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ON THE REFINED CLASS NUMBER FORMULA FOR GLOBAL FUNCTION FIELDS

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Abstract. We investigate a conjecture of Gross regarding a congruence relation of the Stickelberger element. We consider the case when k is a global function field of characteristic p and $\mathrm{Gal}(K/k)$ is an abelian l-group where l is a prime number different from p. Under the additional assumption that k does not contain a primitive l-th root of unity and that the divisor class number of k is prime to l, we prove that the conjecture of Gross holds. This result generalizes the author's previous result on the elementary abelian case (cf. [6]).

1. Introduction

We describe the conjecture briefly, and refer the reader to [5] for details.

Let K/k be a finite abelian extension of global fields with Galois group G. Let S be a finite non-empty set of places of k which contains all archimedean places and all places ramified in K. Furthermore, let T be a finite non-empty set of places of k which is disjoint from S, such that the (S,T)-unit group $U_{S,T}$ is torsion-free. Let $n = |S| - 1$ and let $\widehat{G}$ be the group of complex characters of G.

The Stickelberger element θ_G is the unique element in $\mathbb{Z}[G]$ which satisfies

$$\chi(\theta_G) = L_{S,T}(\chi, 0)$$

for all $\chi \in \widehat{G}$, where $L_{S,T}$ is the S-truncated, T-modified Dirichlet L-function. Gross has conjectured a congruence relation which bears striking resemblance to the analytic class number formula. In order to describe the conjecture we need to introduce some further notation.

Choose an ordered basis $\{u_1, \dots, u_n\}$ of $U_{S,T}$. Pick a place $v_0 \in S$, and for each $v_i \in S \setminus \{v_0\}$, we let $f_i : k^* \to G$ denote the homomorphism induced from local artin map for v_i. We set

$$\det\nolimits_G \lambda := \det\nolimits_{1\le i,j\le n}(f_i(u_j) - 1).$$

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The conjecture of Gross states that

$$\theta_G \equiv m \cdot \det_G \lambda \pmod{I^{n+1}}. \tag{1}$$

Here I is the augmentation ideal of $\mathbb{Z}[G]$ and the integer $m = \pm h_{S,T}$ is the T-modified class number of the S-integers of k whose sign is determined by the (S,T)-version of the analytic class number formula.

Let $\mathrm{Gr}(K/k, S, T)$ denote the congruence relation (1). For the reader's convenience, we list (without proofs) some of the basic facts regarding this conjecture. Consult [1] or [8] for details.

Proposition 1. *(a) If $v \notin S \cup T$ and $S' = S \cup \{v\}$, then $\mathrm{Gr}(K/k, S, T)$ implies $\mathrm{Gr}(K/k, S', T)$.*
(b) Suppose H is a subgroup of G. The natural map $\mathbb{Z}[G] \to \mathbb{Z}[G/H]$ maps θ_G and $\det_G \lambda$ to $\theta_{G/H}$ and $\det_{G/H} \lambda$ respectively, and
$\mathrm{Gr}(K/k, S, T)$ implies $\mathrm{Gr}(K^H/k, S, T)$.
(c) If $n = 0$ then $\mathrm{Gr}(K/k, S, T)$ holds, being equivalent to the analytic class number formula.

We also note that the conjecture has been verified for numerous cases [1, 3, 4, 6, 7].

2. The Main Result

Let $G = G_0 \times G_1 \times \cdots \times G_m$, and set $\mathcal{X} = \{0, \ldots, m\}$. For each $i \in \mathcal{X}$, we have $\mathbb{Z}[G_i] \cong \mathbb{Z} \oplus I_i$ as a direct sum of abelian groups. Here I_i is the augmentation ideal of $\mathbb{Z}[G_i]$.

As G_i is a subgroup of G, $\mathbb{Z}[G_i]$ is naturally embedded in $\mathbb{Z}[G]$, and so is I_i. For each non-empty subset $\mathcal{A}$ of $\mathcal{X}$, we define $I_{\mathcal{A}} := \prod_{i \in \mathcal{A}} I_i \subset \mathbb{Z}[G]$, and we define $I_\emptyset := \mathbb{Z}$. Then we have

$$\mathbb{Z}[G] \cong \bigotimes_{i \in \mathcal{X}} \mathbb{Z}[G_i] \cong \bigotimes_{i \in \mathcal{X}} (\mathbb{Z} \oplus I_i) \cong \bigoplus_{\mathcal{A} \subset \mathcal{X}} I_{\mathcal{A}}.$$

We observe that $I_{\mathcal{A}} \cdot I_{\mathcal{B}} \subset I_{\mathcal{A} \cup \mathcal{B}}$. Therefore $\mathbb{Z}[G]$ is a graded ring with respect to the monoid of subsets of $\mathcal{X}$ with union as monoid operation. Also, we have

$$I = \bigoplus_{\emptyset \neq \mathcal{A} \subset \mathcal{X}} I_{\mathcal{A}},$$

therefore I is a homogeneous ideal of $\mathbb{Z}[G]$ and so is I^n for $n \geq 1$.

Lemma 2. *For each $i \in \mathcal{X}$, let $H_i = G_0 \times \cdots \times G_{i-1} \times G_{i+1} \times \cdots \times G_m$ and let $\phi_i : \mathbb{Z}[G] \longrightarrow \mathbb{Z}[H_i]$ be the map induced by natural projection. Pick an integer r with $0 \leq r \leq m$. If α is an element of $\mathbb{Z}[G]$ with $\phi_i(\alpha) \in I_{H_i}^{r+1}$ for $i = 0, \ldots, r$, then $\alpha \in I^{r+1}$.*

Proof. Write $\alpha = \sum_{\mathcal{A} \subset \mathcal{X}} \alpha_{\mathcal{A}}$. We need to show that $\alpha_{\mathcal{B}} \in I^{r+1}$ for all $\mathcal{B} \subset \mathcal{X}$.

If $\{0, \dots, r\} \subset \mathcal{B}$, then by the definition of $I_{\mathcal{B}}$ it follows that $\alpha_{\mathcal{B}} \in I^{r+1}$. Suppose $i \notin \mathcal{B}$ for some $0 \le i \le r$. It is straightforward to verify that

$$\phi_i : \bigoplus_{\mathcal{A} \subset \mathcal{X}} I_{\mathcal{A}} \longrightarrow \bigoplus_{\mathcal{A} \subset \mathcal{X} \setminus \{i\}} I_{\mathcal{A}}$$

is the projection onto the $\mathcal{A}$-components with $i \notin \mathcal{A}$. As $i \notin \mathcal{B}$, $\alpha_{\mathcal{B}} = \phi_i(\alpha_{\mathcal{B}})$ is the $\mathcal{B}$-component of $\phi_i(\alpha)$. Since $\phi_i(\alpha) \in I_{H_i}^{r+1}$ by assumption and $I_{H_i}^{r+1}$ is a homogeneous ideal of $\mathbb{Z}[H_i]$, $\alpha_{\mathcal{B}} \in I_{H_i}^{r+1}$ as well. If we view H_i as a subgroup of G, we have $I_{H_i} \subset I$ and hence $I_{H_i}^{r+1} \subset I^{r+1}$. Therefore $\alpha_{\mathcal{B}} \in I^{r+1}$. □

Theorem 3. *Let K/k be a finite abelian extension with Galois group $G = G_0 \times G_1 \times \cdots \times G_m$ and let $S = \{v_0, \dots, v_n\}$. Suppose that for each $0 \le i \le n$, its inertia group I_{v_i} of v_i is contained in G_i. Then* $\mathrm{Gr}(K/k, S, T)$ *holds.*

Proof. We prove the theorem by induction on n. When $n = 0$,
$\mathrm{Gr}(K/k, S, T)$ holds as noted in Proposition 1(c).

In the general case, we apply Lemma 2 to $(\theta_G - m \cdot \det_G \lambda)$. As v_i is unramified in the subextension K^{G_i}/k, the induction hypothesis together with Proposition 1(a) implies that the hypothesis of Lemma 2 is satisfied. Hence we conclude that $\mathrm{Gr}(K/k, S, T)$ holds. □

Corollary 4. *Fix a prime number l. For a global function field k, let p be its characteristic, h be its divisor class number and w be the number of roots of unity in k. If l does not divide phw, then* $\mathrm{Gr}(K/k, S, T)$ *holds whenever* $\mathrm{Gal}(K/k)$ *is an abelian l-group.*

Proof. For each positive integer $e \ge 1$, let $k_{S,e}$ be the maximal abelian extension of k unramified outside of S whose Galois group has exponent l^e. Thanks to Proposition 1 (b), we may assume $K = k_{S,e}$. Theorem 5 of the next section ensures that the hypothesis of Theorem 3 is satisfied in this case. □

3. Some Class Field Theory

In this section we use the results from class field theory, and study the structure of G using ideles. The reader may consult [2] for example.

We keep the assumptions of Corollary 4. Let $\mathbb{F}_q$ be the exact field of constants of k, and for each place v of k let $\mathbb{F}_v$ be its residue field. For each finite nonempty set S of places of k and for each integer $e \ge 1$, let $G_{S,e} := \mathrm{Gal}(k_{S,e}/k)$.

Theorem 5. $G_{S,e} \cong \prod_{v \in S} I_v \times \mathbb{Z}/l^e\mathbb{Z}$.

Proof. For each place v of k, let k_v be the completion of k at v, U_v the set of local units in k_v, and $U_v^1 \subset U_v$ the local units which are congruent to $1 \pmod{v}$. Also let $U := \prod_v U_v$ and let $U_S := \prod_{v \notin S} U_v \cdot \prod_{v \in S} U_v^1$.

There is an exact sequence

$$0 \to U/\mathbb{F}_q^* \cdot U_S \to J/k^* \cdot U_S \to J/k^* \cdot U \to 0.$$

We note that the profinite completion of $J/k^* \cdot U_S$ is $\mathrm{Gal}(k_S/k)$ where k_S is the maximal abelian extension of k unramified outside of S and tamely ramified in S. Similarly, the profinite completion of $J/k^* \cdot U$ is $\mathrm{Gal}(k_{unr}/k)$ where k_{unr} is the maximal unramified abelian extension of k.

Let J_0 be the group of ideles of k of degree 0. Then J is isomorphic to $J_0 \times \langle c \rangle$, where c is an idele of degree 1. Therefore we may rewrite the above sequence as

$$0 \to (\prod_{v \in S} \mathbb{F}_v^*)/\mathbb{F}_q^* \to (J_0/k^* \cdot U_S) \times \mathbb{Z} \to (J_0/k^* \cdot U) \times \mathbb{Z} \to 0. \tag{2}$$

Note that for each $v \in S$, the inertia group of v is the image of $\mathbb{F}_v^*$ in the first term of the sequence (2).

As we assume that the order of $J_0/k^* \cdot U$ (which is canonically isomorphic to the divisor class group of k) is not divisible by l, we have $(J_0/k^* \cdot U \times \mathbb{Z}) \otimes \mathbb{Z}/l^e\mathbb{Z} = \mathbb{Z}/l^e\mathbb{Z}$ and $\mathrm{Tor}(J_0/k^* \cdot U \times \mathbb{Z}, \mathbb{Z}/l^e\mathbb{Z}) = 0$. Hence tensoring the exact sequence with $\mathbb{Z}/l^e\mathbb{Z}$ preserves the exactness;

$$0 \to (\prod_{v \in S} \mathbb{F}_v^*/\mathbb{F}_v^{*\,l^e})/\widetilde{\mathbb{F}_q^*} \to (J_0/J_0^{l^e} \cdot k^* \cdot U_S) \times \mathbb{Z}/l^e\mathbb{Z} \to \mathbb{Z}/l^e\mathbb{Z} \to 0,$$

where $\widetilde{\mathbb{F}_q^*}$ is the image of $\mathbb{F}_q^*$ in $\prod_{v \in S} \mathbb{F}_v^*/\mathbb{F}_v^{*\,l^e}$. Therefore $G_{S,e}$, the middle term of the above exact sequence, is isomorphic to $(\prod_{v \in S} \mathbb{F}_v^*/\mathbb{F}_v^{*\,l^e})/\widetilde{\mathbb{F}_q^*} \times \mathbb{Z}/l^e\mathbb{Z}$.

As we assume that k does not contain a primitive l-th root of unity, $\widetilde{\mathbb{F}_q^*} = \{1\}$, and hence $G_{S,e} \cong \prod_{v \in S} \mathbb{F}_v^*/\mathbb{F}_v^{*\,l^e} \times \mathbb{Z}/l^e\mathbb{Z} \cong \prod_{v \in S} I_v \times \mathbb{Z}/l^e\mathbb{Z}$. □

Remark. If we assume that k contains an l-th root of unity, one can prove that $G_{S,e}$ is isomorphic to $\prod_{v \in S'} I_v \times H'$ where $S' = S \setminus \{v_0\}$ for a suitable choice of $v_0 \in S$. Therefore, one may apply Lemma 2 to θ_G to conclude that $\theta_G \in I^n$.

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