Mathematical Research Letters **11**, 589–597 (2004)

UNIQUENESS OF JOSEPH IDEAL

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1. Introduction

Let $\mathfrak{g}$ be a simple complex Lie algebra. By the PBW theorem, the universal enveloping algebra $U(\mathfrak{g})$ has a filtration $U_{n-1}(\mathfrak{g}) \subset U_n(\mathfrak{g})$ so that $U_n(\mathfrak{g})/U_{n-1}(\mathfrak{g})$ is naturally isomorphic to the n-th symmetric power $S^n(\mathfrak{g})$ of $\mathfrak{g}$. For any two-sided ideal $\mathcal{J}$ of $U(\mathfrak{g})$, we can associate an ideal $J = gr(\mathcal{J})$ in the symmetric algebra $S(\mathfrak{g}) = \oplus_{n=0}^{\infty} S^n(\mathfrak{g})$ defined by

$$J = \oplus_{n=1}^{\infty} \mathcal{J}_n / \mathcal{J}_{n-1}$$

where $\mathcal{J}_n = \mathcal{J} \cap U_n(\mathfrak{g})$. The zero set defined by J in $\mathfrak{g} \cong \mathfrak{g}^*$ is the associated variety of $\mathcal{J}$ and will be denoted by $Ass(\mathcal{J})$. If $\mathcal{J}$ is primitive, then $Ass(\mathcal{J})$ is contained in the nilpotent cone of $\mathfrak{g}$.

The Lie algebra $\mathfrak{g}$ has a unique minimal (non-trivial) nilpotent orbit $\mathbf{O}_{\min}$. If α is the highest root, and $(e_\alpha, h_\alpha, e_{-\alpha})$ is the standard $sl(2)$-triple corresponding to α, then $\mathbf{O}_{\min}$ is simply the adjoint orbit of e_α.

In an important paper [J], Joseph constructed a completely prime 2-sided primitive ideal $\mathcal{J}_0$ whose associated variety is $\bar{\mathbf{O}}_{min}$. He also derived a number of properties of $\mathcal{J}_0$; for example he computed its infinitesimal character. We shall refer to $\mathcal{J}_0$ as the Joseph ideal. Joseph also proved that, when $\mathfrak{g}$ is not of type A, $\mathcal{J}_0$ is the unique completely prime two-sided ideal whose associated variety is the closure of the minimal orbit. However, it was noticed by the second author [S] that there is a gap in Joseph's proof, namely in the proof of [J, Lemma 8.8] (see the remark at the end of §2). This is somewhat undesirable, given the subsequent importance of the Joseph ideal in representation theory, especially in the theory of minimal representations.

In this paper, we provide a simple proof of the uniqueness of the Joseph ideal. Our proof is different from the one envisioned in Joseph's original paper [J]. It is possible that the uniqueness can be deduced from some other results in the literature. For example, Braverman and Joseph have informed us that it should be a consequence of the results of their paper [BJ]. There, they considered a one-parameter (non-commutative) deformation of the coordinate ring of $\bar{\mathbf{O}}_{min}$ and showed that there is a unique deformation which gives rise to the Joseph ideal. This gives an alternative construction of the Joseph ideal. However, to show uniqueness, one would need to show that given a completely prime ideal

Received February 18, 2004.

$\mathcal{J}$ with associated variety $\bar{\mathbf{O}}_{min}$, $U(\mathfrak{g})/\mathcal{J}$ belongs to the one-parameter family of deformations considered in [BJ]. This does not seem to be addressed in [BJ], but could perhaps be shown using Proposition 2.1 below.

2. Joseph Ideal

We assume henceforth that $\mathfrak{g}$ is not of type A and begin by describing some results of Kostant and Garfinkle [G].

Let J_0 be the prime ideal in the symmetric algebra $S(\mathfrak{g})$ corresponding to the closure of the minimal nilpotent orbit in $\mathfrak{g}^*$. Kostant has shown that, as a $\mathfrak{g}$-module,

$$S(\mathfrak{g})/J_0 = \sum_{m=0}^{\infty} V(m\alpha),$$

and that J_0 is generated by a $\mathfrak{g}$-submodule $V(0) + W$ in $S^2(\mathfrak{g})$, such that

$$S^2(\mathfrak{g}) = V(2\alpha) \oplus V(0) \oplus W.$$

The structure of W was determined by Garfinkle as follows.

Assume first that $\mathfrak{g}$ is not of type C_n. Let $\mathfrak{m}$ be the centralizer of h_α. It is a reductive subalgebra whose simple roots are precisely the simple roots of $\mathfrak{g}$ perpendicular to α. Write $[\mathfrak{m}, \mathfrak{m}] = \oplus \mathfrak{m}_i$ as a direct sum of simple summands. Let α_i be the highest root of the summand $\mathfrak{m}_i$. Then, if $\mathfrak{g}$ is not of type C_n

$$W = \oplus_i V(\alpha + \alpha_i).$$

Note that each irreducible $\mathfrak{g}$-module $V(\alpha + \alpha_i)$ can be considered, in a canonical fashion, as a submodule of $U_2(\mathfrak{g})$, as it appears with multiplicity one there.

On the other hand, if $\mathfrak{g}$ is of type C_n, then $[\mathfrak{m}, \mathfrak{m}]$ is simple of type C_{n-1}, so that there is a unique α_1 (the highest root of $[\mathfrak{m}, \mathfrak{m}]$). In this case,

$$W = V(\alpha + \alpha_1) \oplus V(\frac{1}{2}(\alpha + \alpha_1)).$$

Now we have a crucial proposition:

Proposition 2.1. *Let $\mathcal{J}$ be a completely prime ideal in $U(\mathfrak{g})$ such that $Ass(\mathcal{J}) = \bar{\mathbf{O}}_{\min}$. Then $\mathcal{J}$ contains W and $\Omega - c$, where Ω is the Caismir element of $U(\mathfrak{g})$ and c is a constant.*

Proof. If $Ass(\mathcal{J}) = \bar{\mathbf{O}}_{\min}$, then J contains a power of J_0. Since J_0^k is generated by the symmetric power $S^k(V(0) + W)$ over $S(\mathfrak{g})$, it follows that the $\mathfrak{g}$-types of J_0^k/J_0^{k+1} are contained in

$$S^k(V(0) + W) \otimes (\oplus_{m=0}^{\infty} V(m\alpha)).$$

Thus, the highest weights of $\mathfrak{g}$-types in $S(\mathfrak{g})/J$ are located on finitely many lines parallel to α. In particular, for all sufficiently large integers n, $V(n\alpha + n\alpha_i)$ does not appear as a submodule of $S(\mathfrak{g})/J$. Since $S(\mathfrak{g})/J$ is isomorphic to $U(\mathfrak{g})/\mathcal{J}$ as a $\mathfrak{g}$-module, $V(n\alpha + n\alpha_i)$ does not appear as a submodule of $U(\mathfrak{g})/\mathcal{J}$ for large n.

Thus, if a is a highest weight vector of an irreducible constituent of $W \subset U_2(\mathfrak{g})$, then $a^n = 0$ (modulo $\mathcal{J}$) for sufficiently large n. Since $\mathcal{J}$ is completely prime, a must be in $\mathcal{J}$, so that the first part holds.

Similarly, since the multiplicity of the trivial $\mathfrak{g}$-module in $U(\mathfrak{g})/\mathcal{J}$ is finite, $P(\Omega)$ is in $\mathcal{J}$ for some polynomial P. Since P can be factored into linear factors, and $\mathcal{J}$ is completely prime, it follows that P can be taken to be of degree one. □

In [J], Joseph constructed a primitive ideal $\mathcal{J}$ which is completely prime with $Ass(\mathcal{J}) = \bar{\mathbf{O}}_{min}$, and computed its infinitesimal character. Using Proposition 2.1, we can now give an alternative description of $\mathcal{J}$.

Proposition 2.2. *(Garfinkle [G]) The Joseph ideal $\mathcal{J}$ is equal to the ideal $\mathcal{J}_0$ generated by W and $\Omega - c_0$, where c_0 is the eigenvalue of Ω for the infinitesimal character that Joseph obtained.*

Proof. Proposition 2.1 implies that the Joseph ideal $\mathcal{J}$ contains W. Since $\Omega - c_0$ is in $\mathcal{J}$ as well, it follows that $\mathcal{J} \supseteq \mathcal{J}_0$ and, for their graded versions, $J \supseteq J_0$. If $J_0 \neq J$, then since J_0 is prime, the associated variety of $\mathcal{J}$ would be a proper invariant subvariety of $\bar{\mathbf{O}}_{min}$. It follows that $Ass(\mathcal{J}) = \{0\}$. This is a contradiction, and thus $J = J_0$, which implies that $\mathcal{J} = \mathcal{J}_0$. The proposition is proved. □

Remarks: Garfinkle's proof of this proposition is based on the uniqueness of the Joseph ideal. The proof of uniqueness, however, has a gap in [J, Lemma 8.8]. More precisely, using the notations of [J], several lines before the end of the proof, there is an equation

$$a^n u_n + \dots u_0 = 0.$$

Since a is in J_0, Joseph concluded that u_0 is in J_0. This is the mistake. Indeed, the above equation only holds *modulo* J. So one can only conclude that u_0 is in $J + J_0 = U(\mathfrak{g})$, unless $J_0 \supseteq J$, which is what Joseph wanted to prove with this argument.

3. Uniqueness of $\mathcal{J}_0$.

In this section we shall prove the uniqueness of $\mathcal{J}_0$ when $\mathfrak{g}$ is not of type A.

Theorem 3.1. *Let $\mathfrak{g}$ be a simple complex Lie algebra which is not of type A. Let $\mathcal{J}$ be a completely prime two-sided ideal in $U(\mathfrak{g})$ such that its associated variety is $\bar{\mathbf{O}}_{\min}$. Then $\mathcal{J}$ is the Joseph ideal $\mathcal{J}_0$.*

Proof. By Proposition 2.1, $\mathcal{J}$ contains W and $\Omega - c$ for some constant c. In view of Garfinkle's results, to show that $\mathcal{J} = \mathcal{J}_0$, we need to show that $c = c_0$, where c_0 is the eigenvalue of Ω for $\mathcal{J}_0$.

Now the value of c is determined by the intersection

$$(\mathfrak{g} + \Omega \cdot \mathfrak{g}) \cap \mathcal{J}.$$

Since $\mathcal{J}$ contains W, to show that $\mathcal{J}$ is equal to $\mathcal{J}_0$ (i.e. $c = c_0$), it suffices to show that

$$\mathfrak{g} \cdot W \cap (\mathfrak{g} + \Omega \cdot \mathfrak{g}) \neq 0.$$

Indeed, this will show that c is independent of $\mathcal{J}$. It will be more convenient to pass to the symmetric algebra $S(\mathfrak{g})$, where we need to prove that

$$\mathfrak{g} \cdot W \supset \Omega \cdot \mathfrak{g}.$$

To do so, we need to recall some results of Chevalley and Kostant [K] on the decomposition of $S(\mathfrak{g})$ with respect to the adjoint action. Every element x in $\mathfrak{g}$ defines a first order differential operator D_x on $S(\mathfrak{g}_{\mathbb{C}})$ by

$$D_x(y) = \langle x, y \rangle$$

where y is in $\mathfrak{g} = S^1(\mathfrak{g})$, and $\langle x, y \rangle$ is the Killing form. Let Z be the subring of invariant polynomials in $S(\mathfrak{g})$. Then

$$Z = \mathbb{C}[\omega_1, \dots, \omega_n]$$

where ω_i are invariant polynomials of degree $d_i + 1$, and d_i are the exponents of $\mathfrak{g}$. We shall assume that $d_1 = 1$, so that $\omega_1 = \Omega$ is the Casimir operator. Recall that $\Omega = \sum_i e_i e_i'$, where (e_i) is any basis of $\mathfrak{g}$, and (e_i') the corresponding dual basis. The following is a reinterpretation of [K]:

Proposition 3.2. *Let Z^+ be the augmentation ideal in Z. Let ω be in Z^+. It defines an element A_ω in $Hom(\mathfrak{g}, U(\mathfrak{g}))$ such that $A_\omega(x) = D_x(\omega)$. Moreover, $\omega \mapsto A_\omega$ defines an isomorphism between Z^+ and $Hom(\mathfrak{g}, S(\mathfrak{g}))$, decreasing degree by one.*

Now note the following proposition:

Proposition 3.3. *Let V be an irreducible constituent of W. Then $\mathfrak{g} \cdot V \subseteq S^3(\mathfrak{g})$ contains a submodule isomorphic to $\mathfrak{g}$.*

Proof. Let D_Ω be the differential operator corresponding to the Casimir element Ω. Invariance of Ω implies that D_Ω is a homomorphism from $S^3(\mathfrak{g})$ to $S^1(\mathfrak{g})$. It suffices to show that D_Ω is non-trivial when restricted to the subspace $\mathfrak{g} \cdot V$.

To see this, observe that for any $p \in S^2(\mathfrak{g})$ and $x \in \mathfrak{g}$,

$$D_\Omega(xp) = 2D_x(p).$$

Thus, given non-zero p, one can always find x such that $D_\Omega(xp) \neq 0$. This proves the proposition. ☐

Remarks: In fact, the multiplicity of $\mathfrak{g}$ in $\mathfrak{g}V$ is 1. To see this, note that one has inclusions:

$$\mathrm{Hom}(\mathfrak{g}V, \mathfrak{g}) \subset \mathrm{Hom}(\mathfrak{g} \otimes V, \mathfrak{g}) = \mathrm{Hom}(V, \mathfrak{g} \otimes \mathfrak{g}),$$

and the latter space has dimension 1, as shown in [BJ, Lemma 4.4].

We can now finish the proof of theorem. Checking out the exponents, Prop. 3.2 implies that

$$\dim Hom(\mathfrak{g}, S^3(\mathfrak{g})) = \begin{cases} 1, \text{ if } \mathfrak{g} \text{ is exceptional;} \\ 2 \text{ if } \mathfrak{g} \text{ has type } B_n,\ C_n\ (n \geq 2) \text{ or } D_n\ (n > 4); \\ 3 \text{ if } \mathfrak{g} \text{ has type } D_4. \end{cases}$$

Observe that this number is equal to the number of irreducible constituents of W.

It follows that for exceptional Lie algebras, $\Omega \cdot \mathfrak{g}$ is the unique summand in $S^3(\mathfrak{g})$ isomorphic to $\mathfrak{g}$. Proposition 3.3 implies that $\mathfrak{g} \cdot V(\alpha + \alpha_i)$ (now α_i is unique) contains $\Omega \cdot \mathfrak{g}$ and therefore the theorem follows for exceptional Lie algebras.

The above argument can be adapted to D_4 by making use of the group of outer automorphisms $\Sigma \cong S_3$ of D_4. The group Σ acts on $Hom(\mathfrak{g}, S^3(\mathfrak{g}))$ which is isomorphic to Z^4 by Prop. 3.2. The latter space can be easily decomposed under the action of Σ. It has a line fixed by Σ, and the two dimensional complement is the irreducible representation r of $\Sigma \cong S_3$. Clearly, the fixed line in $Hom(\mathfrak{g}, S^3(\mathfrak{g}))$ corresponds to $\Omega \cdot \mathfrak{g}$. On the other hand, since Σ permutes the three simple roots α_i, the contribution of $\sum_i \mathfrak{g} \cdot V(\alpha + \alpha_i)$ to $Hom(\mathfrak{g}, S^3(\mathfrak{g}))$ is Σ-invariant. Next, note that D_Ω induces a Σ invariant map from $Hom(\mathfrak{g}, S^3(\mathfrak{g}))$ to $Hom(\mathfrak{g}, S^1(\mathfrak{g}))$. It follows that D_Ω annihilates $r \otimes \mathfrak{g}$. On the other hand, D_Ω does not annihilate $\mathfrak{g} \cdot W$. It follows that $\Omega \cdot \mathfrak{g}$ is contained in $\mathfrak{g} \cdot W$, as desired.

Finally, we are left with the classical cases: C_n $(n \geq 2)$, B_n $(n \geq 2)$ and D_n $(n > 4)$. The argument we provide below is due to N. Wallach. Here, W has 2 irreducible constituents V_1 and V_2, and we need to show that the two copies of $\mathfrak{g}$ in $\mathfrak{g} \cdot V_1$ and $\mathfrak{g} \cdot V_2$ are different. By Prop. 3.2, we know that there are 2 invariant polynomials q_1 and q_2 of degree 4 which give rise to these two copies of $\mathfrak{g}$ in $S^3(\mathfrak{g})$. Thus, we need to show that q_1 is not a multiple of q_2.

How can one obtain the invariant q_i from V_i? Well, the representation V_i possesses an invariant symmetric bilinear form, and thus there is an invariant polynomial Q_i in $S^2(V_i)$. If $\pi_i : S^2(\mathfrak{g}) \longrightarrow V_i$ is the projection, then

$$q_i(x) = Q_i(\pi_i(x^2)) \qquad \text{for } x \in \mathfrak{g}.$$

We do not know how to show that q_i is non-zero from general considerations. However, for the cases at hand, this will be shown in the course of the argument below.

Another description of q_i is as follows. If $\{v_j\}$ is a basis of V_i, and $Q_i = \sum_{j \leq k} a_{jk} v_j v_k$ is an invariant quadratic form on V_i, then q_i is the image of Q_i under the natural map $S^2(S^2(\mathfrak{g})) \longrightarrow S^4(\mathfrak{g})$. In other words, q_i is equal to

Q_i regarded as a quartic polynomial on $\mathfrak{g}$ (by regarding the v_j's as quadratic polynomials on $\mathfrak{g}$). From this second description, it is clear that if $x \in \mathfrak{g}$, then

$$D_x(q_i) = \sum_{j \leq k} a_{jk} \cdot (D_x(v_j)v_k + v_j D_x(v_k)) \in \mathfrak{g} \cdot V_i.$$

This shows that q_i indeed gives rise to a copy of $\mathfrak{g}$ in $\mathfrak{g} \cdot V_i$ (as long as q_i is non-zero).

We are now ready to finish the proof for the classical cases. Let us begin with an observation. If V is the standard representation of a classical $\mathfrak{g}$, so that V possesses a $\mathfrak{g}$-invariant form $\langle -, - \rangle$, then $S^k(V)$ and $\wedge^k V$ inherit a non-zero $\mathfrak{g}$-invariant form:

$$\begin{cases} \langle a_1 a_2 ... a_k, b_1 b_2 .... b_k \rangle_{S^k(V)} = \sum_{\sigma \in S_k} \prod_i \langle a_i, b_{\sigma(i)} \rangle \\ \langle a_1 \wedge a_2 ... \wedge a_k, b_1 \wedge b_2 .... \wedge b_k \rangle_{\wedge^k V} = \sum_{\sigma \in S_k} sign(\sigma) \prod_i \langle a_i, b_{\sigma(i)} \rangle. \end{cases}$$

In the following, we shall use these bilinear forms on the symmetric and exterior algebra.

Now consider the orthogonal case (type B or D). We realize $\mathfrak{g}$ as the Lie algebra of $n \times n$ skew-symmetric matrices, so that $\mathfrak{g} \cong \wedge^2 \mathbb{C}^n$. Let us write down the invariant polynomial q_1. The map

$$\pi_1'(a \otimes b) = \frac{1}{2}(ab + ba), \qquad a,\, b \in \mathfrak{g}$$

defines an equivariant map from $S^2(\mathfrak{g})$ onto the space $S^2(\mathbb{C}^n)$ of $n \times n$ symmetric matrices. Since $S^2(\mathbb{C}^n) \cong \mathbb{C} \oplus V_1$, the projection π_1 is given by:

$$\pi_1(a \otimes b) = \pi_1'(a \otimes b) - \frac{1}{n} Tr(\pi_1'(a \otimes b)) \cdot I_n$$

where I_n is the identity matrix and Tr is the trace map. Since the degree 2 invariant on V_1 is simply the map $X \mapsto Tr(X^2)$, we get:

$$q_1(a) = Tr\left((\pi_1(a \otimes a))^2\right).$$

In particular, if $a_0 = e_1 \wedge e_2 \in \wedge^2 \mathbb{C}^n$, a simple computation shows that

$$q_1(a_0) = \frac{2}{n} \cdot (n-2) \neq 0.$$

On the other hand, the representation V_2 is isomorphic to $\wedge^4 \mathbb{C}^n$ (this is irreducible if $n = 5, 7$ or $n > 8$), and the projection $\pi_2 : S^2(\mathfrak{g}) \longrightarrow \wedge^4 \mathbb{C}^n$ is given by:

$$\pi_2(a \otimes b) = a \wedge b \qquad \text{for } a,\, b \in \mathfrak{g}.$$

Since $\pi_2(a_0 \otimes a_0) = 0$, we deduce that

$$q_2(a_0) = Q_2(\pi_2(a_0 \otimes a_0)) = 0.$$

Now to see that q_2 is non-zero, note that for $x \in \mathfrak{g} = \wedge^2 \mathbb{C}^n$, we have

$$q_2(x) = \langle x \wedge x, x \wedge x \rangle_{\wedge^4 \mathbb{C}^n},$$

and a short computation gives:

$$q_2(e_1 \wedge e_2 + e_3 \wedge e_4) = 1.$$

We thus conclude that q_1 and q_2 are non-zero and linearly independent, as desired.

Finally, we come to the case C_n $(n \geq 2)$, which is somewhat more complicated. Let $\{e_1, e_2, .., e_n, e_{-1}, e_{-2}, ..., e_{-n}\}$ be the standard basis for $\mathbb{C}^{2n}$, equipped with the skew-symmetric form ω defined by $\omega(e_i, e_{-j}) = \delta_{ij}$ and $\omega(e_i, e_j) = \omega(e_{-i}, e_{-j})$ $= 0$. We identify $\mathfrak{g}$ with the space $S^2(\mathbb{C}^{2n})$ of $2n \times 2n$ symmetric matrices.

Let $V_1 = V(\alpha + \alpha_1)$ and $V_2 = V(\frac{1}{2}(\alpha + \alpha_1))$. Note that $\wedge^2\mathbb{C}^{2n} = \mathbb{C}\omega \oplus V_2$ where

$$\omega = \sum_{i=1}^{n} e_i \wedge e_{-i}.$$

Thus, we have

$$S^2(\wedge^2\mathbb{C}^{2n}) \cong \mathbb{C}\omega^2 \oplus \omega \cdot V_2 \oplus S^2(V_2).$$

In particular, $S^2(\wedge^2\mathbb{C}^{2n})$ contains V_1 with multiplicity one but does not contain $V(2\alpha)$.

Now consider the equivariant map

$$\phi : S^2(\mathfrak{g}) = S^2(S^2\mathbb{C}^{2n}) \longrightarrow S^2(\wedge^2\mathbb{C}^{2n})$$

given by

$$(a \cdot b) \otimes (c \cdot d) \mapsto (a \wedge c) \cdot (b \wedge d) + (a \wedge d) \cdot (b \wedge c).$$

We claim that the image of ϕ is isomorphic to $\mathbb{C} \oplus V_1 \oplus V_2$. Indeed, V_1 is in the image of ϕ, since

$$\phi((e_1 \cdot e_2) \otimes (e_1 \cdot e_2)) = -(e_1 \wedge e_2)^2$$

is a highest weight vector of weight $\alpha + \alpha_1$. On the other hand, if

$$\pi : S^2(\wedge^2\mathbb{C}^{2n}) \longrightarrow \wedge^2\mathbb{C}^{2n}$$

is given by:

$$(a \wedge b) \cdot (c \wedge d) \mapsto \omega(a, b) \cdot (c \wedge d) + \omega(c, d) \cdot (a \wedge b),$$

then one easily checks that $\pi \circ \phi$ is a surjective equivariant map from $S^2(\mathfrak{g})$ to $\mathbb{C}\omega \oplus V_2$.

Now let

$$a = \sum_{i=1}^{n} e_i^2 + \sum_{i=1}^{n} e_{-i}^2 \in S^2(\mathbb{C}^{2n}) = \mathfrak{g}.$$

Then a short computation shows that

$$\pi \circ \phi(a^2) = 8\omega$$

Thus, if $\pi_2 : S^2(\mathfrak{g}) \to V_2$ is the projection map, then $\pi_2(a^2) = 0$ so that $q_2(a) = 0$. It is not difficult to check that $q_2 \neq 0$ by evaluating at the element $e_1 e_{-1}$ say. Indeed, one has:

$$\pi \circ \phi((e_1 e_{-1})^2) = -2 e_1 \wedge e_{-1} \in \wedge^2 \mathbb{C}^{2n}$$

and so

$$\pi_2((e_1 e_{-1})^2) = -2 e_1 \wedge e_{-1} + \frac{2}{n}\omega.$$

Then a short computation gives:

$$q_2(e_1 e_{-1}) = \frac{4}{n} \cdot (n-1) \neq 0.$$

It remains to show that $q_1(a) \neq 0$. Let $q \in S^2(\mathfrak{g})$ correspond to the $\mathfrak{g}$-invariant form $\langle -, - \rangle_{S^2(\mathbb{C}^{2n})}$, so that q is a multiple of the Killing form. We have:

$$q = \sum_{i=1}^{n} e_i^2 \cdot e_{-i}^2 + 2 \sum_{i<j} (e_i e_j) \cdot (e_{-i} e_{-j}) - \sum_{i=1}^{n} (e_i e_{-i})^2 - \sum_{i \neq j} (e_i e_{-j})(e_{-i} e_j).$$

One checks that

$$\pi \circ \phi(q) = (4n+2)\omega$$

and thus the element

$$\pi_1(a^2) = \phi(a^2) - \frac{4}{2n+1}\phi(q)$$

lies in $V_1 \subset \mathrm{Image}(\phi)$. Now if $\langle -, - \rangle$ denotes the standard invariant symmetric bilinear form on $S^2(\wedge^2 \mathbb{C}^{2n})$, then a somewhat messy computation, best suppressed here, gives:

$$q_1(a) = \langle \pi_1(a^2), \pi_1(a^2) \rangle \neq 0.$$

This shows that q_1 and q_2 are linearly independent, as desired.

Theorem 3.1 is proved completely. □

Acknowledgments

It is a pleasure to thank Nolan Wallach for his interest in this work and for his help in the proof of the classical case of the main theorem. We also thank A. Joseph and A. Braverman for bringing their paper [BJ] to our attention.

The first author is partially supported by the AMS Centennial fellowship and NSF grant DMS-0352682. The second author is partially supported by NSF grant DMS-0138604.

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