

CONVOLUTION AND FRACTIONAL INTEGRATION WITH MEASURES ON HOMOGENEOUS CURVES IN \mathbb{R}^n

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1. Introduction

In this paper we consider convolution operators on \mathbb{R}^n of the form

$$(1) \quad R_\gamma f(x) := \int_{-1}^1 f(x - h(t)) |t|^{-1+\gamma} dt$$

where $h(t) := (t^{d_1}, \dots, t^{d_n})$ for a strictly increasing sequence of positive integers d_1, \dots, d_n and $0 < \gamma \leq 1$. Let $p_0 := |d|/(\gamma n)$ and $q_0 := |d|/(\gamma(n-1))$, where $|d| := \sum_j d_j$. Let $\mathcal{T} \subset [0, 1] \times [0, 1]$ be the closed trapezoid with vertices $(0, 0)$, $(1, 1)$, $\mathcal{Q} := (p_0^{-1}, q_0^{-1})$, and $\mathcal{Q}' := (1 - q_0^{-1}, 1 - p_0^{-1})$. We will show that R_γ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ if the point (p^{-1}, q^{-1}) is in $\mathcal{T} \setminus \{\mathcal{Q}, \mathcal{Q}'\}$. This result is optimal, except possibly at \mathcal{Q} and \mathcal{Q}' (i.e., R_γ is not bounded when $(p^{-1}, q^{-1}) \notin \mathcal{T}$). In the place of strong estimates at these exceptional points, we will prove restricted weak type estimates.

The method of proof we use was first employed by Christ in [1] to study the case of the curve $h(t) = (t, t^2, \dots, t^n)$ with no fractional integration ($\gamma = 1$). Tao and Wright have established strong (p, q) boundedness for completely general families of smooth curves [9], but in that paper they were unable to show boundedness in endpoint cases. Earlier papers, such as Oberlin [5], [6] and Greenleaf, Seeger, and Wainger [3], have attained boundary estimates (even strong boundary estimates) but are limited to dimensions $n \leq 4$, with work in dimension 4 restricted to the curve $h(t) = (t, t^2, t^3, t^4)$.

2. Preliminaries and the Parameter Sets Ω_k

To prove the desired estimates, it suffices to consider a truncated version of (1) in which t ranges from 0 to 1. The corresponding half $-1 \leq t \leq 0$ can be treated analogously. Also it suffices (by duality and interpolation) to prove R_γ satisfies a restricted weak type (p_0, q_0) estimate. Furthermore, we will take $\gamma = 1$ throughout most of the paper (the case $0 < \gamma < 1$ will be deduced from the case $\gamma = 1$ by interpolation). We have thus reduced matters to the study of the operator $Rf(x) := \int_0^1 f(x - h(t)) dt$.

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To establish the particular restricted weak type estimate of interest, it is necessary and sufficient to show that for any Borel set $E \subset \mathbb{R}^n$ of finite, non-zero measure and any $\alpha > 0$

$$(2) \quad |\{x \in \mathbb{R}^n \mid R\chi_E(x) > \alpha\}| \lesssim (|E|^n \alpha^{-|d|})^{1/(n-1)}$$

(throughout, $u \lesssim v$ means $u \leq Cv$ with C depending only on constants fixed *a priori* such as n , the integers d_j , and the fractional integration parameter γ). From now on, fix $F := \{x \in \mathbb{R}^n \mid 2\alpha \geq R\chi_E(x) > \alpha\}$ and $\beta := \frac{1}{|E|} \int_F R\chi_E$. Since $\beta/\alpha \sim |F|/|E|$, estimate (2) is equivalent to

$$(3) \quad \alpha^{|d|}(\beta/\alpha)^{n-1} \lesssim |E|.$$

We come now to the engine which will drive the rest of the proof. This first lemma is almost the same as lemma 1 in Christ [1]; the difference is that here the sets $\Omega_k \subset \mathbb{R}_+^k$ are tailored to avoid the coordinate hyperplanes $\{t \in \mathbb{R}_+^k \mid t_i = 0\}$. This new feature was not necessary in [1], but will be essential for our purposes.

Lemma 1. *Let $\nu := \max\{\alpha/2, \beta/12\}$, and let $\Phi_k : \mathbb{R}^k \rightarrow \mathbb{R}^n$ be given by $\Phi_k(t_1, \dots, t_k) := h(t_1) - h(t_2) + \dots + (-1)^{k-1}h(t_k)$ for all $1 \leq k \leq 2n$. There exists an $x_0 \in E$, a constant ϵ depending only on n , and sets $\Omega_k \subset [\nu, 1]^k$, $1 \leq k \leq 2n$, such that for each even k*

$$(4) \quad x_0 + \Phi_k(\Omega_k) \subset E$$

$$(5) \quad (t_1, \dots, t_k) \in \Omega_k \Rightarrow (t_1, \dots, t_{k-1}) \in \Omega_{k-1}$$

$$(6) \quad (t_1, \dots, t_{k-1}) \in \Omega_{k-1} \Rightarrow \alpha \geq \int \chi_{\Omega_k}(t_1, \dots, t_k) dt_k \geq \epsilon\alpha$$

$$(7) \quad (t_1, \dots, t_k) \in \Omega_k \Rightarrow |t_k - t_j| \geq \epsilon\alpha \quad \forall j < k,$$

and for each odd k analogous statements hold, replacing E by F and α by β .

Proof. Consider the truncated operator $\tilde{R}f(x) := \int_\nu^1 f(x - h(t)) dt$ and the associated sequences of sets

$$\begin{aligned} E_0 &:= E, \quad F_0 := F \\ E_{j+1} &:= \left\{ x \in E_j \mid \tilde{R}^* \chi_{F_j}(x) > 2^{-j-3}\beta \right\} \\ F_{j+1} &:= \left\{ y \in F_j \mid \tilde{R} \chi_{E_{j+1}}(y) > 2^{-j-3}\alpha \right\} \end{aligned}$$

(In [1], the original operator R is used instead of \tilde{R}). To be of any use, the sets E_j and F_j must be non-empty; we will establish the stronger property

$$(8) \quad \langle \tilde{R} \chi_{E_j}, \chi_{F_j} \rangle \geq 2^{-j-1}\beta|E| \quad \forall j \geq 0$$

by induction. Take $j = 0$, and suppose that $\nu = \alpha/2$. We step through definitions in the usual way:

$$\langle \tilde{R} \chi_E, \chi_F \rangle \geq \langle R \chi_E, \chi_F \rangle - \frac{\alpha}{2}|F| \geq \frac{1}{2} \langle R \chi_E, \chi_F \rangle = \frac{1}{2}\beta|E|.$$

If instead $\nu = \beta/12$, split $E = E_s \cup E_b$, where E_s is the subset of E on which $R^*\chi_F \leq \beta/4$ and E_b is the relative complement of E_s in E . We use the trivial inequality $\langle \chi_E, R^*\chi_F \rangle \leq \frac{\beta}{4}|E_s| + \langle \chi_{E_b}, R^*\chi_F \rangle$ to estimate $\langle \chi_E, \tilde{R}^*\chi_F \rangle$ from below:

$$\langle \chi_E, \tilde{R}^*\chi_F \rangle \geq \frac{2}{3}\langle \chi_{E_b}, R^*\chi_F \rangle \geq \frac{2}{3}\left(\langle \chi_E, R^*\chi_F \rangle - \frac{\beta}{4}|E_s|\right) \geq \frac{1}{2}\beta|E|.$$

As a consequence, it is again true that $\langle \tilde{R}\chi_E, \chi_F \rangle \geq \frac{1}{2}\beta|E|$. This establishes the truth of (8) when $j = 0$. The induction step is straightforward:

$$\begin{aligned} \langle \tilde{R}\chi_{E_{j+1}}, \chi_{F_{j+1}} \rangle &= \langle \tilde{R}\chi_{E_{j+1}}, \chi_{F_j} \rangle - \langle \tilde{R}\chi_{E_{j+1}}, \chi_{F_j \setminus F_{j+1}} \rangle \\ &\geq \langle \chi_{E_{j+1}}, \tilde{R}^*\chi_{F_j} \rangle - 2^{-j-3}\alpha|F| \\ &\geq \langle \chi_{E_j}, \tilde{R}^*\chi_{F_j} \rangle - \langle \chi_{E_j \setminus E_{j+1}}, \tilde{R}^*\chi_{F_j} \rangle - 2^{-j-3}\beta|E| \\ &\geq \langle \tilde{R}\chi_{E_j}, \chi_{F_j} \rangle - 2^{-j-2}\beta|E| \\ &\geq 2^{-j-2}\beta|E|. \end{aligned}$$

This establishes the claim (8).

Now the sets Ω_k may be defined. Fix a point x_0 anywhere in E_n . By definition of E_n and \tilde{R} ,

$$|\{t \in [\nu, 1] \mid x_0 + h(t) \in F_{n-1}\}| > 2^{-n-2}\beta.$$

We may take Ω_1 to be any Borel subset of $\{t \in [\nu, 1] \mid x_0 + h(t) \in F_{n-1}\}$ with measure $2^{-n-2}\beta$. The construction of Ω_2 proceeds iteratively. For each $t_1 \in \Omega_1$, $x_0 + h(t_1)$ is in F_{n-1} ; therefore $\tilde{R}\chi_{E_{n-1}}(x_0 + h(t_1)) > 2^{-n-1}\alpha$. As a result, the set

$$(9) \quad \{(t_1, t_2) \in \Omega_1 \times [\nu, 1] \mid x_0 + \Phi_2(t_1, t_2) \in E_{n-1}, |t_2 - t_1| > 2^{-n-2}\alpha\}$$

must have cross-sections (for each $t_1 \in \Omega_1$) of measure at least $2^{-n-2}\alpha$. The set Ω_2 is now chosen to be a subset of (9) which has cross-sections of measure *precisely* $2^{-n-2}\alpha$ for each $t_1 \in \Omega_1$; with a little care this may be done so that Ω_2 is Borel. For generic odd k , we let Ω_k be a Borel subset of

$$\{t \in \Omega_{k-1} \times [\nu, 1] \mid c_0 + \Phi_k(t) \in F_{n-(k+1)/2}, \forall j < k \mid t_k - t_j > \epsilon_k\beta\}$$

with cross sections (in the k -th direction) of measure $2^{-n-4+(k+1)/2}\beta$. For k even we take Ω_k from

$$\{t \in \Omega_{k-1} \times [\nu, 1] \mid x_0 + \Phi_k(t) \in E_{n-(k/2)}, \forall j < k \mid t_k - t_j > \epsilon_k\alpha\}$$

with cross sections of measure $2^{-n-3+(k/2)}\alpha$. In both cases ϵ_k is suitably chosen to permit our choices. These Ω_k satisfy (4) - (7) by construction. \square

3. An Initial Estimate

It turns out to be quite easy to bound $|E|$ from below with expressions of the form $\alpha^{|d|}(\beta/\alpha)^\rho$. The difficulty is, of course, achieving an optimal ρ . This will be accomplished in two stages. The first set of estimates will be optimal in dimensions 2 and 3, and a second set in the next section will yield optimal exponents for all n .

Theorem 1. *Let $h(t)$ and E, F, α, β be as described above. Then for even n*

$$(10) \quad |E| \gtrsim \alpha^{|d|}(\beta/\alpha)^{1+3+\dots+(n-1)},$$

and for odd n

$$(11) \quad |E| \gtrsim \alpha^{|d|}(\beta/\alpha)^{-1+|d|-2-4-\dots-(n-1)}.$$

Proof. Assume first that n is even. In this case lemma 1 says that $x_0 + \Phi_n(\Omega_n) \subset E$, so

$$(12) \quad |E| \geq \int_{\Phi_n(\Omega_n)} dx \geq (n!)^{-1} \int_{\Omega_n} \left| \det \left(\frac{\partial \Phi_n}{\partial t} \right) \right| (t) dt,$$

using the fact that Φ_n is smooth, and outside a closed, nowhere dense set, the equation $\Phi_n(t) = x$ has at most $n!$ solutions for a given x (this fact will be established shortly).

For $t = (t_1, \dots, t_n) \in \Omega_n$, lemma 1 guarantees that $t_j \geq \alpha/2$ for all j . Provided $i < j$, lemma 1 also guarantees $|t_j - t_i| \gtrsim \alpha$ for j even and $|t_j - t_i| \gtrsim \beta$ for j odd. We apply these inequalities to an estimate of the Jacobian determinant (also to be established)

$$(13) \quad \left| \det \left(\frac{\partial \Phi_n}{\partial t} \right) \right| (t) \gtrsim (\min_i |t_i|)^{|d|-n(n+1)/2} \prod_{i < j} |t_j - t_i|,$$

and conclude $|\det(\frac{\partial \Phi_n}{\partial t})| \gtrsim \alpha^{|d|-n}(\beta/\alpha)^\rho$ on Ω_n , where $\rho = 2 + 4 + \dots + (n - 2)$. An induction argument easily establishes $|\Omega_n| \gtrsim \alpha^{n/2}\beta^{n/2}$. Applying the estimate of the Jacobian determinant and the estimate of the size of $|\Omega_n|$ to the right-hand side of (12) gives (10).

Consider the case when n is odd. Now $\Phi_n(\Omega_n) \subset F$. From (13) and lemma 1 we find $|\det(\frac{\partial \Phi_n}{\partial t})| \gtrsim \beta^{|d|+\rho-n(n+1)/2}\alpha^{\rho-(n-1)/2}$ on Ω_n , where $\rho = 2 + 4 + \dots + (n - 1)$ (here we used the fact that $t_j \geq \beta/12$). In this case we also have $|\Omega_n| \gtrsim \alpha^{(n-1)/2}\beta^{(n+1)/2}$; proceeding as above we find $|F| \gtrsim \alpha^{|d|}(\beta/\alpha)^{|d|-\rho}$. Multiplying both sides by $(\beta/\alpha)^{-1}$ and recalling $|E| \sim |F|(\beta/\alpha)^{-1}$ gives (11). \square

Remark. It is worth noting that (11) is dual to the estimate

$$(14) \quad |E| \gtrsim \alpha^{|d|}(\beta/\alpha)^{2+4+\dots+(n-1)},$$

which is of the form found in Christ [1] and gives (3) when $n = 3$. Because R and R^* are equal (up to conjugation by a reflection), (11) and (14) are equivalent.

Next we justify making the change of variables $x \mapsto \Phi_n(t)$ by demonstrating that almost every x has no more than $n!$ preimages via Φ_n^{-1} . The following lemma establishes this assertion for a broad class of curves $h(t)$.

Lemma 2. *Let h be a C^1 curve in \mathbb{R}^n and suppose that*

$$(15) \quad \det \left(\frac{\partial \Phi_n}{\partial t} \right) (t_1, \dots, t_n) \neq 0$$

when t_1, \dots, t_n are positive and distinct. Then for any $x \in \mathbb{R}^n$, there are no more than $n!$ points t for which $\Phi_n(t) = x$ and t_1, \dots, t_n are positive and distinct.

Proof. We will show that Φ_n is one-to-one on the region

$$\Gamma_n := \{ (t_1, \dots, t_n) \mid 0 < t_{\sigma(1)} < \dots < t_{\sigma(n)} \},$$

where σ is any permutation of $\{1, 2, \dots, n\}$. Let $t, t' \in \Gamma_n$, $t \neq t'$, and for all j from 1 to n let I_j be the smallest closed interval containing both t_j and t'_j . By the fundamental theorem of calculus,

$$\Phi_n(t') - \Phi_n(t) = \sum_j (-1)^{s_j} \int_{I_j} \frac{dh}{dt}(u) du = M(t', t)v$$

where $s_j = j + 1$ if $t'_j \geq t_j$ ($s_j = j$ otherwise), $M(t', t)$ is the matrix whose j -th column is $(-1)^{s_j} \int_{I_j} \frac{dh}{dt}(u) du$, and $v = [1 \cdots 1]^T$. We wish to show $\Phi_n(t') - \Phi_n(t) \neq 0$; one way to accomplish this is to construct a second matrix $\widetilde{M}(t', t)$ such that $M(t', t)v = \widetilde{M}(t', t)v$ and $\det \widetilde{M}(t', t) \neq 0$. Consider the function

$$f(x) := \sum_{j=1}^n (-1)^{s_j} \chi_{I_j}(x).$$

Since $t' \neq t$, f is not identically zero (because t' and t are in the same ordered region Γ_n). Modulo endpoints, there exist pairwise disjoint, open intervals J_k and non-zero integers c_k so that $f(x) = \sum_k c_k \chi_{J_k}(x)$. As there are n intervals I_j , we may demand without loss of generality that there be $2n - 1$ intervals J_k . We may also demand that J_k be to the left of J_{k+1} for all k .

Consider the integer $p_k := |c_{k-1} - c_k|$. If $p_k \neq 0$, there must be at least p_k intervals I_{j_1}, I_{j_2}, \dots which have an endpoint falling somewhere between J_{k-1} and J_k . Keeping track of these endpoints shows $\sum_k p_k \leq 2n - 2$.

If k_1, k_2, \dots, k_L is any subsequence of $1, 2, \dots, (2n - 1)$ chosen so that c_{k_l} alternates in sign, we have $2(L - 1) \leq \sum_l |c_{k_l} - c_{k_{l-1}}| \leq 2(n - 1)$ or $L \leq n$. Thus, the function f cannot be too oscillatory. In particular, we may divide the support of f (again modulo endpoints) into n non-empty, pairwise disjoint intervals F_k on which f is either non-negative or non-positive (and never identically zero).

We now define $\widetilde{M}(t', t)$ by taking its k -column to be $\int_{F_k} \frac{dh}{dt}(u) f(u) du$. The condition $M(t', t)v = \widetilde{M}(t', t)v$ is an immediate consequence of the fact that

the F_k are disjoint and cover the support of f . By multilinearity, taking $u = (u_1, \dots, u_n)$,

$$\det \widetilde{M}(t', t) = (-1)^{n-1} \int_{\prod_k F_k} \left(\det \left(\frac{\partial \Phi_n}{\partial u} \right) (u) \prod_{k=1}^n f(u_k) \right) du.$$

The integrand has constant sign and is non-zero on a set of positive measure, so $\det \widetilde{M}(t', t) \neq 0$. □

The final piece of theorem 1 is to examine the Jacobian determinant of $\Phi_n(t)$. The following lemma is a well-known fact about Vandermonde-type determinants (see for example pp. 200-201 of [10]; it is also readily established by induction on n). In particular, (13) and (15) follow immediately.

Lemma 3. *Let $h(t) = (t^{d_1}, \dots, t^{d_n})$, where $1 \leq d_1 < \dots < d_n$ are integers. Then*

$$(16) \quad \det \left(\frac{\partial \Phi_n}{\partial t} \right) (t) = c\mathcal{P}(t) \prod_{i < j} (t_j - t_i),$$

where \mathcal{P} is a symmetric, homogeneous polynomial with positive integer coefficients and the constant c equals $(-1)^{n-1} \prod_j d_j$.

4. The Main Estimate

When $\beta/\alpha \ll 1$, theorem 1 fails to prove (3). The reason is that lemma 1 does not offer enough control over the distances $|t_i - t_j|$ to prove a sufficient lower bound for the Jacobian determinant of Φ_n . To resolve this problem we return to the ideas of Christ [1] to gain more control over these distances (for example, taking into account that they are not independent of one another).

For bookkeeping purposes, we will construct a partition Λ (hereafter called a band structure following the terminology of [1]) of the indices $\{1, 2, \dots, k\}$ into disjoint subsets (called bands) in such a way that $|t_i - t_j|$ is “big” when i and j are in distinct bands and “small” when i and j are in the same band. Taking a cue from lemma 1, we demand that that no two even indices be in the same band and that no even index be in the same band as the index 1. We will also classify each index $1, \dots, k$ according to its status within its own band; possible types are free, quasi-free, and bound. The classification proceeds according to the following rules:

1. Each band will have precisely one free index. If a band contains an even index or the index 1, then that index will be free (no band can contain two such indices). Otherwise the lowest index in a band is called free.
2. In any band containing exactly two indices, the index not free is quasi-free. A quasi-free index together with its free companion will be called a free/quasi-free pair.
3. All indices not free or quasi-free are bound. A bound index is said to be bound to the unique free index of its band.

If index i is bound to index j , we write $i \Rightarrow j$ and set $B(i) := j$. We will enumerate the bound indices as b_1, b_2, \dots , and f_1, f_2, \dots will be an enumeration of the remaining indices (those either free or quasi-free).

In the next section we will show that there exists $\omega \subset \Omega_k$, for some k with $n \leq k \leq 2n - 2$, and a band structure on $\{1, \dots, k\}$ such that the following are true ($t = (t_1, \dots, t_k)$ is any point in ω):

- (i) $|\omega| \sim |\Omega_k|$.
- (ii) The total number of indices classified either free or quasi-free is n .
- (iii) $|t_i - t_j| \geq \delta\alpha$ when i, j belong to distinct bands.
- (iv) $\epsilon\beta \leq |t_i - t_j| \leq \delta\alpha$ when i, j are a free/quasi-free pair.
- (v) $\epsilon\beta \leq |t_i - t_j| \leq \delta'\alpha$ when i is bound to j .
- (vi) The constants ϵ, δ , and δ'/δ are controlled *a priori* and small relative to unity.

The set ω takes the place of Ω_n in theorem 1 and allows for optimal results.

Theorem 2. *Let $h(t) := (t^{d_1}, t^{d_2}, \dots, t^{d_n})$ and E, F, α, β be as defined. Then for $n \geq 2$, estimate (3) holds, i.e., $|E| \gtrsim \alpha^{|d|}(\beta/\alpha)^{n-1}$.*

Proof. Let us take ω for granted. Consider the case when k is even. As in theorem 1, we have the trivial estimate $|E| \geq |\Phi_k(\omega)|$; however, ω may live in more than n dimensions. Thus, extra care must be taken to compute $|\Phi_k(\omega)|$. Following [1], we make a preliminary change of variables

$$\begin{aligned} \tau_j &= t_{f_j}, & 1 \leq j \leq n \\ s_i &= t_{b_i} - t_{B(b_i)}, & 1 \leq i \leq k - n \end{aligned}$$

and regard Φ_k as a function of (τ, s) , $G_s(\tau) := \Phi_k(t)$. The variable s is fixed and the computations proceed with $G_s(\tau)$ acting as a map from \mathbb{R}^n to itself; in particular, we make the change of variables $x \mapsto G_s(\tau)$ and the estimate $|\det(\frac{\partial G_s}{\partial \tau})| \gtrsim \alpha^{|d|-n}(\beta/\alpha)^{\mathcal{M}}$ on ω (\mathcal{M} is the number of quasi-free indices; we will, of course, prove this). The proof proceeds analogously to theorem 1:

$$\begin{aligned} |E| &\geq |\{G_s(\tau) \mid t(\tau, s) \in \omega\}| \\ &\gtrsim \int_{\{\tau \mid t(\tau, s) \in \omega\}} \left| \det \left(\frac{\partial G_s(\tau)}{\partial \tau} \right) \right| d\tau \\ &\gtrsim \alpha^{|d|-n}(\beta/\alpha)^{\mathcal{M}} |\{\tau \mid t(\tau, s) \in \omega\}|. \end{aligned}$$

Now we average this lower bound for $|E|$ over the ball $\{s \in \mathbb{R}^{k-n} \mid |s_i| \leq \delta'\alpha \forall i\}$; the result is $|E| \gtrsim \alpha^{|d|-k}(\beta/\alpha)^{\mathcal{M}}|\omega|$. When k is even, $|\omega| \sim \alpha^k(\beta/\alpha)^{k/2}$, hence $|E| \gtrsim \alpha^{|d|}(\beta/\alpha)^{\mathcal{M}+k/2}$. A simple counting argument shows that $\mathcal{M}+k/2 \leq n-1$. Thanks to theorem 1 we need only consider the case $\beta \lesssim \alpha$, so (3) follows.

When k is odd, we proceed exactly as above, but conclude instead that $|F| \gtrsim \alpha^{|d|}(\beta/\alpha)^{\mathcal{M}+(k+1)/2}$. This time at least $(k+1)/2$ indices are free, so $\mathcal{M} + (k+1)/2 \leq n$. As a result $|F| \gtrsim \alpha^{|d|}(\beta/\alpha)^n$ and $|E| \gtrsim \alpha^{|d|}(\beta/\alpha)^{n-1}$ as desired. \square

As before, the number of preimages $G_s^{-1}(\{x\})$ must be bounded if the corresponding change of variables is to be justified. Unfortunately, arguments like

lemma 2 no longer work. There is hope, but as in Christ [1] it involves Bezout’s theorem. To be precise, we consider the homogeneous system (in the variables $\tau_0, \tau_1, \dots, \tau_n$)

$$0 = (x_1\tau_0^{d_1}, x_2\tau_0^{d_2}, \dots, x_n\tau_0^{d_n}) - G_{s\tau_0}(\tau_1, \dots, \tau_n).$$

From page 198 of [7], this system either has an infinite number of solutions in $\mathbb{C}\mathbb{P}^n$ or exactly $\prod d_j$ solutions counting multiplicity. So long as x is not in the image (under G_s) of the set where the Jacobian of G_s is degenerate (which must have measure zero), there cannot be an infinite number of solutions. Fixing $\tau_0 = 1$ and restricting to real space, we see that $x = G_s(\tau)$ has at most $\prod_j d_j$ solutions.

The remaining issue to be resolved in this section is to establish a lower bound of the Jacobian determinant of G_s on the set ω . Recall

$$G_s(\tau) = \sum_{j=1}^n \left[(-1)^{f_j-1} h(\tau_j) + \sum_{b_i \Rightarrow f_j} (-1)^{b_i-1} h(\tau_j + s_i) \right].$$

As bound indices are necessarily odd, $(-1)^{b_i-1} = 1$ in all cases. Taking partial derivatives yields

$$\frac{\partial G_s}{\partial \tau_j} = (-1)^{f_j-1} \frac{dh}{dt}(\tau_j) + \sum_{b_i \Rightarrow f_j} \frac{dh}{dt}(\tau_j + s_i).$$

When $s = 0$, $\frac{\partial G_s}{\partial \tau_j}$ is an integer multiple of $\frac{dh}{dt}$; furthermore, it is a non-zero multiple since no free index may have only one bound companion. Let $M_h(\tau)$ be the $n \times n$ matrix whose j -th column is $\frac{dh}{dt}(\tau_j)$ and set $V_h(\tau) := \det M_h(\tau)$. By multilinearity we have

$$(17) \quad \det \left(\frac{\partial G_s}{\partial \tau} \right) (\tau) = \sum_i (-1)^{\nu(i)} V_h(\tau + \epsilon_i);$$

$\sum_i (-1)^{\nu(i)} \neq 0$ follows, once again, from examining the case $s = 0$. We recall (16) and factor V_h . Take, for example, a term from the right-hand side of (17):

$$V_h(\tau + \epsilon_i) = \mathcal{P}(\tau + \epsilon_i) \prod_{j < l} (\tau_l - \tau_j + (\epsilon_i)_l - (\epsilon_i)_j).$$

Observe that $|(\epsilon_i)_l - (\epsilon_i)_j| \leq 2\delta'\alpha$. If it happens that $|\tau_l - \tau_j| < \delta\alpha$, then τ_l and τ_j must correspond (before the linear change of variables) to a free/quasi-free pair, in which case $(\epsilon_i)_l = (\epsilon_i)_j = 0$ since non-zero perturbations arise only from bound indices. Thus, if δ'/δ is sufficiently small, we have

$$\left(1 - \frac{2\delta'}{\delta} \right)^{\frac{n(n-1)}{2}} \leq \prod_{j < l} \frac{\tau_l - \tau_j + (\epsilon_i)_l - (\epsilon_i)_j}{\tau_l - \tau_j} \leq \left(1 + \frac{2\delta'}{\delta} \right)^{\frac{n(n-1)}{2}}.$$

Furthermore, $\tau_l \geq \alpha/2$ for all l , so $(1 - 2\delta')\tau_l \leq \tau_l + (\epsilon_i)_l \leq (1 + 2\delta')\tau_l$ and

$$(1 - 2\delta')^{|d| - \frac{n(n+1)}{2}} \leq \frac{\mathcal{P}(\tau + \epsilon_i)}{\mathcal{P}(\tau)} \leq (1 + 2\delta')^{|d| - \frac{n(n+1)}{2}}.$$

With appropriate *a priori* bounds on δ and δ'/δ we may combine the above inequalities to conclude

$$\left| \det \left(\frac{\partial G_s}{\partial \tau} \right) (\tau) - V_h(\tau) \sum_i (-1)^{\nu(i)} \right| \leq \frac{1}{2} |V_h(\tau)|.$$

On the set ω , $|V_h| \gtrsim \alpha^{|d|-n} (\beta/\alpha)^M$ because $|\tau_l| \geq \alpha/2$ for all l and $|\tau_l - \tau_j| \geq \delta\alpha$ unless τ_l and τ_j correspond to a free/quasi-free pair. Finally, $|\det(\frac{\partial G_s}{\partial \tau})| \sim V_h$ on ω , so the estimate is complete.

The last piece of theorem 2 is to demonstrate the existence of ω and its band structure, which is the task of the next section.

5. Existence of Band Structures

Let σ be a permutation of $\{1, \dots, 2n - 2\}$ for which the set $\tilde{\Omega}_{2n-2} := \{t \in \Omega_{2n-2} \mid t_{\sigma(1)} < \dots < t_{\sigma(2n-2)}\}$ satisfies $|\tilde{\Omega}_{2n-2}| \geq |\Omega_{2n-2}|/(2n - 2)!$. For all $j < 2n - 2$, we restrict the sets Ω_j according to σ :

$$\tilde{\Omega}_j := \{(t_1, \dots, t_j) \in \Omega_j \mid \exists (t_{j+1}, \dots, t_{2n-2}) \text{ s.t. } (t_1, \dots, t_{2n-2}) \in \tilde{\Omega}_{2n-2}\}.$$

Observe that $|\tilde{\Omega}_j| \sim |\Omega_j|$ for all j (when j is even $|\Omega_{j-1}| \lesssim \frac{1}{\alpha} |\Omega_j| \lesssim \frac{1}{\alpha} |\tilde{\Omega}_j| \leq \frac{1}{\alpha} \int_{\Omega_{j-1}} \alpha dt = |\tilde{\Omega}_{j-1}|$ and similarly with β when j is odd). We will work with $\tilde{\Omega}_j$ instead of Ω_j to simplify computations.

Suppose r is an integer no greater than $2n - 2$; let \mathcal{S}_r be the power set of $\{2, \dots, r\}$. For $\Gamma \in \mathcal{S}_r$, define $\tilde{\Omega}_r^\Gamma := \{t \in \tilde{\Omega}_r \mid t_{\sigma(j)} \geq t_{\sigma(j-1)} + \delta\alpha \text{ iff } j \in \Gamma, \sigma(j), \sigma(j-1) \leq r\}$. As \mathcal{S}_r has at most 2^{r-1} elements, there must be at least one Γ so that $|\tilde{\Omega}_r^\Gamma| \geq 2^{1-r} |\tilde{\Omega}_r|$. Given such a $\Gamma := \{L_2, \dots, L_R\}$, $2 \leq L_2 < \dots < L_R \leq r$, we may construct a band structure on $\tilde{\Omega}_r^\Gamma$:

$$(18) \quad \{\{\sigma(1), \sigma(2), \dots, \sigma(L_2 - 1)\}, \{\sigma(L_2), \sigma(L_2 + 1), \dots, \sigma(L_3 - 1)\}, \\ \dots, \{\sigma(L_{R-1}), \dots, \sigma(L_R - 1)\}, \{\sigma(L_R), \dots, \sigma(r)\}\}.$$

For $j < r$, the sets $\tilde{\Omega}_j^\Gamma$ also inherit natural band structures: the band structure of $\tilde{\Omega}_j^\Gamma$ is the restriction of the partition (18) to a partition of $\{1, \dots, j\}$. These bands depend on the parameter δ (the bands will be said to have size δ).

Lemma 4. *Given $\delta_0, (\delta'_0/\delta_0)$ sufficiently small (depending only on n), there exists $\delta \leq \delta_0, \delta' := \delta(\delta'_0/\delta_0), k \in \{n, \dots, 2n - 2\}, \omega \subset \Omega_k$, and a band structure such that*

- (i) $|\omega| \sim |\Omega_k|$.
- (ii) *The number of indices either free or quasi-free is n .*
- (iii) $|t_i - t_j| \geq \delta\alpha$ when i, j belong to distinct bands.
- (iv) $\epsilon\beta \leq |t_i - t_j| \leq \delta\alpha$ when i, j belong to the same band, $i \neq j$.
- (v) $\epsilon\beta \leq |t_i - t_j| \leq \delta'\alpha$ when i is bound to j .

Proof. To begin, let $k_0 := 2n - 2$, $\omega := \tilde{\Omega}_{2n-2}$, take $\delta = \delta_0$. Construct bands of size δ for ω (and restrict ω to $\tilde{\Omega}_{k_0}^\Gamma$ for appropriate Γ). The variables \mathcal{N} and \mathcal{M} will always represent the number of indices designated free and quasi-free, respectively. If δ is sufficiently small, all even indices and the index 1 will be free thanks to lemma 1. Properties (i), (iii), and (iv) also hold and will remain true at all times. To achieve (ii) and (v), we follow the recipe given in [1]:

1. Construct bands of size δ for ω and restrict ω accordingly. When $k_0 = 2n - 2$, there are at least n free indices, so $\mathcal{N} + \mathcal{M} \geq n$. Stepping through the process below, $\mathcal{N} + \mathcal{M}$ can never fall below n . If $\mathcal{N} + \mathcal{M} = n$, go to step 3; otherwise proceed to step 2.
2. Project ω onto $\tilde{\Omega}_{k_0-1}$ and reset k_0 to $k_0 - 1$. The new ω inherits a compatible band structure from the old, and since we removed only one index, $\mathcal{N} + \mathcal{M}$ can decrease by at most one (it could increase by at most one if a bound index becomes quasi-free). Repeat this restriction process for lower and lower k_0 until $\mathcal{N} + \mathcal{M} = n$ (since $\mathcal{N} + \mathcal{M} \leq k_0$, we must reach the goal in a bounded number of repetitions). When this is accomplished, go to step 3.
3. Let $\delta' := \delta(\delta'_0/\delta_0)$, $\tilde{\omega} := \{t \in \omega \mid |t_i - t_j| \leq \delta' \alpha \forall i, j \ i \Rightarrow j\}$. If $|\tilde{\omega}| \geq |\omega|/2$, the set $\tilde{\omega}$ satisfies (i)-(v) with $k := k_0$ and the process is complete. If not, continue to step 4.
4. Replace ω by $\omega \setminus \tilde{\omega}$ and δ by $\delta(\delta'_0/\delta_0)$. Because the process did not terminate at step 3, when we construct new bands of size δ for ω , at least one index which was bound will become free (one of the old bands must break apart). Furthermore, indices which were previously free will remain free, and indices which were quasi-free will remain quasi-free or become free. As a result, $\mathcal{N} + \mathcal{M}$ has increased (it's now strictly greater than n) and the total number of bound indices has decreased. Go back to step 2.

Each time the process passes through step 4, the number of bound variables decreases. If there are no bound variables, the process will terminate at step 3 (since $\tilde{\omega} = \omega$). Therefore, the process must terminate in a bounded number of steps. The final set $\tilde{\omega}$ constructed in step 3 is precisely the set satisfying the conditions of the lemma. \square

6. Fractional Integration

We now consider convolutions with added fractional integration as in (1). As promised, an interpolation argument quickly reduces the argument to the established estimates $\gamma = 1$.

Theorem 3. *Let $0 < \gamma \leq 1$. Then R_γ defined in (1) satisfies a restricted weak type (p, q) estimate with $p = |d|/(\gamma n)$ and $q = |d|/(\gamma(n - 1))$.*

Proof. Observe that when $\gamma = 1/k$ for some integer k , a change of variables $t \mapsto t^k$ reduces this theorem to the case $\gamma = 1$ for some more degenerate curve.

For all other γ , we use analytic interpolation of the restricted weak type estimates

$$\left| \int R_\gamma \chi_E(x) \chi_F(x) dx \right| \lesssim |E|^{Re(\gamma) \frac{n}{1+d}} |F|^{1-Re(\gamma) \frac{n}{1+d}}$$

between the lines $Re(\gamma) = 1/k$ and $Re(\gamma) = 1/(k + 1)$. □

Remark. The limiting case $\gamma = 0$, where we define

$$R_0 f(x) := p.v. \int_{-1}^1 f(x - h(t)) \frac{dt}{t},$$

has long been known to be bounded on L^p for $1 < p < \infty$ as we would expect. See for example [8].

7. Comments

It is straightforward to show that R_γ cannot be of restricted type (p, q) when $(p^{-1}, q^{-1}) \notin \mathcal{T}$. One needs only test on characteristic functions of small balls about the origin and again on non-isotropic boxes, just as done in Christ [1]. The only difference is that the boxes should scale like the curve $h(t)$, i.e., they should have side lengths $(\delta^{d_1}, \dots, \delta^{d_n})$. The unboundedness of R_γ outside \mathcal{T} is also established by Tao and Wright [9], where they construct appropriate test functions for general families of curves.

The proofs presented here are independent of the endpoints of t -integration in (1). In particular, theorem 3 also holds for the global operators

$$R_\gamma f(x) := \int_{-\infty}^{\infty} f(x - h(t)) |t|^{-1+\gamma} dt,$$

as does the dual bound (but R_γ is no longer bounded on L^1 or L^∞).

Theorem 3 also holds in the case of curves $h(t) := (t^{d_1}, \dots, t^{d_n})$ where the d_i are distinct real numbers, and $d_i \geq i$ for each i (some modification of the region of boundedness must occur if the exponents d_i are too small). The only new ingredients are a generalized factorization to replace (16) and an extension of Bezout’s theorem to more general systems such as “fewnomials.” See theorem 3 of chapter 2 in [4] for more about systems of fewnomials.

In low dimensions, the arguments presented here can be applied to C^∞ curves of finite type (curves with n linearly independent derivatives at the origin); the only change is to replace (13) by lemma 2.1 of Christ [2]. In higher dimensions, problems arise because the equation $G_s(\tau) = x$ can have arbitrarily many solutions τ on sets of x with positive measure. What is needed is a proof that only boundedly many of these solutions lie in $\omega \subset \Omega_k$.

We construct an example of a “bad” $G_s(\tau)$ as follows. Let $\psi(t)$ be a smooth function which is identically one on the interval $[3/4, 5/4]$ and vanishing outside $(2/3, 4/3)$. Let

$$\phi(t) := \sum_{k=1}^{\infty} 2^{-2^k} (t - 2^{-k})^3 \psi(2^k t).$$

The function ϕ is C^∞ and vanishes to infinite order at $t = 0$. In addition, ϕ has vanishing second derivative and non-vanishing third derivative at $t = 2^{-k}$ for $k \geq 1$. Consider the plane curve $h(t) := (t, t^2 + \phi(t))$. Following the prescription of section 4, let $G_{(s_1, s_2)}(t_1, t_2) := h(t_1) + h(t_1 + s_1) - h(t_2) - h(t_2 + s_2)$. Direct computation shows that

$$G_{(\epsilon, 0)}\left(u - \frac{\epsilon}{2}, u\right) = \left(0, \frac{\epsilon^2}{2} + \phi\left(u + \frac{\epsilon}{2}\right) - 2\phi(u) + \phi\left(u - \frac{\epsilon}{2}\right)\right)$$

$$\det \frac{\partial G_{(\epsilon, 0)}}{\partial \tau}\left(u - \frac{\epsilon}{2}, u\right) = 2 \frac{d\phi}{dt}\left(u + \frac{\epsilon}{2}\right) - 4 \frac{d\phi}{dt}(u) + 2 \frac{d\phi}{dt}\left(u - \frac{\epsilon}{2}\right).$$

For any $k \geq 1$, there is a solution $u_{k, \epsilon}$ of the equation $G_{(\epsilon, 0)}(u_{k, \epsilon} - \epsilon/2, u_{k, \epsilon}) = (0, \epsilon^2/2)$ which approaches 2^{-k} as $\epsilon \rightarrow 0$; this is shown by applying the intermediate value theorem to the identity

(19)

$$\phi\left(u + \frac{\epsilon}{2}\right) - 2\phi(u) + \phi\left(u - \frac{\epsilon}{2}\right) = \frac{\epsilon^2}{4} \int_{[0, 1]^2} \frac{d^2\phi}{dt^2}\left(u + \frac{\epsilon}{2}(x + y - 1)\right) dx dy$$

and recalling that $\frac{d^2\phi}{dt^2}(2^{-k}) = 0 \neq \frac{d^3\phi}{dt^3}(2^{-k})$. It must also be that as $\epsilon \rightarrow 0$,

$$\frac{1}{\epsilon^2} \det \frac{\partial G_{(\epsilon, 0)}}{\partial \tau}(u_{k, \epsilon} - \epsilon/2, u_{k, \epsilon}) \rightarrow \frac{1}{2} \frac{d^3\phi}{dt^3}(2^{-k}) \neq 0$$

(simply differentiate (19) with respect to u). We conclude that for any fixed $k \geq 1$ and any small, positive ϵ , the equation $G_{(\epsilon, 0)}(\tau) = (0, \epsilon^2/2)$ has a solution $\tau_{k, \epsilon}$ near the point $(2^{-k} - \epsilon/2, 2^{-k})$; furthermore, $G_{(\epsilon, 0)}(\tau)$ is invertible near $\tau_{k, \epsilon}$ when ϵ is sufficiently small. For any natural number N , then, there is an ϵ and an open set of points x (which includes $x = (0, \epsilon^2/2)$) for which $G_{(\epsilon, 0)}(\tau) = x$ has at least N solutions.

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