n)$ such that if

$$|H| \geq \delta |\coprod_R|, \tag{1.15}$$

then

$$\inf_{\widetilde{\coprod}_R} v \geq \epsilon.$$

Here $|H|$ and $|\widetilde{\coprod}_R|$ are measured with respect to the metric $g_{ij}(x,0)$.

Proof. We have the similar situation as in the proof of Theorem 1.1. The argument follows closely as in [Gr1]. Let $h = \log(1/v)$. It is easy to see that $\left(\frac{\partial}{\partial t} - \Delta\right) h = -|\nabla h|^2$. For a cut-off function $\phi(x)$, we have that

$$\begin{aligned}
\frac{\partial}{\partial t}\left(\int_{B_0(x,R)} h_+ \phi^2\, dy\right) &= \int_{B_0(x,R)} (h_+)_t \phi^2\, dy - \int_{B_0(x,R)} h_+ \phi^2 \mathcal{R}\, dy \\
&\leq \int_{B_0(x,R)} (h_+)_t \phi^2\, dy \\
&\leq \int_{B_0(x,R)} (\Delta h_+)\phi^2\, dy - |\nabla h_+|^2 \phi^2\, dy \\
&\leq -\frac{1}{2}\int_{B_0(x,R)} |\nabla h_+|^2 \phi^2\, dy + 2\int_{B_0(x,R)} |\nabla \phi|^2\, dy.
\end{aligned}$$

This is the (4.3) of [Gr1]. The rest of the proof follows verbatim as in the proof of [Gr1, Lemma 4.1].

One has the following immediate corollary of the above theorem.

Corollary 1.2. *Let $v(x,t)$ be a weak positive solution to* (1.4) *on $M \times [0,T]$. Then for any $T \geq t > s > 0$*

$$\frac{v(y,s)}{v(x,t)} \leq \exp\left(C\left(\frac{r^2(x,y)}{t-s} + \frac{t}{s} + 1\right)\right). \tag{1.16}$$

Here $C = C(\gamma) > 0$.

Proof. This was proved, for example in [Mo, page 110-112].

Theorem 1.3. *Let $(M, g_{ij}(x,t))$ be a complete solution to the Ricci flow with bounded curvature. Assume that $g_{ij}(x,t)$ has nonnegative Ricci curvature. Let $H(x,y,t)$ be the minimal positive heat kernel of the heat equation* (1.4). *Then there exist positive constants C_1, C_2 and D_1, D_1 such that*

(1.17)
$$C_1 \frac{1}{V_0(x,\sqrt{t})} \exp\left(-D_2 \frac{r^2(x,y)}{t}\right) \le H(x,y,t) \le C_2 \frac{1}{V_0(x,\sqrt{t})} \exp\left(-D_1 \frac{r^2(x,y)}{t}\right).$$

Here $V_0(x,a)$ and $r(x,y)$ denote the volume of $B_0(x,a)$ and distance between x and y, with respect to $g_{ij}(x,0)$, respectively. $D_1 < \frac{1}{4}$ is a absolute constant. $D_2 = D_2(\gamma)$, $C_i = C_i(n, D_1, D_2, A_0)$.

Proof. It is easy to see that for any $t > 0$,

$$\int_M H(x,y,t)\, dy \le 1.$$

Here dy is the volume element with respect to the metric at time t. Fix a point $z \in M$ and let $u(x,t) = H(x,z,t)$. Then, using the equivalence between the metric $g_{ij}(x,t)$ and $g_{ij}(x,0)$, we can deduce from the above inequality that there exist a constant $C(A_0) > 0$ and a point $y \in B_0(z, 2\sqrt{t})$ such that

$$u(y,2t) \le \frac{C(A_0)}{V_0(z, 2\sqrt{t})}.$$

Applying the Harnack and the volume doubling property we have that

$$u(z,t) \le \frac{C(n,A_0)}{V_0(z,\sqrt{t})}. \tag{1.18}$$

Therefore we have the upper bound for $H(x,x,t)$. The upper bound in (1.17) follows from a general result of Grigor'yan [Gr2, Theorem 1.1]. (The result in [Gr2] is for heat operator with respect to a fixed metric. However, the key of the proof is the existence of backward heat kernel apart from the Harnack inequality, as shown in [LY, L], which can be constructed in our case as shown in [NT1, Theorem 1.2] using the metric shrinking property.) The lower bound can be obtained using the argument in [Gr1, page 73]. Let ϕ be a cut-off function such that $\phi = 1$ on $B_0(y, \frac{1}{2}\sqrt{t})$ and $\phi = 0$ outside $B_0(y, \sqrt{t})$. Now define

$$w(x,s) = \int_M H(x,y,s)\phi(y)\, dy_0$$

for $s \ge 0$ and $w(x,s) \equiv 1$ for $s \le 0$. Then $w(x,s)$ is a solution to the heat equation on $B_0(y, \frac{\sqrt{t}}{2}) \times (-\infty, T)$. Here we have extend the metric to be $g_{ij}(x,0)$

for $s \leq 0$. Applying the Harnack inequality (1.16) we have that

$$\begin{aligned}
1 = u(y,0) &\leq C(n)u(y,\frac{\sqrt{t}}{2}) \\
&= C(n)\int_M H(y,z,\frac{t}{2})\phi(z)\,dz_0 \\
&\leq C(n)\int_{B_0(y,\sqrt{t})} H(y,z,\frac{t}{2})\,dz_0 \\
&\leq C(n)\int_{B_0(y,\sqrt{t})} H(y,y,t)\,dz_0 \\
&\leq C(n)H(y,y,t)V_0(y,\sqrt{t}).
\end{aligned}$$

This gives the lower bound for $H(x,x,t)$. The general form in (1.17) is just another application of the Harnack inequality, or Corollary 1.2.

Remark 1.2. *In* [Sa]*, the above Theorem 1.2 and Theorem 1.3 were proved for the parabolic operator of type* $\frac{\partial}{\partial t} - L$*, with*

$$Lf = m^{-1}\,div\,(m\mathcal{A}(\nabla f))\,,$$

where m *is a measure independent of* t*,* $\mathcal{A}$ *is a measurable section of* $End(T_M)$ *which is uniformly equivalent to the identity. The time dependent Laplacian operator can only expressed in the above form with time dependent measure* $\sqrt{\det(g_{ij}(x,t))}dx_1 \wedge \cdots \wedge dx_n$*. Therefore one can not just apply the results of* [Sa] *directly. One can also prove the above theorems following the iteration procedure of Moser as in* [Sa]*. The iteration procedure in* [Gr1] *was told by experts to be closer to the one of De Giorgi.*

2. A maximum principle for tensors and its applications

In this section we shall prove a maximum principle for the symmetric tensors satisfying (1.5) under the assumption that $(M, g_{ij}(x,t))$ has bounded nonnegative sectional curvature. Since the argument is very close to that in [NT3] we will be sketchy here.

Let η_{ij} be a symmetric tensor satisfying (1.5). The basic assumption on η is that there exists a constant $a > 0$ such that

$$\int_M \|\eta\|(x,0)\exp\left(-ar^2(x)\right)\,dx < \infty \tag{2.1}$$

and

$$\liminf_{r\to\infty}\int_0^T\int_{B_0(o,r)} \|\eta\|^2(x,t)\exp\left(-ar^2(x)\right)\,dx\,dt < \infty. \tag{2.2}$$

Here $\|\eta\|(x,t)$ is the norm of $\eta_{ij}(x,t)$ with respect to metrics $g_{ij}(x,t)$. But $B_0(0,r)$ is the ball with respect to the initial metric $g_{ij}(x,0)$ and $r(x)$ is the distance from x to a fixed point $o \in M$ with respect to the initial metric. Due to the fact that the maximum principle for the heat equation does not hold on complete manifolds in general, one needs some growth conditions on the solutions to make it true. The condition (2.2) is optimal by comparing to the example given in [J, page 211-213]. The above mentioned classical example is a solution to the heat equation on $\mathbb{R} \times [0,\infty)$, which has zero initial data. The violation of the uniqueness implies the failure of the maximum principle for the sub-solutions. The example has growth, as $|x| \to \infty$, just faster than $\exp(ar^2(x))$. The condition (2.1) is needed to ensure that the equation (1.5) does have a solution indeed. It is also in the sharp form.

Before we state our result, let us first fix some notations. Let $\varphi : [0,\infty) \to [0,1]$ be a smooth function so that $\varphi \equiv 1$ on $[0,1]$ and $\varphi \equiv 0$ on $[2,\infty)$. For any $x_0 \in M$ and $R > 0$, let $\varphi_{x_0,R}$ be the function defined by

$$\varphi_{x_0,R}(x) = \varphi\left(\frac{r(x,x_0)}{R}\right).$$

Again, $r(x,y)$ denotes the distance function of the initial metric. Let $f_{x_0,R}$ be the solution of

$$\left(\frac{\partial}{\partial t} - \Delta\right) f = -f$$

with initial value $\varphi_{x_0,R}$, given by

$$f_{x_0,R}(x,t) = \int_M H(x,y,t)\varphi_{x_0,R}(y)dy_0.$$

It is easy to see that f is defined for all t and positive, and it is bounded for $t > 0$.

We shall establish the following maximum principle.

Theorem 2.1. *Let $(M, g_{ij}(x,t))$ be a complete noncompact Riemannian manifolds satisfying (1.1)–(1.3), with nonnegative sectional curvature. Let $\eta(x,t)$ be a symmetric tensor satisfying (1.5) on $M \times [0,T]$ with $0 < T < \frac{1}{40a}$ such that $||\eta||$ satisfies (2.1) and (2.2). Suppose at $t = 0$, $\eta_{ij} \geq -bg_{ij}(x,0)$ for some constant $b \geq 0$. Then there exists $0 < T_0 < T$ depending only on T and a so that the following are true.*

(i) *$\eta_{ij}(x,t) \geq -be^{(4n\sqrt{A_0}+1)t}g_{ij}(x,t)$ for all $(x,t) \in M \times [0,T_0]$.*

(ii) *For any $T_0 > t' \geq 0$, suppose that there is a point x' in M^m and there exist constants $\nu > 0$ and $R > 0$ such that the sum of the first k eigenvalues $\lambda_1, \ldots, \lambda_k$ of η_{ij} satisfies*

$$\lambda_1 + \cdots + \lambda_k \geq -kb + \nu k \varphi_{x',R}$$

for all x at time t'. Then for all $t > t'$ and for all $x \in M$, the sum of the first k eigenvalues of $\eta_{ij}(x,t)$, with respect to $g_{ij}(x,t)$, satisfies

$$\lambda_1 + \cdots + \lambda_k \geq -kbe^{(4n\sqrt{A_0}+1)t} + \nu k f_{x',R}(x, t-t').$$

Proof. We only prove (ii). The proof of (i) is by the exactly same, if not easier, argument. First we let

$$h(x,t) = \int_M H(x,y,t)\|\eta\|(y,0)\, dy_0. \tag{2.3}$$

It is easy to see that $h(x,t)$ is a solution to (1.4). Using Corollary 1.2, the assumption (2.2) and the maximum principle of [NT1] we have that

$$\exp\left(-2\sqrt{A_0}t\right)\|\eta\|(x,t) \leq h(x,t). \tag{2.4}$$

Denote by $A_0(o, r_1, r_2)$ the annulus $B_0(o, r_2) \setminus B_0(o, r_1)$. Here $o \in M$ is a fixed point $B_0(o,r)$ is the ball as before. For any $R > 0$, let σ_R be a cut-off function which is 1 on $A_0(o, \frac{R}{4}, 4R)$ and 0 outside $A_0(o, \frac{R}{8}, 8R)$. We define

$$h_R(x,t) = \int_M H(x,y,t)\sigma_R(y)||\eta||(y,0) dy_0.$$

Then h_R satisfies the heat equation with initial data $\sigma_R||\eta||$. By Lemma 2.2 of [NT3], noticing that the only thing used in Lemma 2.2 is the heat kernel upper bound estimate, we have that there exists a function $\tau(r) > 0$ with $\lim_{r\to\infty} \tau(r) = 0$ and $T_0 > 0$ such that for all $(x,t) \in A_o(\frac{R}{2}, 2R) \times [0, T_0]$

$$h(x,t) \leq h_R(x,t) + \tau(R)$$

and for any $r > 0$

$$\lim_{R\to\infty} \sup_{B_0(o,r)\times[0,T_0]} h_R = 0.$$

Now let

$$\tilde{h}_R(x,t) = \exp\left((4n\sqrt{A_0}+1)t\right)\left(h_R(x,t) + \tau(R)\right).$$

By (2.4) we have that

$$\|\eta\|(x,t) \leq \tilde{h}_R(x,t)$$

for $(x,t) \in A_0(\frac{R}{2}, 2R) \times [0, T_0]$. We can also construct a positive function $\phi(x,t) > 0$ (using the representation through heat kernel) satisfying the heat equation

$$\left(\frac{\partial}{\partial t} - \Delta\right)\phi = (4n\sqrt{A_0}+1)\phi.$$

The big coefficient $(4n\sqrt{A_0}+1)$ is to dominate the negative input from the time differentiation of metrics. We can make $\phi(x,t) \geq \exp(c(r^2(x)+1))$ as in [NT2, Lemma 1.1]. Now we prove theorem for the case $\nu = 1$. Let $\psi = -f + \epsilon\phi + \tilde{h}_R + \exp((4n\sqrt{A_0}+1)t)b$ and

$$(\eta_R)_{ij}(x,t) = \eta_{ij}(x,t) + \psi g_{ij}(x,t)$$

where $f = f_{x_0,R}(x,t)$ defined right before the statement of the theorem. It is easy to see that the first k eigenvalues of $(\eta_R)_{ij}(x,0)$ is positive due to the assumption and positivity of ϕ. The functions ϕ and $\tilde{h}_R$ are constructed so that $\eta_R \geq 0$ on $\partial B_0(o,R) \times [0,T_0]$. Now if the sum of the first k eigenvalues is not nonnegative on $B_0(o,R) \times [0,T_0]$ we can apply the maximum to the first such instance, namely to $(x_0,t_0) \in B_0(o,R) \times [0,T_0]$, where the sum of the first k-eigenvalues of η_R reach zero for the first time inside $B_0(o,R)$. By choosing the normal coordinate near x_0, such that η_R is diagonal at x_0 and the j-th coordinate direction is the j-th eigen-direction, we have that

$$\begin{aligned}
0 &\geq \left(\frac{\partial}{\partial t} - \Delta\right)\left(\sum_{i,j=1}^{k} (\eta_R)_{ij} g^{ij}\right) \\
&= \sum_{i,j=1}^{k} \left(\left(\frac{\partial}{\partial t} - \Delta\right)(\eta_R)_{ij}\right) g^{ij} + 2\sum_{i,j=1}^{k} (\eta_R)_{ij} R^{ij} \\
&\geq \sum_{i,j=1}^{k} \left[2R_{ipjq}\eta_{pq} - R_{ip}\eta_{pj} - R_{pj}\eta_{ip} + \left(\left(\frac{\partial}{\partial t} - \Delta\right)\psi\right) g_{ij} - 2\psi R_{ij}\right] g^{ij}.
\end{aligned}$$

Observe that under the assumption $R_{ijij} \geq 0$, and under the above choice of the orthogonal frame such that the tensor η_{ij} is also diagonal at the fixed point (x_0,t_0) with its eigenvalues λ_i of η ordered as $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. Then we have that

$$\begin{aligned}
\sum_{i,j=1}^{k} \Big[2R_{ipjq}\eta_{pq} - R_{ip}\eta_{pj} - R_{pj}\eta_{ip}\Big] g^{ij} \\
&= 2\left(\sum_{i=1}^{k}\sum_{p=1}^{n} R_{ipip}\lambda_p - \sum_{i=1}^{k} R_{ii}\lambda_i\right) \\
&= 2\left(\sum_{i=1}^{k}\sum_{p=1}^{n} R_{ipip}\lambda_p - \sum_{i=1}^{k}\sum_{p=1}^{n} R_{ipip}\lambda_i\right) \\
&= 2\left(\sum_{i=1}^{k}\sum_{p=k+1}^{m} \lambda_p R_{ipip} - \sum_{i=1}^{k}\sum_{p=k+1}^{m} R_{ipip}\lambda_i\right) \\
&= 2\left(\sum_{i=1}^{k}\sum_{p=k+1}^{m} R_{ipip}(\lambda_p - \lambda_i)\right) \\
&\geq 0.
\end{aligned}$$

Then we have that

$$\begin{aligned}
0 &\geq k\left[\left(\frac{\partial}{\partial t} - \Delta\right)\psi\right] - 2\psi\sum_{ij}^{k} R_{ij}g^{ij} \\
&\geq k\left[\epsilon\phi + \tilde{h}_R + \exp((4n\sqrt{A_0}+1)t)b\right] > 0.
\end{aligned}$$

The contradiction shows that the sum of the first k-eigenvalue for η_R is nonnegative on $B_0(o,R)\times[0,T_0]$. The theorem follows by letting $R\to\infty$ and $\epsilon\to 0$.

The similar maximum principle for the scalar heat equations is relatively easy to prove. They also require an assumption as (2.2). The time dependent case was first proved in [NT1] following the original argument for the time-independent case in [L]. As an application we have the following approximation result on continuous convex functions.

Theorem 2.2. *Let $(M, g_{ij}(x,t))$ be as above. Let $u(x)$ be a Lipschitz continuous convex function satisfying*

$$|u|(x) \leq C\exp\left(ar^2(x)\right) \tag{2.5}$$

for some positive constants C and a. Let $v(x,t)$ be the solution to the time-dependent heat equation (1.4). *There exists $T_0>0$ depending only on a and there exists $T_0>T_1>0$ such that the following are true.*

(i) *For $0<t\leq T_0$, $v(\cdot,t)$ is a smooth convex function (with respect to $g_{ij}(x,t)$).*

(ii) *Let*

$$\mathcal{K}(x,t) = \{w \in T_x^{1,0}(M)|\ v_{ij}(x,t)w^i = 0, \text{ for all } j\}$$

be the null space of $v_{ij}(x,t)$. *Then for any* $0 < t < T_1$, $\mathcal{K}(x,t)$ *is a distribution on* M. *Moreover the distribution is invariant in time as well as under the parallel translation.*

In order to prove the above theorem we need the following approximation result due to Greene-Wu [GW3, Proposition 2.3].

Lemma 2.1. *Let* u *be a convex function on* M. *Assume that* u *is Lipschitz with Lipschitz constant* 1. *For any* $b > 0$, *there is a* C^∞ *convex function* w *such that*

(i) $|w(x) - w(y)| \leq r(x, y)$;
(ii) $|w - u| \leq b$ *on* M; *and*
(iii) $w_{ij} \geq -bg_{ij}$ *on* M.

Proof of Theorem 2.2. In order to apply Theorem 2.1 to current theorem one needs to verify that $\eta_{ij} = v_{ij}(x,t)$ satisfies the assumption (2.1) and (2.2). Lemma 2.1 provides a smoothing approximation of u, therefore we only need to verify the assumption for smooth u which satisfies $u_{ij} \geq -bg_{ij}$ for some $b > 0$. The Lemma 3.1 of [NT3] can be applied to current situation to serve this purpose. One just needs to observe that (i) of Lemma 3.1 in [NT3] follows from the representation formula via the heat kernel, (ii) of Lemma 3.1 in [NT3] only uses nonnegative Ricci and the argument in the proof of (iii) of Lemma 3.1 in [NT3] can be transplanted without any changes to the time-dependent heat operator. After that, one can conclude that $v_{ij}(x,t) \geq 0$. By Theorem 2.1 one can easily infer that the null space of $v_{ij}(x,y)$ must be of constant rank for some small interval $[0, T_1]$. (See also the forth-coming book [Cetc] for the detailed proof of this point.) The fact that it is invariant under the parallel translation follows exactly same as in [NT3]. The time-invariance of the null space was sketched in [H2]. See also [Ca]. For a clear and rigorous proof please wait and see [Cetc].

The following is the main result on the structure of solutions to the Ricci flow preserving the nonnegativity of the sectional curvature.

Theorem 2.3. *Let* $(M, g_{ij}(x,t))$ *be solution to the* (1.1) *satisfying* (1.3) *with nonnegative sectional curvature. Denote* $(\tilde{M}, \tilde{g}_{ij}(x,t))$ *the universal cover of* $(M, g_{ij}(x,t))$. *Then* $\tilde{M}$ *splits isometrically as* $\tilde{M} = \tilde{N} \times \tilde{M}_1$, *where* $\tilde{N}$ *is a compact manifold with nonnegative sectional curvature.* $\tilde{M}_1$ *is diffeomorphic to* $\mathbb{R}^k$. *For the metric on* $\tilde{M}_1$, *obtained by restricting* $\tilde{g}_{ij}(x,t)$ *onto* $\tilde{M}_1$ *with* $t > 0$, *there is a smooth strictly convex exhaustion function on* $\tilde{M}_1$. *Moreover, the soul of* $\tilde{M}_1$ *is a point and the soul of* $\tilde{M}$ *is* $\tilde{N} \times \{o\}$, *if* o *is a soul of* $\tilde{M}_1$.

Proof. By lifting everything to its universal cover we can assume that M is simply-connected. Let $\mathcal{B}$ be the Busemann function on M, with respect to the initial metric $g_{ij}(x,0)$ and a fixed point in M. As it was proved in [CG, GW2] that $\mathcal{B}$ is a convex Lipschitz function with Lipschitz constant 1. Also it is an exhaustion function on M. In fact $\mathcal{B}(x) \geq cr(x)$ when $r(x)$ is sufficient large, for some $C > 0$, where $r(x)$ is the distance function to a fixed point $o \in M$. Let $v(x,t)$ be the solution of (1.4) with $v(x,0) = \mathcal{B}(x)$. Under the assumption that $R_{ijij} \geq 0$ is preserved under the Ricci flow (1.1), we know that $v(x,t)$ is convex by Theorem 2.2. Applying Theorem 2.2 again we know that the null space of $v_{ij}(x,t)$ is a parallel distribution on M. By the simply-connectedness of M and the De Rham's decomposition theorem we know that M splits as $M = N' \times M_1'$, where on M_1, $(v_{ij}(x,t)) > 0$ as a tensor, and $v_{ij} \equiv 0$ on N'. Since $v(x,t)$ is strictly convex and exhaustive on M_1', by Theorem 3 (a) of [GW2] we know that M_1' is diffeomorphic to $\mathbb{R}^{k'}$, where $k' = \dim(M_1')$. We claim that N' is compact. Otherwise, v is not constant since v is exhaustive on N' (v is an exhaustion function on M by Corollary 1.4 of [NT3]). Using the fact that $v_{ij} \equiv 0$ on N', the gradient of v is a parallel vector field, which gives the splitting of N' as $N' = N'' \times \mathbb{R}$, such that v is constant on N''. By the exhaustion of v again we conclude that N'' is compact. Also v is a linear function on the flat factor $\mathbb{R}$. But we already know that v is exhaustive, which implies that $v \to +\infty$ on both ends of $\mathbb{R}$. This is a contradiction. This proves that N' is compact. Let $\tilde{N} = N'$ and $\tilde{M}_1 = M_1'$ we have the splitting for $(M, g_{ij}(x,t_1))$ for some $t_1 > 0$. It is also clear that there exists strictly convex exhaustion function on $\tilde{M}_1$. As for the splitting at $t = 0$ we can obtain by the limiting argument. First we have the isometric splitting $M = \tilde{N} \times \tilde{M}_1$ as above for some fixed $t_1 > 0$. On the other hand, by Theorem 2.2 (see also [H2, Lemma 8.2]) we know that the distribution given by the null space of v_{ij} is also invariant in time. Therefore, the splitting $M = \tilde{N} \times \tilde{M}_1$ also holds for $0 < t \leq t_1$. Now just taking limit as $t \to 0$ we have the metric splitting of $(M, g_{ij}(x,0))$ as $\tilde{N} \times \tilde{M}_1$. (At $t = 0$ the distribution may not be the null spaces of the Hessian of $\mathcal{B}$. In fact the Hessian of $\mathcal{B}$ may not even be defined. However, a parallel translation invariant distribution does split the manifold by De Rham decomposition.)

As a consequence of the fact that there exist strictly convex exhaustion function on $(\tilde{M}_1, g_{ij}(t_1)|_{\tilde{M}_1})$, we know that the soul of $\tilde{M}_1$ (with respect to $g_{ij}(t_1)$) is a point. The reason is as follows. First the restriction of $v(x,t)$ to its soul will be constant since the soul is a compact totally geodesic submanifold. On the other hand $v(x,t)$ is strictly convex if the soul, which is a totally geodesic submanifold, has positive dimension. The contradiction implies that the soul of $\tilde{M}_1$ is a point for $t > 0$. For the case $t = 0$ the result follows by the topological consideration. Assume that the soul of $(\tilde{M}_1, g_{ij}(0))|_{\tilde{M}_1}$is not a point. Denote the soul by $\mathcal{S}(\tilde{M}_1)$. Then since $\mathcal{S}(\tilde{M}_1)$ is the homotopy retraction of $\tilde{M}_1$ we know that $H_s(\tilde{M}_1) =$, where $s = \dim(\mathcal{S}(\tilde{M}_1)) \geq 1$. On the other hand since we already know that $\tilde{M}_1$ is diffeomorphic to $\mathbb{R}^k$. Thus $H_s(\tilde{M}_1) = \{0\}$, which is a contradiction. Therefore we know that the soul of $\tilde{M}_1$ with respect to the initial metric is also a point. The claim that the soul of M is just $\tilde{N} \times \{o\}$ follows from the following simple lemma.

Lemma 2.2. *Let N be a compact Riemannain manifolds with nonnegative sectional curvature. Let M_1 be a complete noncompact Riemannian manifold with nonnegative sectional curvature. Let $M = N \times M_1$. Then the soul of M, $\mathcal{S}(M) = N \times \mathcal{S}(M_1)$, where $\mathcal{S}(M_1)$ is a soul of M_1.*

Proof. For any point $z \in M$ we write $z = (x, y)$ according to the product. First of all, it is easy to see that $N \times \mathcal{S}(M_1)$ is totally geodesic. It is also totally convex since any geodesic $\gamma(s)$ on M can be written as $(\gamma_1(s), \gamma_2(s))$, where $\gamma_i(s)$ are geodesics in the factor. Therefore, due to the fact $\mathcal{S}(M_1)$ is totally convex we know that $\gamma(s)$ lies inside $N \times \mathcal{S}(M_1)$ if its two end points do.

Let $\gamma(s)$ be any geodesic ray issued from $p \in M$. Write $p = (x_0, y_0)$ according to the product. Since N is compact we have that for the projection $\gamma(s) = (\gamma_1(s), \gamma_2(s))$, $\gamma_1(s) = x_0$ and $\gamma_2(s)$ is a ray in M_1. Let $\mathcal{B}^\gamma$ be the Busemann function with respect to γ. We claim that $\mathcal{B}^\gamma(x, y) = \mathcal{B}^{\gamma_2}(y)$, where $\mathcal{B}^{\gamma_2}(y)$ is the Busemann function of γ_2 in M_1. Once we have the claim we conclude that the level set of $\mathcal{B}^\gamma$ is just $N\times$ the level set of $\mathcal{B}^{\gamma_2}$ in M_1 and the half space $H^\gamma = \{z \in M \,|\, \mathcal{B}^\gamma(z) \le 0\}$, as proved in [LT, Proposition 2.1], $H^\gamma = N \times H^{\gamma_2}$. Since this is true for any ray we have that $C = \bigcap_\gamma H^\gamma$ is given by $N \times C_{M_1}$, where C_{M_1} denote the corresponding totally convex compact subset in M_1 cutting out similarly by H^{γ_2}. As in [CG], if the compact totally convex subset C has non-empty boundary we define $C^a = \{z | d(z, \partial C) \ge a\}$. It is easy to see that $C^a = N \times C^a_{M_1}$. In particular, this implies that the soul of M is $N \times \mathcal{S}(M_1)$ since the soul of M is constructed by retracting C^a iteratively.

Now we verify the claim $\mathcal{B}^\gamma(x, y) = \mathcal{B}^{\gamma_2}(y)$. By the definition we have that

$$\begin{aligned}
\mathcal{B}^\gamma(x, y) &= \lim_{s\to\infty} s - d((x, y), \gamma(s)) \\
&= \lim_{s\to\infty} s - \sqrt{d^2_N(x, x_0) + d^2_{M_1}(y, \gamma_2(s))} \\
&= \lim_{s\to\infty} (s - d_{M_1}(y, \gamma_2(s)) + \left(\sqrt{d^2_{M_1}(y, \gamma_2(s))} - \sqrt{d^2_N(x, x_0) + d^2_{M_1}(y, \gamma_2(s))}\right) \\
&= \lim_{s\to\infty} (s - d_{M_1}(y, \gamma_2(s)) - \frac{d_N(x, x_0)}{\sqrt{d^2_{M_1}(y, \gamma_2(s))} + \sqrt{d^2_N(x, x_0) + d^2_{M_1}(y, \gamma_2(s))}} \\
&= \lim_{s\to\infty} (s - d_{M_1}(y, \gamma_2(s)) \\
&= \mathcal{B}^{\gamma_2}(y).
\end{aligned}$$

This completes the proof of the lemma.

Remark 2.1. *Combining with Theorem 5.2 of* [NT3]*, the proof of Theorem 2.3 (Lemma 2.2) implies that if the M is a complete Kähler manifolds with nonnegative sectional curvature, whose universal cover does not contain the Euclidean factor, then the soul of M is either a point or the compact factor which is a compact Hermitian symmetric spaces. In particular, the result holds if the Ricci* *curvature of M is positive somewhere. Therefore one can view the result such as Theorem 4.2 or Theorem 5.1 as a complex analogue of the 'soul theorem' when only the nonnegativity of the bisectional curvature is assumed.*

Since the Ricci flow preserves the nonnegativity of the curvature operator (if the curvature is uniformly bounded by [H2]) we have the following corollary on the structure of complete simply-connected Riemannian manifolds with nonnegative curvature operator.

Corollary 2.1. *Let M be a complete simply-connected Riemannian manifold with bounded nonnegative curvature operator. Then M is a product of a compact Riemannian manifold with nonnegative curvature operator with a complete noncompact manfold which is diffeomorphic to $\mathbb{R}^k$.*

Remark 2.2. *The compact factor in the above result has been classified in* [CY] *to be the product of compact symmetric spaces, Kähler manifolds biholomorphic to the complex projective spaces and the manifolds homeomorphic to spheres. See also* [CC] *and* [GaM]*. More importantly, it relies crucially on the result of* [H2]*,* [MM] *and* [SiY]*. Some literature attribute the result to* [GM]*. But* [GM] *just proved the cohomology groups with coefficient $\mathbb{R}$ (of compact M^n with nonnegative curvature operator) are the same as the sphere S^n. It seems that the classification was quite far from finished without the crucial later work in* [MM] *and* [SiY]*.*

The above Corollary 2.1 was proved earlier in [N] *by Noronha without assuming the curvature tensor being bounded. Our method here has this restriction on boundedness of the curvature since we have to use the short time existence result of Shi in* [Sh2] *on the Ricci flow. In the case of dimension three, the same result holds if one assumes that the sectional curvature is nonnegative, which is same as the curvature operator being nonnegative. However in* [Sh1] *the stronger result was proved even for nonnegative Ricci curvature case. The proof in* [Sh1] *appeals to the previous deep results of Hamilton* [H1, H2] *and Schoen-Yau* [SY]*.*

3. Examples

As another application of Theorem 2.3 we give examples of complete Riemannian manifolds with nonnegative sectional curvature on which the Ricci flow (at least the solution satisfying (1.3)) does not preserve the nonnegativity of the sectional curvature. These manifolds can be constructed as follows. Let $G = SO(n+1)$ with the standard bi-invariant metric and $H = SO(n)$ be its close subgroup. Then H has action on G (as translation) as well as its standard action on $P = \mathbb{R}^n$ (as rotation). Let $M = G \times P/H$. Topologically M is just the tangent bundle over S^n since $H \to G \to G/H = S^n$ is just the corresponding principle bundle over S^n. The construction is due to Cheeger and Gromoll [CG] where the examples were to illustrate their structure theorem therein. About these examples the following are known (cf. [CG]): The metric on M has nonnegative sectional curvature due to the fact that the metric is constructed as the base of a Riemannian submersion; There is also another Riemannian submersion π_* from T(S^n) to S^n with fiber given by $\pi(g\times P)$, where π is the first submersion map from $G\times P$ to M (in general, there always exists a Riemannian submersion from M to its soul according to a result of Perelman [P1]); The fiber (which is given by $\pi(g \times P)$) of this submersion $\pi_* : M \to S^n$ is totally geodesic; The fibers are not flat. Namely the metric on each tangent space $T_p(S^n)$ is not the standard flat metric; M has the unique soul $\mathcal{S}(M) = \pi(G \times \{0\})$ and the metric on M is not of product even locally.

Proposition 3.1. *For the example manifolds above, the Ricci flow with* (1.3) *does not preserve the nonnegativity of the sectional curvature.*

Proof. First M is simply-connected by the exact sequence of the fibration $F \to M \to S^n$ with $F = \mathbb{R}^n$. Assume that the Ricci flow preserves the nonnegativity of the sectional curvature. If the manifold has bounded curvature, then by Theorem 2.3, we know that $M = N \times M_1$, where M_1 is diffeomorphic to $\mathbb{R}^k$. This contradicts to the fact that the metric on M is not locally product (for most cases, it already contradicts to the fact that the tangent bundle $T(S^n)$ is non-trivial topologically). In order to apply Theorem 2.3 we need to verify the curvature of the initial metric is uniformly bounded. In the following we focus on the case $M = SO(3) \times P/H$. The general case follows from a similar consideration.

As we know from [CG, page 442 and CE, page 146-147], the metric is so defined that $\pi : SO(3)\times\mathbb{R}^2 \to T(S^2)$ is a Riemannian submersion, where $SO(3)$ is the equipped with the bi-invariant metric. Since the Riemannian submersion increases the curvature, we know that the metric constructed in this way has nonnegative sectional curvature. The metric can also be described using the second submersion π_* from $T(S^2) \to S^2$ such that for any point in the fiber if the tangent direction is horizontal we use the metric from S^2 and for the vertical direction we use the metric given by

$$dr^2 + \frac{r^2}{1+r^2}d\theta^2.$$

Here (r, θ) is the polar coordinates for $\mathbb{R}^2$. This expression was claimed in [CE, page 146]. For the sake of the completeness we indicate the calculation here. Similar to the situation considered in [C] (see also [CGL]) we can use $\frac{\partial}{\partial s}$ to denote the component of the Killing vector field of action $SO(2)$ in $SO(3)$. The normalized Killing vector field is given by

$$W = \frac{1}{\sqrt{1+r^2}}\left(\frac{\partial}{\partial\theta} + \frac{\partial}{\partial s}\right).$$

Since

$$\begin{aligned}\mathcal{H}(\frac{\partial}{\partial\theta}) &= \frac{\partial}{\partial\theta} - \langle\frac{\partial}{\partial\theta}, W\rangle W\\ &= \frac{\partial}{\partial\theta} - \frac{r^2}{1+r^2}W\end{aligned}$$

the metric on the base of $\frac{\partial}{\partial\theta}$ is given by

$$\|\frac{\partial}{\partial\theta}\|_M^2 = \|\mathcal{H}(\frac{\partial}{\partial\theta})\|^2 = \frac{r^2}{1+r^2}.$$

Here $\mathcal{H}(\frac{\partial}{\partial\theta})$ denotes the horizontal lift (projection) of $\frac{\partial}{\partial\theta}$. This description make it easy to verify that the curvature is uniformly bounded. In order to calculate the curvature we need the formula of [O'N] on the submersion. The Corollary 1 of [O'N, page 465] says that

$$\text{(3.1)} \qquad \begin{aligned} &(a)\, K(P_{vw}) = \hat{K}(P_{vw}) - \frac{\langle T_v v, T_w w\rangle - \|T_v w\|^2}{\|v\wedge w\|^2} \\ &(b)\, K(P_{xv})\|x\|^2\|v\|^2 = \langle (\nabla_x T)_v v, x\rangle + \|A_x v\| - \|T_v x\|^2 \\ &(c)\, K(P_{xy}) = K_*(P_{x_* y_*}) - \frac{3\|A_x y\|^2}{\|x\wedge y\|^2},\ \text{where } x_* = \pi_*(x), \end{aligned}$$

where x, y are horizontal and v, w are vertical. Here A and T are the second fundamental form type tensor for the Riemannian submersion $\pi_* : T(S^2) \to S^2$ with the property that $T \equiv 0$ is the fiber of the submersion is totally geodesic. $\hat{K}(\cdot)$ denotes the sectional curvature of the fiber and $K_*(\cdot)$ denotes the sectional curvature of the base respectively. (One should consult [O'N] for more details on the definition of these operators and their geometric meanings.) Since the fiber of π_* is totally geodesic (cf. [CG, page 442]), $T \equiv 0$, we have the simplified formula

$$\text{(3.2)} \qquad \begin{aligned} &(a)\, K(P_{vw}) = \hat{K}(P_{vw}) \\ &(b)\, K(P_{xv})\|x\|^2\|v\|^2 = \|A_x v\|^2 \\ &(c)\, K(P_{xy}) = K_*(P_{x_* y_*}) - \frac{3\|A_x y\|^2}{\|x\wedge y\|^2},\ \text{where } x_* = \pi_*(x). \end{aligned}$$

By (c) and the nonnegativity of $K(P_{xy})$ we have that $K(P_{xy})$ is uniformly bounded. The curvature of the fiber can be calculated directly. In fact in terms of the polar coordinates on the fiber it is given by

$$\frac{3}{(1+r^2)^2}.$$

Therefore we have that $K(P_{vw})$ is also uniformly bounded. The only thing need to be checked is the mixed curvature $K(P_{xv})$. By the definition of A we know that

$$A_x v = \mathcal{H}\nabla_x V$$

where $\mathcal{H}$ is the horizontal projection and V is any arbitrary extension of v. For a unit horizonal vector E we have

$$\langle A_x v, E\rangle = -\langle v, \nabla_x E\rangle.$$

Therefore it is enough to show that the right hand side is bounded. Since, by the first submersion consideration using the quotient, we know that $K(P_{xy})$ is nonnegative. Therefore by (c) of (3.2),

$$\|A_x y\|^2 \leq \frac{1}{3} K_*(P_{x_* y_*}) \|x \wedge y\|^2.$$

This shows that $|\langle v, \nabla_x E\rangle|$ is uniformly bounded.

For the sake of the completeness we also include a proof of the fact that the fiber of π_* is totally geodesic since there is no written proof in the literature. Recall that $M = SO(n+1) \times P/SO(n)$. Here $SO(n)$ is viewed as close subgroup of $SO(n+1)$ by the inclusion:

$$A \to \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}.$$

We have the involution ζ which is given by

$$\zeta = \begin{pmatrix} 1 & 0 \\ 0 & -I \end{pmatrix}.$$

ζ acts on $SO(n+1)$ by $A \to \zeta A \zeta$. It is easy to see that the fixed point set of ζ is $SO(n)$. Now we consider the action of ζ on $SO(n+1) \times P$ as $(g, x) \to (\zeta g \zeta, x)$. It is easy to see that this action is commutative with the action of $SO(n)$ since for any $h \in SO(n)$ $\zeta h = h\zeta$. Therefore the action descends to M. It is easy to see that the fixed point of this action is $\pi(e, P)$. This implies that the fiber (of the submersion π_*) $\pi(e, P)$ is totally geodesic since it is the fixed point set of an isometry. The other fiber can be verified similarly since for any point $p \in S^n$ there is also an involution fixes p.

Remark 3.1. *Since we used Theorem 2.3 in Proposition 3.1 above, the examples only apply to the solution to Ricci flow satisfying* (1.3)*. This includes the solution provided by Shi's existence theorem. It is still unknown that the solution to Ricci flow satisfying* (1.3) *is unique or not. Whether or not there exist similar compact examples also worths further investigation.*

Acknowledgement

The author would like to thank Professors Ben Chow, Peter Li, Miles Simon, H. Wu and S.-T. Yau for their interests in this work. Discussions with Professors Tom Ilmanen and Jiaping Wang are very helpful. We would like to thank them too. Special thanks go to Professor Luen-Fai Tam for many helpful suggestions. The paper is not possible without the previous collaboration [NT3] with him. Finally we would like to thank the referee for her/his helpful comments.

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