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SLOPES OF 2-ADIC OVERCONVERGENT MODULAR FORMS WITH SMALL LEVEL

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In memory of my grandfathers,
Raymond Ransom Winterbone (1928–2003)
and
C. W. J. "Jack" Kilford (1922–2004).

Abstract. Let τ be the primitive Dirichlet character of conductor 4, let χ be the primitive even Dirichlet character of conductor 8 and let k be an integer. We show that the U_2 operator acting on cuspidal overconvergent modular forms of weight $2k-1$ and character τ has slopes in the arithmetic progression $\{2, 4, \dots, 2n, \dots\}$, and the U_2 operator acting on cuspidal overconvergent modular forms of weight k and character $\chi \cdot \tau^k$ has slopes in the arithmetic progression $\{1, 2, \dots, n, \dots\}$.

We also show that the characteristic polynomials of the Hecke operators U_2 and T_p acting on the space of classical cusp forms of weight k and character either τ or $\chi \cdot \tau^k$ split completely over $\mathbf{Q}_2$.

1. Introduction

Definition 1. *Let f be a normalised cuspidal modular eigenform with q-expansion at ∞ given by $\sum_{n=1}^{\infty} a_n q^n$. The (p)-*slope *of f is defined to be the p-valuation of a_p; we normalise the p-valuation of p to be 1. If we do not specify p, then we mean the 2-slope.*

In this paper, we prove the following theorem on the slopes of classical modular cusp forms:

Theorem 2. *Let τ be the nontrivial character of conductor 4, and let k be a positive integer. The slopes of the U_2 operator acting on $S_{2k-1}(\Gamma_0(4), \tau)$ are*

$$2, 4, 6, \dots, 2k-4.$$

Let χ be the even primitive Dirichlet character of conductor 8. The slopes of the U_2 operator acting on $S_k(\Gamma_0(8), \chi \cdot \tau^k)$ are

$$1, 2, 3, \dots, k-2.$$

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As a corollary of this theorem, we also prove the following result about the field over which cusp forms of weight k and character $\chi \cdot \tau^k$ or τ are defined:

Corollary 3. *Let k be a positive integer and let S be either $S_{2k-1}(\Gamma_0(4), \tau)$ or $S_k(\Gamma_0(8), \chi \cdot \tau^k)$.*

The Fourier coefficients of a normalised eigenform in S are elements of $\mathbf{Q}_2$.

This corollary gives a partial answer to an extension of Questions 4.3 and 4.4 of Buzzard [2], which give a conjectural bound on the degree of the field of definition of certain spaces of modular forms over $\mathbf{Q}_p$.

2. Previous work

Matthew Emerton determines in his thesis [12] the smallest slope for the spaces of modular cuspforms $S_k(\Gamma_0(2^n), \theta)$, where θ is a primitive Dirichlet character of conductor 2^n.

Theorem 4 (Emerton [12], Proposition 5.1)**.** *Let m be a positive integer greater than 1, and let θ be a primitive Dirichlet character of conductor 2^m such that $\theta(-1) = (-1)^k$. The smallest slope of the U_2 operator acting on cuspforms of weight k and character θ is 2^{3-m}.*

If we look at the character of conductor 4 and the odd character of conductor 8, there is a CM modular form which is defined over the field $\mathbf{Q}$. We quote a result of Schoeneberg, proved in Ogg [17]:

Theorem 5 (Ogg [17], Theorem VI.22)**.** *Let i be the square root of -1, and let k be an positive integer greater than 1 and congruent to $1 \mod 4$. Then there is a normalised cuspidal modular eigenform in $S_k(\Gamma_0(4), \tau)$ with q-expansion*

$$f_k(q) = \frac{1}{4} \sum_{m,n \in \mathbf{Z}} (m + n \cdot i)^{k-1} \cdot q^{m^2+n^2} = q + (-1)^{\frac{k-1}{4}} 2^{\frac{k-1}{2}} q^2 + \cdots$$

Let l be a positive odd integer greater than 1. Then there is a normalised cuspidal modular eigenform in $S_l(\Gamma_0(8), \tau \cdot \chi)$ with q-expansion

$$g_l(q) = \frac{1}{2} \sum_{m,n \in \mathbf{Z}} (m + 2n \cdot i)^{l-1} \cdot q^{m^2+2n^2} = q + (-1)^{\frac{l-1}{2}} 2^{l-1} q^2 + \cdots$$

We see by inspection that the slope of f_k is $(k-1)/2$, and that the slope of g_l is $l-1$. Hence we can, in certain cases, determine the smallest slope and another classical slope of U_2 acting on modular newforms of level $\Gamma_1(4)$ or $\Gamma_1(8)$ using previously known results.

Lawren Smithline has also proved results about the slopes of classical modular forms, and the techniques used in [20] are similar to those in this paper.

Theorem 6 (Smithline [20], Corollary 6.1.3.3)**.** *Let v be a non-negative integer and let $k = 2 \cdot 3^{v+1}$. Then there are exactly 3^v classical modular eigenforms of weight k and level 3 with 3-slope $3^{v+1} - 1$.*

Buzzard and Calegari [4] have proved the following theorem on the slopes of overconvergent modular cusp forms:

Theorem 7 (Buzzard-Calegari [4], Corollary 1)**.** *Let v_p be the p-valuation on $\mathbf{Q}_p$, normalised such that $v_p(p) = 1$. The slopes of the U_2 operator on the space of overconvergent cusp forms of weight 0 are given by*

$$1 + 2v_2\left(\frac{(3n)!}{n!}\right).$$

Jacobs [14] has proved, using similar techniques, the following theorem:

Theorem 8 (Jacobs, [14])**.** *Let k be a positive integer, and let θ be a primitive Dirichlet character of conductor 9 such that $\theta(-1) = (-1)^k$.*

The slopes of the operator U_3 acting on $S_k(\Gamma_0(18), \theta)^{2-new}$ are given by the arithmetic progression $\{1/2, 3/2, 5/2, \dots\}$.

3. Overconvergent modular forms

A famous quote of Jacques Hadamard [13] says that "the shortest and best way between two truths of the real domain often passes through the imaginary one." It seems that often the best way to prove results like Theorem 2 about classical modular forms is to prove a theorem for the overconvergent modular forms and then derive the theorem for classical modular forms as a consequence. We therefore recall the definition of the 2-adic overconvergent modular forms, first by defining overconvergent modular forms of weight 0, and then by deriving the definition for forms with weight and character.

Following Katz [15], section 2.1, we recall that, for C an elliptic curve over an $\mathbf{F}_2$-algebra R, there is a mod 2 modular form $A(C)$ called the *Hasse invariant*, which has the q-expansion over $\mathbf{F}_2$ equal to 1.

We consider the Eisenstein series of weight 4 and tame level 1 defined over $\mathbf{Z}$, with q-expansion

$$E_4(q) := 1 + 240 \sum_{n=1}^{\infty} \left(\sum_{0<d|n} d^3 \right) \cdot q^n.$$

We see that E_4 is a lifting of $A(C)^4$ to characteristic 0, as the reduction of E_4 to characteristic 2 has the same q-expansion as $A(C)^4$, and therefore $E_4 \mod 2$ and $A(C)^4$ are both modular forms of level 1 and weight 4 defined over $\mathbf{F}_2$, with the same q-expansion.

If C is an elliptic curve defined over $\overline{\mathbf{Z}}_2$, then the value of $E_4(C)$ is not a number, but an element of $H^0(C, \Omega_C^{\otimes 4})$, the de Rham cohomology over $\overline{\mathbf{Z}}_2$. However, this module is a free rank one $\overline{\mathbf{Z}}_2$-module, and so the valuation $v_2(E_4(C))$ is well-defined. This will allow us to define the ordinary locus of $X_0(2^m)$ and certain neighbourhoods of it. We follow the work of Coleman [9], and first define structures on the modular curve $X_1(2^m)$.

Definition 9 (Coleman [9], page 448)**.** *Consider $X_1(2^m)_{/\mathbf{Q}_2}$ as a rigid analytic space, and let t be a point of $X_1(2^m)$.*

If t is a point of $X_1(2^m)$ which corresponds to a cusp, then we define the valuation $v(E_4(t))$ to be 0, following [3]*, section 4.*

If t is a non-cuspidal point, then it corresponds in a moduli-theoretic way to a pair (C, P), where C is an elliptic curve defined over a field extension of $\mathbf{Q}_2$ and P is a point on C of exact order 2^m.

We define the ordinary locus of $X_1(2^m)$ to be the set of points t of $X_1(2^m)$ such that $v_2(E_4(t)) = 0$, and define $Z_1(2^m)$ to be the rigid connected component of the ordinary locus in $X_1(2^m)$ which contains the cusp ∞. This is a rigid analytic space.

In [11], page 36, it is shown that $Z_1(2^m)$ is an affinoid subdomain of the rigid space $X_1(2^m)_{/\mathbf{Q}_2}$.

We will perform calculations in later sections on the modular curve $X_0(2^m)$. As our references [9] and [11] both work with $X_1(2^m)$, we recall the definition of $X_0(2^m)$ to show the applicability of their results to our specific situation.

Definition 10. *Consider $X_1(2^m)$ as a modular curve. We see that the group $G := (\mathbf{Z}/2^m\mathbf{Z})^\times$ acts upon the non-cuspidal points of $X_1(2^m)$, by the following action: if $a \in (\mathbf{Z}/2^m\mathbf{Z})^\times$, then the action of a sends the pair (C, P) to (C, aP). This action extends to the cuspidal points of $X_1(2^m)$, and it sends cusps to cusps.*

We will define the modular curve $X_0(2^m)_{/\mathbf{Q}_2}$ to be the quotient of $X_1(2^m)$ by $(\mathbf{Z}/2^m\mathbf{Z})^\times$.

We note that the action of the group G does not change the valuation of $E_4(E)$ for a given elliptic curve E. We define $Z_0(2^m)$ to be the rigid connected component of the ordinary locus in $X_0(2^m)$ which contains the cusp ∞. It is a rigid analytic space.

We will now define strict affinoid neighbourhoods of $Z_0(2^m)$.

Definition 11 (Coleman [9], Section B2)**.** *We think of $X_0(2^m)$ as a rigid space over $\mathbf{Q}_2$, and we let $t \in X_0(2^m)(\overline{\mathbf{Q}}_2)$ be a point, corresponding either to an elliptic curve defined over a finite extension of $\mathbf{Q}_2$, or to a cusp. Let w be a rational number, such that $0 < w < \min(2^{2-m}/3, 1/4)$.*

We define $Z_0(2^m)(w)$ to be the connected component of the affinoid

$$\{t \in X_0(2^m) : \ v_2(E_4(t)) \leq 4w\}$$

which contains the cusp ∞.

The condition involves $4w$ rather than w because we are working with a lifting of the fourth power of the Hasse invariant. Also, note that as E_4 is a lifting of the mod 2 modular form A^4, and that any another lifting of A^4 would be of the form $E_4 + 2F$, where F is a classical modular form, then this valuation is well-defined if $0 \leq v(E_4(t)) < 1$. This corresponds to the condition $0 \leq w < 1/4$ in Definition 11.

We can now define overconvergent modular forms of weight 0.

Definition 12 (Coleman [8], page 397)**.** *Let w be a rational number, such that $0 < w < \min(2^{2-m}/3, 1/4)$. Let $\mathcal{O}$ be the structure sheaf of $Z_0(2^m)(w)$. We call sections of $\mathcal{O}$ on $Z_0(2^m)(w)$* w-overconvergent 2-adic modular forms of weight 0 and level $\Gamma_0(2^m)$. *If a section f of $\mathcal{O}$ is a w-overconvergent modular form, then we say that f is an* overconvergent 2-adic modular form.

Let K be a complete subfield of $\mathbf{C}_2$, and define $Z_0(2^m)(w)_{/K}$ to be the affinoid over K induced from $Z_0(2^m)(w)$ by base change from $\mathbf{Q}_2$. The space

$$M_0(2^m, w; K) := \mathcal{O}(Z_0(2^m)(w)_{/K})$$

of w-overconvergent modular forms of weight 0 and level $\Gamma_0(2^m)$ is a K-Banach space.

We now use non-cuspidal modular forms of the desired weight and character to define overconvergent modular forms with non-zero weight, so we recall the definition of Eisenstein series.

Notation 13. *We define τ to be the nontrivial Dirichlet character of conductor 4 and we define χ to be the nontrivial even Dirichlet character of conductor 8. We will denote an arbitrary primitive Dirichlet character by θ.*

Let N be a positive integer, let $\theta : (\mathbf{Z}/N\mathbf{Z})^\times \to \mathbf{C}^\times$ be a primitive Dirichlet character, and let k be an integer such that $\theta(-1) = (-1)^k$. Recall from Washington [22, page 30] that the *extended Bernoulli numbers* $B_{k,\theta}$ are defined by

$$\sum_{a=1}^{N} \frac{\theta(a) \cdot t \cdot \exp(at)}{\exp(Nt) - 1} = \sum_{k=0}^{\infty} B_{k,\theta} \cdot \frac{t^i}{i!}$$

We define the normalised Eisenstein series $E^*_{k,\theta}$ to be the modular form with q-expansion at infinity given by

$$E^*_{k,\theta}(q) := \frac{-B_{k,\theta}}{2k} + \sum_{n=1}^{\infty} \left(\sum_{\substack{0<d|n \\ (d,p)=1}} \theta(d) \cdot d^{k-1} \right) \cdot q^n,$$

where $B_{k,\theta}$ is the extended Bernoulli number attached to k and θ. This notation follows that defined in Emerton [12], Proposition 3.11.

There is an operator V on the space of modular forms $S_k(\Gamma_0(N), \theta)$, which injects into $S_k(\Gamma_0(2N), \theta)$; its effect on q-expansions is to send q to q^2. We define $V^*_{k,\theta}(q)$ to be $V(E^*_{k,\theta}(q))$.

Definition 14. *Let w be a real number such that $0 < w < \min(2^{2-m}/3, 1/4)$. Let k be an integer and let θ be a character such that $\theta(-1) = (-1)^k$, and let K be a complete subfield of $\mathbf{C}_2$. Let $E^*_{k,\theta}$ be the Eisenstein series of weight k and character θ. Let $\mathcal{M}_0(2^m, w; K)$ be the vector space of q-expansions of overconvergent 2-adic modular forms of weight 0; this is a subspace of $K[[q]]$.*

The space of overconvergent 2-adic modular forms of weight k and character θ is given by

$$\mathcal{M}_{k,\theta}(2^m, w; K) := E^*_{k,\theta} \cdot \mathcal{M}_0(2^m, w; K).$$

This is a Banach space over K.

Let p be an odd prime. There are continuous Hecke operators T_p which act on $\mathcal{M}_{k,\theta}(2^m, w; K)$; the operator U or U_2, which is defined on q-expansions as

$$U_2\left(\sum_{n=0}^{\infty} a_n q^n\right) = \sum_{n=0}^{\infty} a_{2n} q^n \tag{1}$$

is also *compact* and therefore has a spectral theory[1].

We note that the Eisenstein series $E^*_{k,\theta}$ is an eigenfunction of U_2, with eigenvalue 1. We also note that $U_2(V(E^*_{k,\theta})) = E^*_{k,\theta}$, because the action on q-expansions is to send q to q^2 and then back to q. However, $V(U_2(E^*_{k,\theta})) = V^*_{k,\theta}$, because $U_2(E^*_{k,\theta}) = E^*_{k,\theta}$; this means that the order in which V and U_2 are applied matters.

As a consequence of results of Coleman, we have the following theorem:

Theorem 15 (Coleman [9], Theorem B3.2)**.** *Let w be a real number such that $0 < w < \min(2^{2-m}/3, 1/4)$, let k be an integer and let θ be a character such that $\theta(-1) = (-1)^k$.*

The characteristic polynomial of U_2 acting on overconvergent 2-adic modular forms of weight k and character θ is independent of the choice of w.

This theorem allows us to choose a convenient value of w and prove results for that w, and guarantees that these results will hold for any w.

The connected component in Definition 11 is hard to work with. We will therefore rewrite it in terms of modular functions of level greater than 1, to prove the following theorem:

Theorem 16. *Let $N = 4$ or 8, and define w_0 to be $N/3$. The space of w_0-overconvergent modular forms of weight 0 and level N, with coefficients in $\mathbf{Q}_2(2^{4/N})$, is a Tate algebra in one variable over $\mathbf{Q}_2(2^{4/N})$.*

Proof. We have given a valuation on the points t of the rigid space $X_0(2^m)$, based on the lifting of the Eisenstein series E_4. We recall that the modular j-invariant is defined to be $j := E_4^3/\Delta$. Therefore, we see that, if the elliptic curve corresponding to t has good reduction, then $\Delta(t)$ has valuation 0, and therefore that

$$v_2(t) = \frac{1}{4} v_2(E_4(t)) = \frac{1}{12} v_2((E_4)(t)^3) = \frac{1}{12} v_2(j(t)).$$

[1] The reference [19] calls these operators "completely continuous."

From Lemma 2.3 of Emerton [12], we see that there is a modular function j_8 which is a uniformiser on $X_0(8)$. It has q-expansion at ∞

$$j_8 = \frac{1}{q\prod_{n=1}^{\infty}(1+q^n)^4(1+q^{2n})^2(1+q^{4n})^4} = \left(\frac{\Delta(q)^2\Delta(q^4)}{\Delta(q^2)\Delta(q^8)^2}\right)^{1/12}.$$

Also, $j_8(\infty)=\infty$.

There is another modular function j_{16} which is a uniformiser on $X_0(16)$, with q-expansion at ∞ given by

$$j_{16} = \frac{1}{q\prod_{n=1}^{\infty}(1+q^n)^2(1+q^{2n})(1+q^{4n})(1+q^{8n})^2} = \left(\frac{\Delta(q^8)\Delta(q)^2}{\Delta(q^{16})^2\Delta(q^2)}\right)^{1/24}.$$

We see also that $j_{16}(\infty)=\infty$.

By an explicit calculation of q-expansions, using the formulae in Chapter 2 of [12], we see that

$$j = \frac{(j_8^4+256j_8^3+5120j_8^2+32768j_8+65536)^3}{j_8^8\cdot(j_8^2+16j_8+64)\cdot(j_8+4)}$$

and

$$\frac{1}{j_8} = \frac{1}{j_{16}} + \frac{2}{j_{16}^2}.$$

Because we know that $j_8(\infty)=\infty$, the connected component of $Z_0(8)$ which contains ∞ is of the form $v_2(j_8)<D$ for some rational number D. We see that, if $v_2(j_8)<2$, then $v_2(j_8)=v_2(j)$. This means that we have shown that

$$Z_0(8)(w) = \{t\in X_0(8):\ v_2(j_8(t))\le 12w\}\ \text{ for } 0<w<1/6.$$

Similarly, we see that the connected component of $Z_0(16)$ which contains ∞ is of the form $v_2(j_8)<D$ for some rational number D. We see that, if $v_2(j_{16})<1$, then $v_2(j_{16})=v_2(j_8)$, and therefore that $v_2(j_{16})=v_2(j)$. This means that we have shown that

$$Z_0(16)(w) = \{t\in X_0(8):\ v_2(j_{16}(t))\le 12w\}\ \text{ for } 0<w<1/12.$$

We now define another modular function on $X_0(2^m)$, in terms of Eisenstein series. Define modular functions on $X_0(8)$ and $X_0(16)$ by

$$z_4 := \frac{E^*_{1,\tau}/V^*_{1,\tau}-1}{2} = \frac{2}{j_8+2} \tag{2}$$

and

$$z_8 := \frac{E^*_{1,\chi\tau}/V^*_{1,\chi\tau}-1}{\sqrt{2}} = \frac{\sqrt{2}}{j_{16}+2}, \tag{3}$$

where we choose and fix a square root of 2 in $\mathbf{C}_2$.

We also have the identity

$$z_4 = \frac{\sqrt{2}z_8}{1+2z_8^2}. \tag{4}$$

These identities can all be verified by explicit calculation. Let w be a rational number such that $0 < w < 1/6$. Then using the formulae above, we see that

$$Z_0(8)(w) = \{t \in X_0(8) : \ v(z_4(t)) \geq 1 - 12w\} .$$

We now choose $w = 1/12$, to obtain

$$Z_0(8)(1/12) = \{t \in X_0(8) : \ v(z_4(t)) \geq 0\} .$$

Now, the rigid functions on the closed disc over $\mathbf{Q}_2$ with centre 0 and radius 1 are defined to be power series of the form

$$\sum_{n \in \mathbf{N}} a_n z^n \ : \ a_n \in \mathbf{Q}_2, \ a_n \to 0.$$

Therefore, the 1/12-overconvergent modular forms of level $\Gamma_0(4)$ and weight 0 are

$$\mathbf{Q}_2\langle z_4 \rangle,$$

which is what we wanted to show. We now follow the same procedure for $X_0(16)$. Let w be a rational number such that $0 < w < 1/12$. Then using the formulae above, we see that

$$Z_0(16)(w) = \{t \in X_0(16) : \ v(z_8(t)) \geq 1/2 - 12w\} .$$

We now choose $w = 1/24$, to obtain

$$Z_0(16)(1/24) = \{t \in X_0(16) : \ v(z_8(t)) \geq 0\} .$$

Now, the rigid functions on the closed disc over $\mathbf{Q}_2$ with centre 0 and radius 1 are defined to be power series of the form

$$\sum_{n \in \mathbf{N}} a_n z^n \ : \ a_n \in \mathbf{Q}_2(\sqrt{2}), \ a_n \to 0.$$

Therefore, the 1/24-overconvergent modular forms of level $\Gamma_0(8)$ and weight 0 are

$$\mathbf{Q}_2(\sqrt{2})\langle z_8 \rangle,$$

so we have shown that these spaces of modular forms are Tate algebras in one variable. □

We now show that the odd powers of z_N are sent to 0 under the U_2 operator.

Lemma 17. *Let $N = 4$ or 8 and let i be a non-negative integer. Then*

$$U_2(z_N^{2i+1}) = 0, \ \text{and} \ U_2(z_N^{2i}) = \left(U_2(z_N^2)\right)^i . \tag{5}$$

Proof. For this proof, we will take θ to be a primitive Dirichlet character of conductor N, such that $\theta(-1) = (-1)^k$.

We recall from the discussion after equation (1) that the Eisenstein series $E^*_{k,\theta}$ is an eigenfunction with eigenvalue 1 for U_2, and that $U_2(V(E^*_{k,\theta})) = E^*_{k,\theta}$. Then we see that we have (for $\mu = 2$ or $\sqrt{2}$):

$$\begin{aligned} U_2(z_N) &= U_2\left(\frac{E^*_{k,\theta}/V^*_{k,\theta} - 1}{\mu}\right) = \frac{1}{\mu} \cdot U_2\left(\frac{E^*_{k,\theta} - V^*_{k,\theta}}{V^*_{k,\theta}}\right) \\ &= \frac{1}{\mu E^*_{k,\theta}} \cdot U_2(E^*_{k,\theta} - V^*_{k,\theta}) = \frac{1}{\mu E^*_{k,\theta}} \cdot (E^*_{k,\theta} - E^*_{k,\theta}) = 0. \end{aligned}$$

Hence we see that z_N has only odd power q-expansion coefficients, and that therefore z_N^2 has only even power q-expansion coefficients. Let i be a non-negative integer. Then z_N^{2i+1} has only odd power q-coefficients. Hence for all i, we see that

$$U_2\big(z_N^{2i+1}\big) = 0.$$

Because we have just shown that z_N has only odd power q-coefficients, we see that

$$z_N = qF(q^2) = qV(F(q)),$$

for some power series $F(q)$. Therefore we have

$$U_2(z_N^{2i}) = U_2(q^{2i}V(F(q)^{2i})) = U_2(V(q^i F(q)^{2i})),$$

and hence we see that

$$U_2(z_N^{2i}) = q^i F(q)^{2i} = \big(qF(q)^2\big)^i = U_2(z_N^2)^i,$$

which proves the Lemma. □

We have written down the overconvergent modular forms as an explicit Banach space. This means that we can write down its *Banach basis*: the set $\{z_4, z_4^2, z_4^3, \dots\}$ forms a Banach basis for the overconvergent modular forms of weight 0 and level $\Gamma_0(4)$ and the functions $\{z_8, z_8^2, z_8^3, \dots\}$ form a Banach basis for the overconvergent modular forms of weight 0 and level $\Gamma_0(8)$. These Banach bases are composed of weight 0 modular functions — we want to be able to consider the action of the U_2 operator on overconvergent modular forms with non-zero "weight-character" (k, θ) (here, as in the Lemma, θ has conductor 4 or 8 and $\theta(-1) = (-1)^k$). Using an observation from the work of Coleman [9], we will be able to move between weight-character $(0, 1)$ and weight-character (k, θ) via multiplication by a suitable quotient of modular forms.

Let F be an overconvergent modular form of weight-character (k, θ) which has nonzero constant term, and let z be an overconvergent modular function of weight 0. In particular, we note that F may have negative weight. From the discussion in Coleman [9, page 450] we see that the pullback $\overline{U}_2$ of the U_2 operator acting on overconvergent modular forms of weight-character (k, θ) to weight 0 is $1/F \cdot U_2(z \cdot F)$.

Now by equation 3.3 of [10] we have that $U_2(z \cdot V(F)) = U_2(z) \cdot F$. We therefore consider the modular form $H = V(G)$, and substitute H for F in the formula we have just derived for $U_2(z \cdot V(F))$, to obtain:

$$\overline{U}_2(z \cdot V(G)) = \frac{1}{V(G)} \cdot U_2(z \cdot V(G)) = \frac{G}{V(G)} \cdot U_2(z).$$

We now let k be an integer, and s be either 0 or 1.

We now choose $G = (E^*_{1,\chi\tau})^k \cdot (E^*_{1,\tau})^s$, so the characteristic power series of U_2 acting on overconvergent modular forms of weight-character $(k+s, \chi^k \cdot \tau^{k+s})$ is the same as the composition of multiplication by $(E^*_{1,\chi\tau}/V^*_{1,\chi\tau})^k \cdot (E^*_{1,\tau}/V^*_{1,\tau})^s$ with U_2 acting on overconvergent modular forms of weight-character 0. We record this as the following Definition.

Definition 18 (The twisted U_2 operator)**.** *Let k be an integer, and s be either 0 or 1.*

The twisted U_2 operator acting on forms of weight-character $(k+s, \chi^k \cdot \tau^{k+s})$ is defined to be the composition of multiplication by $(E^*_{1,\chi\tau}/V^*_{1,\chi\tau})^k \cdot (E^*_{1,\tau}/V^*_{1,\tau})^s$ with U_2 acting on overconvergent modular forms of weight-character 0.

We can consider the action of this twisted U_2 operator on these spaces of overconvergent modular forms.

Definition 19 (The matrix of the twisted U_2 operator)**.** *Let k be an integer and let s be 0 or 1.*

Let $M = (m_{i,j})$ be the infinite compact matrix of the twisted U_2 operator acting on overconvergent modular forms of weight-character $(k+s, \chi^k \cdot \tau^{k+s})$, where $m_{i,j}$ is defined to be the coefficient of z_N^i in the z_N-expansion of

$$U_2(z_N^j) \cdot (E^*_{1,\chi\tau}/V^*_{1,\chi\tau})^k \cdot (E^*_{1,\tau}/V^*_{1,\tau})^s.$$

We note that the entries of M are functions of $k, s, \tau,$ and χ.

We know that U_2 is a compact operator, so we can show that the trace, determinant and characteristic power series of M are all well-defined. We will use a theorem of Serre to prove our theorem on the slopes of U_2 acting on M.

Theorem 20 (Serre [19], Proposition 7)**.**

1. *Let M_n be an $n \times n$ matrix defined over a finite extension of $\mathbf{Q}_2$. Let $det(1 - tM_n) = \sum_{i=0}^n c_i t^i$. Let M_m be the matrix formed by the first m rows and columns of M_n.*

   Assume that there exists a constant $r \in \mathbf{Q}^\times$ such that

   (a) *For all positive integers m such that $1 \le m \le n$, the valuation of $det(M_m)$ is $r \cdot \frac{m(m+1)}{2}$.*

   (b) *The valuation of elements in column j is at least $r \cdot j$.*

   Then we have that, for all positive integers m such that $1 \le m \le n$, $v_2(c_m) = r \cdot \frac{m(m+1)}{2}$.
2. *Let M_∞ be a compact infinite matrix (that is, the matrix of a compact operator). If M_m is a series of finite matrices which tend to M_∞, then the finite characteristic power series $det(1 - tM_m)$ converge coefficientwise to $det(1 - tM_\infty)$, as $m \to \infty$.*

We now quote a result of Coleman that tells us that overconvergent modular forms of small slope are in fact classical modular forms:

Theorem 21 (Coleman [8], Theorem 1.1)**.** *Let k be a non-negative integer. Every 2-adic overconvergent modular eigenform of weight k with slope strictly less than $k-1$ is a classical modular form.*

We will now state the theorem on the slopes of overconvergent modular forms of weight-character $(2k-1, \tau)$ and weight-character $(k, \chi \cdot \tau^k)$. This, combined with Theorem 21 and a knowledge of the dimensions of spaces of classical cusp forms, will suffice to prove Theorem 2 and Corollary 3.

Theorem 22. *Let k be an integer, let τ be the primitive Dirichlet character of conductor 4 and let χ be the even primitive Dirichlet character of conductor 8.*

1. *The slopes of overconvergent 2-adic cuspidal eigenforms of weight $2k-1$ and character τ are $\{2, 4, 6, \ldots, 2n, \ldots\}$.*
2. *The slopes of overconvergent 2-adic cuspidal eigenforms of weight k and character $\chi \cdot \tau^k$ are $\{1, 2, 3, \ldots, n, \ldots\}$.*

We now recall without proof a theorem of Cohen and Oesterlé:

Theorem 23 (Cohen-Oesterlé [7], Théorème 1)**.** *Let θ be a primitive Dirichlet character of conductor $2^m > 2$, and let k be a positive integer. We assume that $\theta(-1) = (-1)^k$.*

The dimension of the space of cuspidal modular forms of weight-character (k, θ) is

$$2^{m-3}(k-1) - 1.$$

Proof of Theorem 2. We see that, for $m = 2$ or 3, the slopes of the first $2^{m-3}(k-1) - 1$ overconvergent modular forms of level $\Gamma_0(2^m)$ and primitive Dirichlet character of conductor 2^m are

$$2^{3-m}, 2^{4-m}, \ldots, k-1-2^{3-m}.$$

Using Theorem 21, we see that all of these slopes are classical, because $k-1-2^{3-m} < k-1$. □

We will now recall a fact from Ribet [18], page 21, to prove Corollary 3.

Proof of Corollary 3. Let f be a (nonzero) normalised classical modular eigenform of integer weight k, level $\Gamma_0(4)$ or $\Gamma_0(8)$ with (primitive Dirichlet) character θ of conductor 4 or 8 respectively. Let σ be an element of $\mathrm{Gal}(\overline{\mathbf{Q}_2}/\mathbf{Q}_2)$. Then we have that

$$\sigma(f) := \sum_{n=1}^{\infty} \sigma(a_n) q^n$$

is a classical cuspidal modular eigenform of weight-character (k, θ), because θ takes values in $\mathbf{Q}_2$, and hence is invariant under conjugation.

We see that the valuation of $\sigma(a_2)$ is the same as that of a_2, because the characteristic polynomial of a_2 is stable under conjugation by σ. Therefore, $\sigma(f)$ is an eigenform of weight-character (k, θ) with the same slope as f. Hence $\sigma(f) = f$, because there is only one classical eigenform of weight-character (k, θ) which has any given slope, by Theorem 21 and Theorem 22. This means that $\sigma(f) = f$ for all σ. Therefore $a_n \in \mathbf{Q}_2$ for all positive integers n. □

4. Proving that Theorem 20 can be applied to the matrix M of the U_2 operator

Proof of Theorem 22. For this section, θ will be a primitive Dirichlet character of conductor $N = 4$ or 8. We will define k to be either an odd integer, if $N = 4$, or an arbitrary integer if $N = 8$. We define s to be 1 if k is even, and 0 if k is odd. We recall that we have defined τ to be the primitive odd Dirichlet character of conductor 4, and χ to be the primitive even Dirichlet character of conductor 8.

We would like to apply Theorem 20 to prove Theorem 22 on the slopes of the U_2 operator on forms of weight-character (k, θ). The matrix M which we defined earlier (Definition 19) is the matrix of the twisted U_2 operator of weight-character (k, θ), which we defined in Definition 18.

As the calculations which will show this are quite complicated, we will give a plan for the proof here.

Plan for the proof of Theorem 22. In this section, we will show that we can apply Theorem 20, which will prove Theorem 22. First we fix an arbitrary positive integer n, an integer k and a primitive Dirichlet character θ of conductor N such that $\theta(-1) = (-1)^k$.

We will begin with the matrix M_{2n}; the matrix formed by the first $2n$ rows and $2n$ columns of M, the matrix of the twisted U_2 operator acting on forms of weight-character (k, θ) defined in Definition 18. The proof will then proceed in the following way:

1. *Define the matrix* $(O_n)_{i,j} := (M_{2n})_{2i,2j}$*; this matrix consists of the elements of* M_{2n} *in even numbered columns and even-numbered rows.*
2. *Let* α *be 2 if* $N = 4$*, and a square root* $\sqrt{2}$ *of 2 if* $N = 8$*. Define the matrix* $D(\beta)$ *to be the diagonal matrix with* β^i *in the* i*th row and the* i*th column. We define the matrix* $O'_n := D(\alpha^{-1}) \cdot O_n \cdot D(\alpha)$*.*
3. *We then show that the valuation of elements in the* i*th column of* O'_n *are* $8i/N$*; this verifies condition* (b) *of Theorem 20, with* $r = 8/N$*.*
4. *We finally show that* O'_n *has determinant of valuation* $8/N \cdot n(n+1)/2$*, by considering the matrix* $P_n := D(\alpha^{-2}) \cdot O'_n$*. By showing that* P_n *has determinant of valuation 0, it can be seen that the valuation of the determinant of* O'_n *is the valuation of the determinant of* $D(\alpha^2)$*, which is* $8/N \cdot n(n+1)/2$*. This will verify condition* (a) *of Theorem 20, again with* $r = 8/N$*.*

At each step of this plan, we must show that the characteristic polynomial of the new matrix defined is the same as that of M_n. In the last step, we will show that P_n has unit determinant by reducing it modulo a prime ideal above 2 and showing that this reduction has determinant 1. This means that we must prove that P_n has coefficients which are in the ring $\mathbf{Z}_2[\alpha]$.

4.1. Defining O_n. We first note that the odd-numbered columns of the matrix M are identically zero, because we have shown that $U_2(z_N^{2a+1}) = 0$, where a is a non-negative integer, and the columns of M are defined to be the coefficients of z_N in the z_N power series expression of

$$U_2(z_4^{2a+1}) \cdot (E^*_{1,\tau}/V^*_{1,\tau})^k \text{ or } U_2(z_8^{2a+1}) \cdot (E^*_{1,\chi\tau}/V^*_{1,\chi\tau})^k \tag{6}$$

if $N = 4$ or $N = 8$ and k is odd, or

$$U_2(z_N^{2a+1}) \cdot (E^*_{1,\chi\tau}/V^*_{1,\chi\tau})^{k-1} \cdot E^*_{1,\tau}/V^*_{1,\tau} \tag{7}$$

if $N = 8$ and k is even. Therefore, we will fix a positive integer n and define another matrix which has the same characteristic power series as the finite matrix M_n. We will then apply Theorem 20 to this new matrix.

In equation (7), we choose this particular multiplier when k is even and $N = 8$ so that the modular form $(E^*_{1,\chi\tau})^{k-1} E^*_{1,\tau}$ has character of conductor exactly 8. This would not be the case if we used $(E^*_{1,\chi\tau})^k$. We also note that

$$E^*_{1,\tau}/V^*_{1,\tau} = 1 + \frac{2\sqrt{2}z_8}{1+2z_8^2};$$

this follows from identity (4). Using this identity, we can compute the z_8-expansion of the product in equation (7).

We consider the matrix O_n, defined by

$$(O_n)_{i,j} := (M_{2n})_{2i,2j}\,, \text{ where } 1 \le i,j \le n.$$

This has the same characteristic power series as M_{2n} because M_{2n} only has entries on the even-numbered columns; we consider the finite characteristic power series of O_n because we will apply the theorem of Serre (Theorem 20) to use information about the finite truncations M_{2n} to prove results about the infinite characteristic power series of M. We now show that the matrices O_n have determinant of valuation $8/N \cdot n(n+1)/2$, in order to be able to use Theorem 20.

4.2. Defining O'_n. To do this we will pre- and post-multiply the matrix O_n by diagonal matrices to obtain a matrix O' which has elements of valuation at least $8i/N$ in column i. Let $D(\alpha)$ be the diagonal matrix with α^i in the i^{th} row and i^{th} column. We let α be 2 if $N = 4$ and a square root of 2 if $N = 8$, and we define

$$O'_n = D(\alpha^{-1}) \cdot O_n \cdot D(\alpha).$$

We see by the definition of characteristic power series that the characteristic power series of O_n and of O'_n are the same.

4.3. Checking the valuation of elements in the ith column. We will now show that the valuation of the elements in the i^{th} column of O' is at least $8i/N$, by considering the power series that gives the i^{th} column. There are identities of modular functions

$$U_2(z_4^2) = \frac{2z_4}{(1+2z_4)^2} \quad \text{and} \quad U_2(z_8^2) = \frac{\sqrt{2}z_8}{1+2z_8^2},$$

where we have chosen and fixed a square root of 2 in $\mathbf{Q}_2(\sqrt{2})$.

The i^{th} element of the $(2j)^{th}$ column of the matrix M of the twisted U_2 operator on overconvergent modular forms of weight-character $(k+s, \chi^{k+s}\tau^s)$ is given by the coefficient of z_4^i in

$$f(z_4) := U_2(z_4^{2j}) \cdot (E^*_{1,\tau}/V^*_{1,\tau})^k = \left(\frac{2z_4}{(1+2z_4)^2}\right)^j \cdot (1+2z_4)^k \text{ if } N=4$$

—this follows from Lemma 17 and equation (6) — or the coefficient of z_8 in

$$\begin{aligned} g(z_8) \quad &:= \quad U_2(z_8^{2j}) \cdot (E^*_{1,\chi\tau}/V^*_{1,\chi\tau})^{k-s} \cdot (E^*_{1,\tau}/V^*_{1,\tau})^s \\ &= \quad \left(\frac{\sqrt{2}z_8}{1+2z_8^2}\right)^j \cdot (1+\sqrt{2}z_8)^{k-s} \cdot \left(1+\frac{2\sqrt{2}z_8}{1+2z_8^2}\right)^s \text{ if } N=8; \end{aligned}$$

— this follows from Lemma 17 and equation (7).

The definition of the matrix O_n when $N = 4$ (the i,jth entry of O_n is the $2i, 2j$th entry of M_{2n}) means that the i,jth entry of O_n is given by the coefficient of z_4^{2i} in $f(z_4)$.

The i,jth entry of O_n is also given by the coefficient of z_4^{2i} in $1/2(f(z_4) + f(-z_4))$; we note that this sum has only even powers of z_4 appearing in it. This sum is

$$\frac{f(z_4)+f(-z_4)}{2} = \frac{1}{2}(2z_4)^j \cdot \left(\frac{(1+2z_4)^k}{(1+2z_4)^{2j}} - \frac{(1-2z_4)^k}{(1-2z_4)^{2j}}\right)$$

Similarly, the definition of the matrix O_n when $N = 8$ means that the i,jth entry of O_n is given by the coefficient of z_8^{2i} in $g(z_8)$.

The i,jth entry of O_n is also given by the coefficient of z_8^{2i} in $1/2(g(z_8) + g(-z_8))$; again, we note that this sum has only even powers of z_8 appearing in it. This sum is given by

$$\frac{(\sqrt{2}z_8)^j}{2}\left(\frac{(1+\sqrt{2}z_8)^{k-s}}{(1+2z_8^2)^j}\cdot\left(1+\frac{2\sqrt{2}z_8}{1+2z_8^2}\right)^s - \frac{(1-\sqrt{2}z_8)^{k-s}}{(1+2z_8^2)^j}\cdot\left(1-\frac{2\sqrt{2}z_8}{1+2z_8^2}\right)^s\right).$$

The effect of the conjugation of O_n by the matrices $D(\alpha)$ and $D(\alpha^{-1})$ is that the i^{th} element of the j^{th} column of the matrix $O'_n = D(\alpha^{-1}) \cdot O_n \cdot D(\alpha)$ is given by the coefficient of z_N^{2i} in either

$$4^j \cdot \left(\frac{z_4}{(1+z_4)^2}\right)^j \cdot (1+z_4)^k \tag{8}$$

or

$$\frac{(4z_4)^j}{2} \cdot \left(\frac{(1+z_4)^k}{(1+z_4)^{2i}} - \frac{(1-z_4)^k}{(1-z_4)^{2i}} \right) \tag{9}$$

if $N = 4$ or the coefficient of z_N^{2i} in either

$$2^j \cdot \left(\frac{z_8}{1+z_8^2} \right)^j \cdot (1+z_8)^{k-s} \cdot \left(1 + \frac{2z_8}{1+z_8^2} \right)^s \tag{10}$$

or

$$\frac{(2z_8)^j}{2} \left(\frac{(1+z_8)^{k-s}}{(1+z_8^2)^i} \cdot \left(1 + \frac{2z_8}{1+z_8^2} \right)^s - \frac{(1-z_8)^{k-s}}{(1+z_8^2)^i} \cdot \left(1 - \frac{2z_8}{1+z_8^2} \right)^s \right), \tag{11}$$

if $N = 8$.

We see from the formulae (8) and (10) that all of the elements of the i^{th} column of O'_n have valuation at least $8i/N$, because there is a multiplier of the correct valuation in front of the rational function of z_N. Therefore we have shown that assumption (b) of Theorem 20 is satisfied.

We will show that the matrix O'_n has determinant with valuation $8/N \cdot n(n+1)/2$ — assumption (a) of Theorem 20 — by showing that it is the product of two matrices, one of which is the diagonal matrix $D(\alpha^2)$, and one of which is the matrix P_n defined in the Plan. We now define P_n and show that it has unit determinant.

4.4. Defining P_n. We define the matrix P_n to be $D(\alpha^2)^{-1} \cdot O'_n$. The entries of P_n are given by $P_{i,j} = \alpha^{-2i} \cdot O'_{i,j}$ and are therefore elements of the ring of integers of $\mathbf{Q}_2(\sqrt{2})$, because the valuation of elements in the i^{th} column of O'_n is at least i. Therefore, we can define the matrix P' by reducing the entries of P_n modulo a prime above 2; if P'_n has determinant 1 in $\mathbf{F}_2$ then O' has determinant of valuation $8/N \cdot n(n+1)/2$.

In fact, the elements of P_n can be obtained easily from the elements of M_n. Let α be 2 if $N = 4$ and a square root of 2 if $N = 8$; the formula for an element of P_n in terms of an element of M_{2n} is

$$(P_n)_{i,j} = \alpha^{-2i} \cdot (M_{2n})_{2i,2j}.$$

We see from the formulae (9) and (11) for the columns of O'_n that the i^{th} element of the $(2l+1)^{th}$ column of the mod 2 matrix P_n in weight k is given by the coefficient of x in

$$c_{2l+1} = \frac{x^{l+1} y^{2l+1} \cdot (1+x)^{(k-1-s)/2}}{(1+x)^{2l+1}},$$

and the i^{th} element of the $(2l)^{th}$ column of the matrix P_n in weight k is given by the coefficient of x in

$$c_{2l} = \frac{x^l y^{2l} \cdot (1+x)^{(k-1-s)/2}}{(1+x)^{2l}}.$$

We obtain these formulae by considering the formulae given before for the columns of the matrix O'_n, and noticing that, if a is a non-negative integer, we can write $(1+z_N)^{2a+1} = (1+z_N)\cdot(1+z_N^2)^a$ in characteristic 2. This allows us to work out what the even-indexed entries of the matrix M_{2n} are. Notice also that the part of $g(z_8)$ which is raised to the power s reduces to 1 modulo 2, as it has numerator $2z_8$. This is why there is a $-s$ in the power of $(1+x)$ in the numerator.

4.5. Showing that P_n has unit determinant. I would like to thank Robin Chapman [6] for the idea behind the following proof. To show that the $n \times n$ mod 2 matrix has determinant 1, we will show that the elements $C := \left\{c'_1 = c_1 \cdot (1+x)^N, \ldots, c'_N = c_N \cdot (1+x)^N\right\}$ are a basis of the $\mathbf{F}_2$-algebra $\mathbf{F}_2[x]/(x^{N+1})$. Because $(1+x)$ is a unit in the ring, we may multiply each c_i by $(1+x)^{N-i}$ to make the calculations easier. We see that there are exactly the right number of elements, so we must check that they are linearly independent.

We write $\sum_{i=1}^{N} \lambda_i c'_i = 0$. We will show that all of the $\lambda_i = 0$, so that the columns are linearly independent. We see that the element $c'_1 = x(1+x)^{N-1}$ is the only element of the set C which has an x^N term. Therefore $\lambda_1 = 0$. Also, we see that $c'_2 = x(1+x)^{N-2}$ is now the only nonzero term with an x term, so therefore $\lambda_2 = 0$.

By continuing this process, we show that each λ_i must be zero. Therefore the set C is composed of linearly independent elements and hence it is a basis. So the determinant of the mod 2 matrix is 1. Hence we have proved assumption (a) of Theorem 20 and therefore Theorem 22. □

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