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A CHARACTERISTIC ZERO HILBERT-KUNZ CRITERION FOR SOLID CLOSURE IN DIMENSION TWO

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Abstract. Let I denote a homogeneous R_+-primary ideal in a two-dimensional normal standard-graded domain over an algebraically closed field of characteristic zero. We show that a homogeneous element f belongs to the solid closure $I^\star$ if and only if $e_{HK}(I) = e_{HK}((I,f))$, where e_{HK} denotes the Hilbert-Kunz multiplicity of an ideal, introduced here in characteristic zero in the graded dimension two case. This provides a version in characteristic zero of the well-known Hilbert-Kunz criterion for tight closure in positive characteristic.

Introduction

Let $(R, \mathfrak{m})$ denote a local Noetherian ring or an $\mathbb{N}$-graded algebra of dimension d of positive characteristic p. Let I denote an $\mathfrak{m}$-primary ideal, and set $I^{[q]} = (f^q : f \in I)$ for a prime power $q = p^e$. Then the Hilbert-Kunz function of I is given by

$$e \longmapsto \lambda(R/I^{[p^e]}),$$

where λ denotes the length. The Hilbert-Kunz multiplicity of I is defined as the limit

$$e_{HK}(I) = \lim_{e \to \infty} \lambda(R/I^{[p^e]})/p^{ed}.$$

This limit exists as a positive real number, as shown by Monsky in [9]. It is an open question whether this number is always rational.

The Hilbert-Kunz multiplicity is related to the theory of tight closure. Recall that the tight closure of an ideal I in a Noetherian ring of characteristic p is by definition the ideal

$$I^* = \{f \in R : \exists c \text{ not in any minimal prime} : cf^q \in I^{[q]} \text{ for almost all } q = p^e\}.$$

For an analytically unramified and formally equidimensional local ring R the equation $e_{HK}(I) = e_{HK}(J)$ holds if and only if $I^* = J^*$ holds true for ideals $I \subseteq J$ (see [6, Theorem 5.4]). Hence $f \in I^*$ if and only if $e_{HK}(I) = e_{HK}((I,f))$. This is the Hilbert-Kunz criterion for tight closure in positive characteristic.

The aim of this paper is to give a characteristic zero version of this relationship between Hilbert-Kunz multiplicity and tight closure for R_+-primary homogeneous ideals in a normal two-dimensional graded domain R. There are several

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notions for tight closure in characteristic zero, defined either by reduction to positive characteristic or directly. We will work with the notion of solid closure (see [5]). In dimension two, the containment in the solid closure $f \in (f_1, \ldots , f_n)^\star$ means that the open subset $D(\mathfrak{m}) \subset \operatorname{Spec} A$ is not an affine scheme, where $A = R[T_1, \ldots , T_n]/(f_1T_1 + \ldots + f_nT_n + f)$ is the so-called forcing algebra, see [1, Proposition 1.3].

The definition of the Hilbert-Kunz multiplicity in positive characteristic does not suggest at first sight an analogous notion in characteristic zero. However, a bridge is provided by the following result of [2], which gives an explicit formula for the Hilbert-Kunz multiplicity and proves its rationality in dimension two (the rationality of the Hilbert-Kunz multiplicity for the maximal ideal was also obtained independently in [10]).

Theorem 1. Let R denote a two-dimensional standard-graded normal domain over an algebraically closed field of positive characteristic, $Y = \operatorname{Proj} R$. Let $I = (f_1, \ldots , f_n)$ denote a homogeneous R_+-primary ideal generated by homogeneous elements f_i of degree $d_i, i = 1, \ldots , n$. Then the Hilbert-Kunz multiplicity of the ideal I equals

$$e_{HK}(I) = \frac{\deg(Y)}{2}(\sum_{k=1}^{t} r_k\nu_k^2 - \sum_{i=1}^{n} d_i^2)\,.$$

Here the numbers r_k and ν_k come from the strong Harder-Narasimhan filtration of the syzygy bundle $\operatorname{Syz}(f_1^q, \ldots , f_n^q)(0)$ given by the short exact sequence

$$0 \longrightarrow \operatorname{Syz}(f_1^q, \ldots , f_n^q)(0) \longrightarrow \bigoplus_{i=1}^{n} \mathcal{O}(-qd_i) \stackrel{f_1^q,\ldots,f_n^q}{\longrightarrow} \mathcal{O}_Y \longrightarrow 0\,.$$

This syzygy bundle is a locally free sheaf on the smooth projective curve $Y = \operatorname{Proj} R$, and its strong Harder-Narasimhan filtration is a filtration $\mathcal{S}_1 \subset \ldots \subset \mathcal{S}_t = \operatorname{Syz}(f_1^q, \ldots , f_n^q)(0)$ such that the quotients $\mathcal{S}_k/\mathcal{S}_{k-1}$ are strongly semistable, meaning that every Frobenius pull-back is semistable. Such a filtration exists for q big enough by a theorem of Langer, [8, Theorem 2.7]. Then we set $r_k = \operatorname{rk}(\mathcal{S}_k/\mathcal{S}_{k-1})$ and $\nu_k = -\mu(\mathcal{S}_k/\mathcal{S}_{k-1})/q \deg(Y)$, where μ denotes the slope.

To define the Hilbert-Kunz multiplicity in characteristic zero we now take the right hand side of the above formula as our model.

Definition 1. Let R denote a two-dimensional normal standard-graded K-domain over an algebraically closed field K of characteristic zero. Let $I = (f_1, \ldots , f_n)$ be a homogeneous R_+-primary ideal given by homogeneous ideal generators f_i of degree d_i. Let $\mathcal{S}_1 \subset \ldots \subset \mathcal{S}_t = \operatorname{Syz}(f_1, \ldots , f_n)(0)$ denote the Harder-Narasimhan filtration of the syzygy bundle on $Y = \operatorname{Proj} R$, set $\mu_k = \mu(\mathcal{S}_k/\mathcal{S}_{k-1})$ and $r_k = \operatorname{rk}(\mathcal{S}_k/\mathcal{S}_{k-1})$. Then the Hilbert-Kunz multiplicity of I is by definition

$$e_{HK}(I) = \frac{\deg(Y)}{2}(\sum_{k=1}^{t} r_k(\frac{\mu_k}{\deg(Y)})^2 - \sum_{i=1}^{n} d_i^2) = \frac{\sum_{k=1}^{t} r_k\mu_k^2 - \deg(Y)^2\sum_{i=1}^{n} d_i^2}{2\deg(Y)}\,.$$

It is easy to show that this definition does not depend on the chosen ideal generators and is therefore an invariant of the ideal, see [2, Proposition 4.9]. With this invariant we can in fact give the following Hilbert-Kunz criterion for solid closure in characteristic zero in dimension two (see Theorem 3.3):

Theorem 2. Let K denote an algebraically closed field of characteristic zero, let R denote a standard-graded two-dimensional normal K-domain. Let I be a homogeneous R_+-primary ideal and let f denote a homogeneous element. Then f is contained in the solid closure, $f \in I^\star$, if and only if $e_{HK}(I) = e_{HK}((I, f))$.

To prove this theorem it is convenient to consider more generally for a locally free sheaf $\mathcal{S}$ on a smooth projective curve Y the expression

$$\mu_{HK}(\mathcal{S}) = \sum_{k=1}^{t} r_k \mu_k^2 \,,$$

where r_k and μ_k are the ranks and the slopes of the semistable quotient sheaves in the Harder-Narasimhan filtration of $\mathcal{S}$. We call this number the Hilbert-Kunz slope of $\mathcal{S}$. With this notion the Hilbert-Kunz multiplicity of an ideal $I = (f_1, \ldots, f_n)$ is related to the Hilbert-Kunz slope of the syzygy bundle by

$$e_{HK}((f_1, \ldots, f_n)) = \frac{1}{2\deg(Y)} \big(\mu_{HK}(\mathrm{Syz}(f_1, \ldots, f_n)(0)) - \mu_{HK}(\bigoplus_{i=1}^{n} \mathcal{O}(-d_i))\big) \,.$$

With this notion we will in fact prove the following theorem, which implies Theorem 2 (see Theorem 2.6).

Theorem 3. Let Y denote a smooth projective curve over an algebraically closed field of characteristic 0. Let $\mathcal{S}$ denote a locally free sheaf on Y and let $c \in H^1(Y, \mathcal{S})$ denote a cohomology class giving rise to the extension $0 \to \mathcal{S} \to \mathcal{S}' \to \mathcal{O}_Y \to 0$ and the affine-linear torsor $\mathbb{P}(\mathcal{S}'^\vee) - \mathbb{P}(\mathcal{S}^\vee)$. Then $\mathbb{P}(\mathcal{S}'^\vee) - \mathbb{P}(\mathcal{S}^\vee)$ is an affine scheme if and only if $\mu_{HK}(\mathcal{S}') < \mu_{HK}(\mathcal{S})$.

1. The Hilbert-Kunz slope of a vector bundle

We recall briefly some notions for locally free sheaves (or vector bundles), see [7] or [4]. Let Y denote a smooth projective curve over an algebraically closed field and let $\mathcal{S}$ denote a locally free sheaf of rank r. Then $\deg(\mathcal{S}) = \deg(\bigwedge^r \mathcal{S})$ is called the degree of $\mathcal{S}$ and $\mu(\mathcal{S}) = \deg(\mathcal{S})/r$ is called the slope of $\mathcal{S}$. If $\mu(\mathcal{T}) \leq \mu(\mathcal{S})$ holds for every locally free subsheaf $\mathcal{T} \subseteq \mathcal{S}$, then $\mathcal{S}$ is called semistable. In general there exists the so-called Harder-Narasimhan filtration. This is a filtration of locally free subsheaves $\mathcal{S}_1 \subset \ldots \subset \mathcal{S}_t = \mathcal{S}$ such that the quotient sheaves $\mathcal{S}_k/\mathcal{S}_{k-1}$ are semistable locally free sheaves with decreasing slopes $\mu_1 > \ldots > \mu_t$. The Harder-Narasimhan filtration is uniquely determined by these properties. $\mathcal{S}_1$ is called the maximal destabilizing subsheaf, $\mu_1 = \mu_{\max}(\mathcal{S})$ is called the maximal slope of $\mathcal{S}$ and $\mu_t = \mu_{\min}(\mathcal{S})$ is called the minimal slope of $\mathcal{S}$. If $\mathcal{S} \to \mathcal{T}$ is a non-trivial sheaf homomorphism, then $\mu_{\min}(\mathcal{S}) \leq \mu_{\max}(\mathcal{T})$.

We begin with the definition of the Hilbert-Kunz slope of $\mathcal{S}$.

Definition 1.1. Let $\mathcal{S}$ denote a locally free sheaf on a smooth projective curve over an algebraically closed field of characteristic 0. Let $\mathcal{S}_1 \subset \ldots \subset \mathcal{S}_t = \mathcal{S}$ denote the Harder-Narasimhan filtration of $\mathcal{S}$, set $r_k = \operatorname{rk}(\mathcal{S}_k/\mathcal{S}_{k-1})$ and $\mu_k = \mu(\mathcal{S}_k/\mathcal{S}_{k-1})$. We define the Hilbert-Kunz slope of $\mathcal{S}$ by

$$\mu_{HK}(\mathcal{S}) = \sum_{k=1}^{t} r_k \mu_k^2 = \sum_{k=1}^{t} \frac{\deg(\mathcal{S}_k/\mathcal{S}_{k-1})^2}{r_k} .$$

The only justification for considering this number is Theorem 3.3 below. We gather together some properties of this notion in the following proposition.

Proposition 1.2. *Let $\mathcal{S}$ denote a locally free sheaf on a smooth projective curve over an algebraically closed field of characteristic* 0*. Then the following hold true.*

(i) *If $\mathcal{S}$ is semistable, then $\mu_{HK}(\mathcal{S}) = \deg(\mathcal{S})^2/\operatorname{rk}(\mathcal{S})$.*
(ii) *Let $\mathcal{T} \subset \mathcal{S}$ denote a locally free subsheaf occurring in the Harder-Narasimhan filtration of $\mathcal{S}$. Then $\mu_{HK}(\mathcal{S}) = \mu_{HK}(\mathcal{T}) + \mu_{HK}(\mathcal{S}/\mathcal{T})$.*
(iii) *We have $\mu_{HK}(\mathcal{S} \oplus \mathcal{T}) = \mu_{HK}(\mathcal{S}) + \mu_{HK}(\mathcal{T})$.*
(iv) *$\mu_{HK}(\mathcal{S}) = \mu_{HK}(\mathcal{S}^\vee)$.*
(v) *Let $\mathcal{L}$ denote an invertible sheaf. Then*
$$\mu_{HK}(\mathcal{S} \otimes \mathcal{L}) = \mu_{HK}(\mathcal{S}) + 2\deg(\mathcal{S})\deg(\mathcal{L}) + \operatorname{rk}(\mathcal{S})\deg(\mathcal{L})^2 .$$
(vi) *Let $\varphi : Y' \to Y$ denote a finite morphism between smooth projective curves of degree n. Then $\mu_{HK}(\varphi^*(\mathcal{S})) = n^2\mu_{HK}(\mathcal{S})$.*

Proof. (i) and (ii) are clear from the definition. (iii). The maximal destabilizing subsheaf of $\mathcal{S} \oplus \mathcal{T}$ is either $\mathcal{S}_1 \oplus 0$, $0 \oplus \mathcal{T}_1$ or $\mathcal{S}_1 \oplus \mathcal{T}_1$. Hence the result follows from (ii) by induction on the rank of $\mathcal{S} \oplus \mathcal{T}$.

(iv). Let $0 = \mathcal{S}_0 \subset \mathcal{S}_1 \subset \ldots \subset \mathcal{S}_t = \mathcal{S}$ denote the Harder-Narasimhan filtration of $\mathcal{S}$. Set $\mathcal{Q}_k = \mathcal{S}/\mathcal{S}_k$. This gives a filtration $0 \subset \mathcal{Q}_{t-1}^\vee \subset \ldots \subset \mathcal{Q}_1^\vee \subset \mathcal{Q}_0^\vee = \mathcal{S}^\vee$. From $0 \to \mathcal{S}_k/\mathcal{S}_{k-1} \to \mathcal{S}/\mathcal{S}_{k-1} \to \mathcal{S}/\mathcal{S}_k \to 0$ we get $0 \to \mathcal{Q}_k^\vee \to \mathcal{Q}_{k-1}^\vee \to \mathcal{Q}_{k-1}^\vee/\mathcal{Q}_k^\vee \cong (\mathcal{S}_k/\mathcal{S}_{k-1})^\vee \to 0$. Hence the filtration is the Harder-Narasimhan filtration of $\mathcal{S}^\vee$ and the result follows from $\mu(\mathcal{Q}_{k-1}^\vee/\mathcal{Q}_k^\vee) = -\mu(\mathcal{S}_k/\mathcal{S}_{k-1})$.

(v). The Harder-Narasimhan filtration of $\mathcal{S} \otimes \mathcal{L}$ is $\mathcal{S}_1 \otimes \mathcal{L} \subset \ldots \subset \mathcal{S}_t \otimes \mathcal{L}$ and $\mu(\mathcal{S}_k \otimes \mathcal{L}/\mathcal{S}_{k-1} \otimes \mathcal{L}) = \mu((\mathcal{S}_k/\mathcal{S}_{k-1}) \otimes \mathcal{L}) = \mu(\mathcal{S}_k/\mathcal{S}_{k-1}) + \mu(\mathcal{L})$. Therefore

$$\begin{aligned}
\mu_{HK}(\mathcal{S} \otimes \mathcal{L}) &= \sum_{k=1}^{t} r_k \mu_k(\mathcal{S} \otimes \mathcal{L})^2 \\
&= \sum_{k=1}^{t} r_k(\mu_k + \deg(\mathcal{L}))^2 \\
&= \sum_{k=1}^{t} r_k(\mu_k^2 + 2\mu_k \deg(\mathcal{L}) + \deg(\mathcal{L})^2)
\end{aligned}$$

$$= \quad \mu_{HK}(\mathcal{S}) + 2\deg(\mathcal{L}) \sum_{k=1}^{t} r_k \mu_k + \deg(\mathcal{L}^2) \sum_{k=1}^{t} r_k \, .$$

This is the stated result, since $\deg(\mathcal{S}) = \sum_{k=1}^{t} r_k \mu_k$ and $\mathrm{rk}(\mathcal{S}) = \sum_{k=1}^{t} r_k$.

(vi). The pull-back of a semistable sheaf under a separable morphism is again semistable, and the pull-back of the Harder-Narasimhan filtration is the Harder-Narasimhan filtration of $\varphi^*(\mathcal{S})$. Hence the result follows from $\deg(\varphi^*(\mathcal{S})) = n \deg(\mathcal{S})$. □

Lemma 1.3. *The Hilbert-Kunz multiplicity of a locally free sheaf $\mathcal{S}$ has the property that $\mu_{HK}(\mathcal{S}) \geq \deg(\mathcal{S})^2/\mathrm{rk}(\mathcal{S})$, and equality holds if and only if $\mathcal{S}$ is semistable.*

Proof. We have to show that

$$\sum_{k=1}^{t} r_k \mu_k^2 \geq \deg(\mathcal{S})^2/\mathrm{rk}(\mathcal{S}) = (r_1\mu_1 + \ldots + r_t\mu_t)^2/(r_1 + \ldots + r_t)$$

or equivalently that

$$(r_1 + \ldots + r_t)(\sum_{k=1}^{t} r_k \mu_k^2) \geq (r_1\mu_1 + \ldots + r_t\mu_t)^2 \, .$$

The left hand side is $\sum_{k=1}^{t} r_k^2 \mu_k^2 + \sum_{i \neq k} r_i r_k \mu_k^2$ (we sum over ordered pairs), and the right hand side is $\sum_{k=1}^{t} r_k^2 \mu_k^2 + \sum_{i \neq k} r_i r_k \mu_i \mu_k$. Hence left hand minus right hand is

$$\sum_{i \neq k} r_i r_k \mu_k^2 - \sum_{i \neq k} r_i r_k \mu_i \mu_k$$

So this follows from $0 \leq (\mu_i - \mu_k)^2 = \mu_i^2 + \mu_k^2 - 2\mu_i\mu_k$ for all pairs $i \neq k$. Equality holds if and only if $\mu_i = \mu_k$, but then $t = 1$ and $\mathcal{S}$ is semistable. □

Remark 1.4. Lemma 1.3 implies that the number $\mu_{HK}(\mathcal{S}) - \frac{\deg(\mathcal{S})^2}{\mathrm{rk}(\mathcal{S})} \geq 0$, and $= 0$ holds exactly in the semistable case. It follows from Proposition 1.2 (v) that this number is invariant under tensoring with an invertible sheaf.

Proposition 1.5. *Let $\mathcal{S}$ and $\mathcal{T}$ denote two locally free sheaves on Y. Then*

$$\mu_{HK}(\mathcal{S} \otimes \mathcal{T}) = \mathrm{rk}(\mathcal{T})\mu_{HK}(\mathcal{S}) + \mathrm{rk}(\mathcal{S})\mu_{HK}(\mathcal{T}) + 2\deg(\mathcal{S})\deg(\mathcal{T}) \, .$$

Proof. Let r_i, μ_i, $i \in I$, and r_j, μ_j, $j \in J$, (I and J disjoint) denote the ranks and slopes occurring in the Harder-Narasimhan filtration of $\mathcal{S}$ and $\mathcal{T}$ respectively. It is a non-trivial fact (in characteristic zero!) that the tensor product of two semistable bundle is again semistable, see [7, Theorem 3.1.4]. From this it follows that the semistable quotients of the Harder-Narasimhan filtration of $\mathcal{S} \otimes \mathcal{T}$ are given as $(\mathcal{S}_i/\mathcal{S}_{i-1}) \otimes (\mathcal{T}_j/\mathcal{T}_{j-1})$ of rank $r_i \cdot r_j$ and slope $\mu_i + \mu_j$. Therefore the Hilbert-Kunz slope is

$$\mu_{HK}(\mathcal{S} \otimes \mathcal{T}) \quad = \quad \sum_{i,j} r_i r_j (\mu_i + \mu_j)^2$$

$$
\begin{aligned}
&= \sum_{i,j} r_i r_j \mu_i^2 + \sum_{i,j} r_i r_j \mu_j^2 + 2\sum_{i,j} r_i r_j \mu_i \mu_j \\
&= (\sum_j r_j)(\sum_i r_i \mu_i^2) + (\sum_i r_i)(\sum_j r_j \mu_j^2) + 2(\sum_i r_i \mu_i)(\sum_j r_j \mu_j) \\
&= \operatorname{rk}(\mathcal{T})\mu_{HK}(\mathcal{S}) + \operatorname{rk}(\mathcal{S})\mu_{HK}(\mathcal{T}) + 2\deg(\mathcal{S})\deg(\mathcal{T})
\end{aligned}
$$

□

2. A Hilbert-Kunz criterion for affine torsors

In this section we consider a locally free sheaf $\mathcal{S}$ on a smooth projective curve Y together with a cohomology class $c \in H^1(Y, \mathcal{S}) \cong \operatorname{Ext}(\mathcal{O}_Y, \mathcal{S})$. Such a class gives rise to an extension $0 \to \mathcal{S} \to \mathcal{S}' \to \mathcal{O}_Y \to 0$. Of course $\deg(\mathcal{S}') = \deg(\mathcal{S})$ and $\operatorname{rk}(\mathcal{S}') = \operatorname{rk}(\mathcal{S}) + 1$. We shall investigate the relationship between $\mu_{HK}(\mathcal{S})$ and $\mu_{HK}(\mathcal{S}')$.

Lemma 2.1. *Let Y denote a smooth projective curve over an algebraically closed field. Let $\mathcal{S}$, $\mathcal{T}$ and $\mathcal{Q}$ denote locally free sheaves on Y. Then the following hold.*

- (i) *Let $\varphi : \mathcal{T} \to \mathcal{S}$ denote a sheaf homomorphism, $c \in H^1(Y, \mathcal{T})$ with corresponding extension $\mathcal{T}'$, let $\mathcal{S}'$ denote the extension of $\mathcal{S}$ corresponding to $\varphi(c) \in H^1(Y, \mathcal{S})$. Then there is a sheaf homomorphism $\varphi' : \mathcal{T}' \to \mathcal{S}'$ extending φ.*
- (ii) *Suppose that $0 \to \mathcal{T} \to \mathcal{S} \to \mathcal{Q} \to 0$ is a short exact sequence, and $c \in H^1(Y, \mathcal{T})$. Then $\mathcal{T}' \subseteq \mathcal{S}'$ and $\mathcal{S}'/\mathcal{T}' \cong \mathcal{S}/\mathcal{T}$.*
- (iii) *Suppose that $0 \to \mathcal{T} \to \mathcal{S} \to \mathcal{Q} \to 0$ is a short exact sequence, and $c \in H^1(Y, \mathcal{S})$. Then $\mathcal{S}' \to \mathcal{Q}' \to 0$ and $\mathcal{Q}' \cong \mathcal{S}'/\mathcal{T}$.*
- (iv) *If $\mathcal{S}$ is semistable of degree 0 and $c \in H^1(Y, \mathcal{S})$, then also $\mathcal{S}'$ is semistable.*

Proof. The cohomology class c is represented by the Čech cocycle $\check{c} \in H^0(U_1 \cap U_2, \mathcal{S})$, where $Y = U_1 \cup U_2$ is an affine covering. Then $\mathcal{S}'$ arises from $\mathcal{S}_1' = \mathcal{S}|_{U_1} \oplus \mathcal{O}$ and $\mathcal{S}_2' = \mathcal{S}|_{U_2} \oplus \mathcal{O}$ by glueing $\mathcal{S}_1'|U_1 \cap U_2 \cong \mathcal{S}_2'|U_1 \cap U_2$ via $(s,t) \mapsto (s + t\check{c}, t)$. The natural mappings $\mathcal{T}_i' \to \mathcal{S}_i'$, $i = 1, 2$, glue together to a morphism $\mathcal{T}' \to \mathcal{S}'$. The injectivity and surjectivity transfer from φ to φ', since these are local properties. (ii) and (iii) then follow from suitable diagrams.

(iv). Suppose that $\mathcal{F} \subseteq \mathcal{S}'$ is a semistable subsheaf of positive slope. Then the induced mapping $\mathcal{F} \to \mathcal{O}$ is trivial and therefore $\mathcal{F} \subseteq \mathcal{S}$, which contradicts the semistability of $\mathcal{S}$. □

Let $\mathcal{S}_1 \subset \ldots \subset \mathcal{S}_t = \mathcal{S}$ denote the Harder-Narasimhan filtration of $\mathcal{S}$ and $c \in H^1(Y, \mathcal{S})$. If the image of c in $H^1(Y, \mathcal{S}/\mathcal{S}_{t-1})$ is zero, then c stems from a class $c_{t-1} \in H^1(Y, \mathcal{S}_{t-1})$. So we find inductively a class $c_n \in H^1(Y, \mathcal{S}_n)$ mapping to c and such that the image in $H^1(Y, \mathcal{S}_n/\mathcal{S}_{n-1})$ is not zero (or c itself is 0). This yields extensions $\mathcal{S}_k'$ of $\mathcal{S}_k$ for $k \geq n$. It is crucial for the behavior of $\mathcal{S}'$ whether $\mu(\mathcal{S}_n/\mathcal{S}_{n-1}) \geq 0$ or < 0. The following Proposition deals with the case $\mu(\mathcal{S}_n/\mathcal{S}_{n-1}) \geq 0$.

Proposition 2.2. *Let $\mathcal{S}_1 \subset \ldots \subset \mathcal{S}_t = \mathcal{S}$ be the Harder-Narasimhan filtration of $\mathcal{S}$ and let $c \in H^1(Y, \mathcal{S})$. Let n be such that the image of c in $H^1(Y, \mathcal{S}_k/\mathcal{S}_{k-1})$ is 0 for $k > n$ but such that the image in $H^1(Y, \mathcal{S}_n/\mathcal{S}_{n-1})$ is $\neq 0$. Suppose that $\mu(\mathcal{S}_n/\mathcal{S}_{n-1})$ is ≥ 0. Let i be the biggest number such that $\mu(\mathcal{S}_i/\mathcal{S}_{i-1}) \geq 0$ (hence $n \leq i$).*

(i) *Suppose that $\mu_i > 0$. Then the Harder Narasimhan filtration of $\mathcal{S}'$ is*

$$\mathcal{S}_1 \subset \ldots \subset \mathcal{S}_i \subset \mathcal{S}'_i \subset \mathcal{S}'_{i+1} \subset \ldots \subset \mathcal{S}' .$$

(ii) *Suppose that $\mu_i = 0$. Then the Harder-Narasimhan filtration of $\mathcal{S}'$ is*

$$\mathcal{S}_1 \subset \ldots \subset \mathcal{S}_{i-1} \subset \mathcal{S}'_i \subset \mathcal{S}'_{i+1} \subset \ldots \subset \mathcal{S}' .$$

Proof. (i). The quotients of the filtration are $\mathcal{S}_k/\mathcal{S}_{k-1}$, $k \leq i$, which have positive slope, $\mathcal{S}'_i/\mathcal{S}_i \cong \mathcal{O}_Y$, and $\mathcal{S}'_k/\mathcal{S}'_{k-1} \cong \mathcal{S}_k/\mathcal{S}_{k-1}$ (Lemma 2.1(ii)) for $k > i$, which have negative slope. These quotients are all semistable and the slope numbers are decreasing.

(ii). The quotients $\mathcal{S}_k/\mathcal{S}_{k-1}$ are semistable with decreasing positive slopes for $k = 1, \ldots, i-1$. The quotients $\mathcal{S}'_k/\ \mathcal{S}'_{k-1} \cong \mathcal{S}_k/\mathcal{S}_{k-1}$ are semistable with decreasing negative slopes for $k = i+1, \ldots, t$. The quotient $\mathcal{S}'_i/\mathcal{S}_{i-1}$ is isomorphic to $(\mathcal{S}_i/\mathcal{S}_{i-1})'$ by Lemma 2.1(iii), hence semistable of degree 0 by Lemma 2.1(iv). □

In the rest of this section we study the remaining case, that $\mu(\mathcal{S}_n/\mathcal{S}_{n-1}) < 0$. In this case it is not possible to describe the Harder-Narasimhan filtration of S' explicitly. However we shall see that in this case the Hilbert-Kunz slope of $\mathcal{S}'$ is smaller than the Hilbert-Kunz slope of $\mathcal{S}$. We need the following two lemmata.

Lemma 2.3. *Let $\mathcal{T}$ denote a locally free sheaf on Y with Harder-Narasimhan filtration $\mathcal{T}_k$, $\mu_k = \mu(\mathcal{T}_k/\mathcal{T}_{k-1})$ and $r_k = \operatorname{rk}(\mathcal{T}_k/\mathcal{T}_{k-1})$. Let*

$$(\tau_i) = (\mu_1, \ldots, \mu_1, \mu_2, \ldots, \mu_2, \mu_3, \ldots, \mu_{t-1}, \mu_t, \ldots, \mu_t)$$

denote the slopes where each μ_k occurs r_k-times. Let $\mathcal{S} \subseteq \mathcal{T}$ denote a locally free subsheaf of rank r and let σ_i, $i = 1, \ldots, r$ denote the corresponding numbers for $\mathcal{S}$. Then $\sigma_i \leq \tau_i$ for $i = 1, \ldots, r$.

Moreover, if $\mathcal{S}$ is saturated (meaning that the quotient sheaf is locally free) and if no subsheaf $\mathcal{S}_j$ of the Harder-Narasimhan filtration of $\mathcal{S}$ occurs in the Harder-Narasimhan filtration of $\mathcal{T}$, then $\sigma_i \leq \tau_{i+1}$ for $i = 1, \ldots, r$.

Proof. Let i, $i = 1, \ldots, r$ be given and let j be such that $\operatorname{rk}(\mathcal{S}_{j-1}) < i \leq \operatorname{rk}(\mathcal{S}_j)$, hence $\sigma_i = \mu_j(\mathcal{S}) = \mu(\mathcal{S}_j/\mathcal{S}_{j-1})$. We may assume that $i = \operatorname{rk}(\mathcal{S}_j)$. Let k be such that $\operatorname{rk}(\mathcal{T}_{k-1}) < i \leq \operatorname{rk}(\mathcal{T}_k)$. Therefore $\mathcal{S}_j \not\subseteq \mathcal{T}_{k-1}$, and the induced morphism $\mathcal{S}_j \to \mathcal{T}/\mathcal{T}_{k-1}$ is not trivial. Hence $\sigma_i = \mu_j(\mathcal{S}) = \mu_{\min}(\mathcal{S}_j) \leq \mu_{\max}(\mathcal{T}/\mathcal{T}_{k-1}) = \mu_k(\mathcal{T}) = \tau_i$.

Now suppose that $\sigma_i > \tau_{i+1}$. Then necessarily $\sigma_i > \sigma_{i+1}$ and $\tau_i > \tau_{i+1}$ by what we have already proven. Therefore $i = \operatorname{rk}(\mathcal{S}_j) = \operatorname{rk}(\mathcal{T}_k)$. If $\mathcal{S}_j \subseteq \mathcal{T}_k$, then they are equal, since both sheaves are saturated of the same rank, but this is excluded by the assumptions. Hence $\mathcal{S}_j \not\subseteq \mathcal{T}_k$ and $\mathcal{S}_j \to \mathcal{T}/\mathcal{T}_k$ is non-trivial. Therefore $\sigma_i = \mu_{\min}(\mathcal{S}_j) \leq \mu_{\max}(\mathcal{T}/\mathcal{T}_k) = \mu_{k+1}(\mathcal{T}) = \tau_{i+1}$. □

Remark 2.4. If the numbers τ_i are given as in the previous lemma, then $\deg(\mathcal{T}) = \sum_i \tau_i$ and $\mu_{HK}(\mathcal{T}) = \sum_i \tau_i^2$.

Lemma 2.5. *Let $\alpha_1 \leq \ldots \leq \alpha_r$ and $\beta_1 \leq \ldots \leq \beta_{r+1}$ denote non-negative real numbers such that $\alpha_i \geq \beta_{i+1}$ for $i = 1, \ldots, r$ and $\sum_{i=1}^r \alpha_i = \sum_{i=1}^{r+1} \beta_i$. Then $\sum_{i=1}^{r+1} \beta_i^2 \leq \sum_{i=1}^r \alpha_i^2$ and equality holds if and only if $\alpha_i = \beta_{i+1}$.*

Proof. Let $\alpha_i = \beta_{i+1} + \delta_i$, $\delta_i \geq 0$. From $\sum_{i=1}^r \alpha_i = \sum_{i=1}^r \delta_i + \sum_{i=1}^r \beta_{i+1} = \sum_{i=1}^{r+1} \beta_i$ we get $\beta_1 = \sum_{i=1}^r \delta_i$ $(\leq \beta_2)$. The quadratic sums are

$$\sum_{i=1}^{r} \alpha_i^2 = \sum_{i=2}^{r+1} \beta_i^2 + \sum_{i=1}^{r} \delta_i^2 + 2 \sum_{i=1}^{r} \delta_i \beta_{i+1}$$

and

$$\sum_{i=1}^{r+1} \beta_i^2 = (\sum_{i=1}^{r} \delta_i)^2 + \sum_{i=2}^{r+1} \beta_i^2 = 2 \sum_{i<j} \delta_i \delta_j + \sum_{i=1}^{r} \delta_i^2 + \sum_{i=2}^{r+1} \beta_i^2 \, .$$

So we have to show that $\sum_{i<j} \delta_i \delta_j \leq \sum_{j=1}^r \delta_i \beta_{i+1}$. But this is clear from $\sum_{i<j} \delta_j \leq \sum_{j=1}^r \delta_j \leq \beta_2 \leq \beta_{i+1}$ for all $i = 1, \ldots, r$. Equality holds if and only if $\delta_i = 0$. □

A cohomology class $H^1(Y, \mathcal{S})$ corresponds to a geometric $\mathcal{S}$-torsor $T \to Y$. This is an affine-linear bundle on which $\mathcal{S}$ acts transitively. A geometric realization is given as $T = \mathbb{P}(\mathcal{S}'^\vee) - \mathbb{P}(\mathcal{S}^\vee)$. The global cohomological properties of this torsor are related to the Hilbert-Kunz slope in the following way.

Theorem 2.6. *Let Y denote a smooth projective curve over an algebraically closed field of characteristic 0. Let $\mathcal{S}$ denote a locally free sheaf on Y and let $c \in H^1(Y, \mathcal{S})$ denote a cohomology class given rise to the extension $0 \to \mathcal{S} \to \mathcal{S}' \to \mathcal{O}_Y \to 0$ and the affine-linear torsor $\mathbb{P}(\mathcal{S}'^\vee) - \mathbb{P}(\mathcal{S}^\vee)$. Then the following are equivalent.*

(i) *There exists a locally free quotient $\varphi : \mathcal{S} \to \mathcal{Q} \to 0$ such that $\mu_{\max}(\mathcal{Q}) < 0$ and the image $\varphi(c) \in H^1(Y, \mathcal{Q})$ is non-trivial.*
(ii) *The torsor $\mathbb{P}(\mathcal{S}'^\vee) - \mathbb{P}(\mathcal{S}^\vee)$ is an affine scheme.*
(iii) *The Hilbert-Kunz slope drops, that is $\mu_{HK}(\mathcal{S}') < \mu_{HK}(\mathcal{S})$.*

Proof. The equivalence (i) $\Leftrightarrow$ (ii) was shown in [3, Theorem 2.3]. The implication (iii) $\Rightarrow$ (i) follows from Proposition 2.2: for if (i) does not hold, then we are in the situation of Proposition 2.2 that $\mu(\mathcal{S}_n / \mathcal{S}_{n-1}) \geq 0$. The explicit description of the Harder-Narasimhan filtration of $\mathcal{S}'$ gives in both cases that $\mu_{HK}(\mathcal{S}') = \mu_{HK}(\mathcal{S})$.

So suppose that (i) holds. This means that there exists a subsheaf $\mathcal{S}_n \subseteq \mathcal{S}$ occurring in the Harder-Narasimhan filtration of $\mathcal{S}$ such that c stems from $c_n \in H^1(Y, \mathcal{S}_n)$ and such that its image in $H^1(Y, \mathcal{S}/\mathcal{S}_{n-1})$ is non-trivial with $\mu_{\max}(\mathcal{S}/\mathcal{S}_{n-1}) = \mu(\mathcal{S}_n/\mathcal{S}_{n-1}) = \mu_n < 0$.

Let $\mathcal{T}_1 \subset \ldots \subset \mathcal{T}_t = \mathcal{S}'$ denote the Harder-Narasimhan filtration of $\mathcal{S}'$ with slopes $\mu_k = \mu(\mathcal{T}_k/\mathcal{T}_{k-1})$ and ranks $r_k = \operatorname{rk}(\mathcal{T}_k/\mathcal{T}_{k-1})$. Suppose that the maximal slope $\mu(\mathcal{T}_1)$ is positive. Then the induced mapping $\mathcal{T}_1 \to \mathcal{S}'/\mathcal{S} \cong \mathcal{O}_Y$ is trivial, and $\mathcal{T}_1 \subseteq \mathcal{S}$. This is then also the maximal destabilizing subsheaf of $\mathcal{S}$, since $\mu_{\max}(\mathcal{S}) \leq \mu_{\max}(\mathcal{S}') = \mu(\mathcal{T}_1)$. Therefore $\mu_{HK}(\mathcal{S}) = \mu_{HK}(\mathcal{T}_1) + \mu_{HK}(\mathcal{S}/\mathcal{T}_1)$ and $\mu_{HK}(\mathcal{S}') = \mu_{HK}(\mathcal{T}_1) + \mu_{HK}(\mathcal{S}'/\mathcal{T}_1)$ by Proposition 1.2(ii). Since $\mathcal{S}'/\mathcal{T}_1$ is the extension of $\mathcal{S}/\mathcal{T}_1$ defined by the image of the cohomology class in $H^1(Y, \mathcal{S}/\mathcal{T}_1)$ (Lemma 2.1(iii)), we may mod out $\mathcal{T}_1$. Note that this does not change the condition in (i). Hence we may assume inductively that $\mu_{\max}(\mathcal{S}) \leq 0$ and $\mu_{\max}(\mathcal{S}') \leq 0$.

Now suppose that $\mathcal{T}_1$ has degree 0. Again, if $\mathcal{T}_1 \subseteq \mathcal{S}$, then this is also the maximal destabilizing subsheaf of $\mathcal{S}$, and we can mod out $\mathcal{T}_1$ as before. So suppose that $\mathcal{T}_1 \to \mathcal{O}_Y$ is non-trivial. Then this mapping is surjective, let $\mathcal{K} \subset \mathcal{S}$ denote the kernel. This means that the extension defined by $c \in H^1(Y, \mathcal{S})$ comes from the extension given by $0 \to \mathcal{K} \to \mathcal{T}_1 \to \mathcal{O}_Y \to 0$, and $\tilde{c} \in H^1(Y, \mathcal{K})$. $\mathcal{K}$ is semistable, since its degree is 0 and $\mu_{\max}(\mathcal{S}) \leq 0$. But then the image of c is 0 in every quotient sheaf of $\mathcal{S}$ with negative maximal slope, which contradicts the assumptions. Therefore we may assume that $\mu_{\max}(\mathcal{S}') < 0$.

We want to apply Lemma 2.3 to $\mathcal{S} \subset \mathcal{S}' = \mathcal{T}$. Assume that $\mathcal{S}$ and $\mathcal{S}'$ have a common subsheaf occuring in both Harder-Narasimhan filtrations. Then they have the same maximal destabilizing subsheaf $\mathcal{F} = \mathcal{S}_1 = \mathcal{T}_1$, which has negative degree. If c comes from $\tilde{c} \in H^1(Y, \mathcal{F})$, then $\mathcal{F} \subset \mathcal{F}' \subseteq \mathcal{S}'$ and $\mu(\mathcal{F}) = \deg(\mathcal{F})/\operatorname{rk}(\mathcal{F}) < \deg(\mathcal{F})/(\operatorname{rk}(\mathcal{F})+1) = \mu(\mathcal{F}')$, which contradicts the maximality of $\mathcal{F}$. Hence the image of c in $H^1(Y, \mathcal{S}/\mathcal{F})$ is not zero and we can mod out $\mathcal{F}$ as before.

Therefore we may assume that $\mathcal{S}$ and $\mathcal{S}'$ do not have any common subsheaf in their Harder-Narasimhan filtrations. Then Lemma 2.3 yields that $\sigma_i \leq \tau_{i+1}$, and all these numbers are ≤ 0 and moreover $\tau_i < 0$. Lemma 2.5 applied to $\alpha_i = -\sigma_i$ and $\beta_i = -\tau_i$ yields that $\sum_{i=1}^{r} \sigma_i^2 \geq \sum_{i=1}^{r+1} \tau_i^2$, and $>$ holds since $\tau_1 \neq 0$. □

Remark 2.7. Suppose that $\mathcal{S}$ is a semistable locally free sheaf of negative degree, and let $c \in H^1(Y, \mathcal{S})$ with corresponding extension $\mathcal{S}'$. Then Theorem 2.6 together with Lemma 1.3 yield the inequalities

$$\frac{\deg(\mathcal{S})^2}{r+1} \leq \mu_{HK}(\mathcal{S}') \leq \frac{\deg(\mathcal{S})^2}{r} .$$

If $\mathcal{S}'$ is also semistable, then we have equality on the left.

3. A Hilbert-Kunz criterion for solid closure

We come now back to our original setting of interest, that of a two-dimensional normal standard-graded domain R over an algebraically closed field K. A homogeneous R_+-primary ideal $I = (f_1, \ldots, f_n)$ gives rise to the syzygy bundle $\operatorname{Syz}(f_1, \ldots, f_n)(0)$ on $Y = \operatorname{Proj} R$ defined by the presenting sequence

$$0 \longrightarrow \operatorname{Syz}(f_1, \ldots, f_n)(m) \longrightarrow \bigoplus_{i=1}^{n} \mathcal{O}_Y(m - d_i) \stackrel{f_1, \ldots, f_n}{\longrightarrow} \mathcal{O}_Y(m) \longrightarrow 0 .$$

Another homogeneous element f of degree m yields an extension

$$0 \longrightarrow \mathrm{Syz}(f_1, \dots, f_n)(m) \longrightarrow \mathrm{Syz}(f_1, \dots, f_n, f)(m) \longrightarrow \mathcal{O}_Y \longrightarrow 0$$

which corresponds to the cohomology class $\delta(f) \in H^1(Y, \mathrm{Syz}(f_1, \dots, f_n)(m))$ coming from the presenting sequence via the connecting homomorphism

$$\delta : H^0(Y, \mathcal{O}_Y(m)) = R_m \to H^1(Y, \mathrm{Syz}(f_1, \dots, f_n)(m)) \,.$$

The Hilbert-Kunz multiplicities of the ideals and the Hilbert-Kunz slopes of the syzygy bundles are related in the following way.

Lemma 3.1. *Let K denote an algebraically closed field of characteristic 0. Let R denote a standard-graded two-dimensional normal K-domain, $Y = \mathrm{Proj}\, R$. Let I be a homogeneous R_+-primary ideal and let f denote a homogeneous element of degree m. Then the Hilbert-Kunz multiplicities $e_{HK}(I) = e_{HK}((I, f))$ are equal if and only if the Hilbert-Kunz slopes of the corresponding syzygies bundles $\mu_{HK}(\mathrm{Syz}(f_1, \dots, f_n)(m)) = \mu_{HK}(\mathrm{Syz}(f_1, \dots, f_n, f)(m))$ are equal.*

Proof. Let μ_k and r_k ($\tilde{\mu}_k$ and $\tilde{r}_k$) denote the ranks and the slopes in the Harder-Narasimhan filtration of $\mathrm{Syz}(f_1, \dots, f_n)(0)$ (of $\mathrm{Syz}(f_1, \dots, f_n, f)(0)$ respectively). For the Hilbert-Kunz multiplicities of the ideals $(f_1, \dots, f_n)$ and $(f_1, \dots, f_n, f)$ we have to compare

$$e_{HK}(I) = \frac{1}{2\deg(Y)} \big(\sum_{k=1}^{t} r_k \mu_k^2 - \deg(Y)^2 \sum_{i=1}^{n} d_i^2 \big)$$

and

$$e_{HK}((I, f)) = \frac{1}{2\deg(Y)} \big(\sum_{k=1}^{\tilde{t}} \tilde{r}_k \tilde{\mu}_k^2 - \deg(Y)^2 (m^2 + \sum_{i=1}^{n} d_i^2) \big) \,.$$

The extension defined by $c = \delta(f) \in H^1(Y, \mathrm{Syz}(f_1, \dots, f_n)(m))$ is

$$0 \longrightarrow \mathcal{S} = \mathrm{Syz}(f_1, \dots, f_n)(m) \longrightarrow \mathcal{S}' = \mathrm{Syz}(f_1, \dots, f_n, f)(m) \longrightarrow \mathcal{O}_Y \longrightarrow 0$$

and the Hilbert-Kunz slopes of these sheaves are due to Proposition 1.2 (v) (since $\deg(\mathrm{Syz}(f_1, \dots, f_n)(0)) = -\deg(Y) \sum_{i=1}^{n} d_i$)

$$\mu_{HK}(\mathcal{S}) = \sum_{k=1}^{t} r_k \mu_k^2 + 2(-\sum_{i=1}^{n} d_i \deg(Y)) m \deg(Y) + (n-1) m^2 \deg(Y)^2$$

and $\mu_{HK}(\mathcal{S}') =$

$$\begin{aligned} &= \sum_{k=1}^{\tilde{t}} \tilde{r}_k \tilde{\mu}_k^2 + 2(-(\sum_{i=1}^{n} d_i + m) \deg(Y)) m \deg(Y) + n m^2 \deg(Y)^2 \\ &= \sum_{k=1}^{\tilde{t}} \tilde{r}_k \tilde{\mu}_k^2 - 2(\sum_{i=1}^{n} d_i) m \deg(Y)^2 + (n-1) m^2 \deg(Y)^2 - m^2 \deg(Y)^2 \,. \end{aligned}$$

So the difference is in both cases (up to the factor $1/2\deg(Y)$)

$$\sum_{k=1}^{\tilde{t}} \tilde{r}_k \tilde{\mu}_k^2 - \sum_{k=1}^{t} r_k \mu_k^2 - m^2 \deg(Y)^2 .$$

Therefore $e_{HK}(I) = e_{HK}((I,f))$ if and only if

$$\mu_{HK}(\mathrm{Syz}(f_1,\ldots,f_n)(m)) = \mu_{HK}(\mathrm{Syz}(f_1,\ldots,f_n,f)(m)) .$$

$\square$

Remark 3.2. Let $0 \to \mathcal{S} \to \mathcal{T} \to \mathcal{Q} \to 0$ denote a short exact sequence of locally free sheaves. Then the alternating sum of the Hilbert-Kunz slopes, that ist $\mu_{HK}(\mathcal{S}) - \mu_{HK}(\mathcal{T}) + \mu_{HK}(\mathcal{Q})$ does not change when we tensor the sequence with an invertible sheaf. This follows from Proposition 1.2(v). For an extension $0 \to \mathcal{S} \to \mathcal{S}' \to \mathcal{O}_Y \to 0$ this number is ≥ 0 by Theorem 2.6, and we suspect that this is true in general. From the presenting sequence $0 \to \mathrm{Syz}(f_1,\ldots,f_n)(0) \to \bigoplus_{i=1}^n \mathcal{O}(-d_i) \to \mathcal{O}_Y \to 0$ it follows via $e_{HK}(I) = \frac{1}{2\deg(Y)}(\mu_{HK}(\mathrm{Syz}(f_1,\ldots,f_n)(0)) - \mu_{HK}(\bigoplus_{i=1}^n \mathcal{O}(-d_i))$ that the Hilbert-Kunz multiplicity of an ideal is always nonnegative. In fact $I = R$ is the only ideal with $e_{HK}(I) = 0$. This follows from Theorem 3.3 below, since $1 \notin I^\star$ for $I \neq R$.

We come now to the main result of this paper. Recall that the solid closure of an $\mathfrak{m}$-primary ideal $I = (f_1,\ldots,f_n)$ in a two-dimensional normal excellent domain R is given by the condition that $f \in (f_1,\ldots,f_n)^\star$ if and only $D(\mathfrak{m}) \subset \mathrm{Spec}\, R[T_1,\ldots,T_n]/(f_1T_1 + \ldots + f_nT_n + f)$ is not an affine scheme. In positive characteristic this is the same as tight closure, see [5, Theorem 8.6]. In the case of an R_+-primary homogeneous ideal in a standard-graded normal K-domain this is equivalent to the property that the torsor $\mathbb{P}(\mathcal{S}'^\vee) - \mathbb{P}(\mathcal{S}^\vee)$ over the corresponding curve $Y = \mathrm{Proj}\, R$ is not affine (see [1, Proposition 3.9]). This relates solid closure to the setting of the previous section.

Theorem 3.3. *Let K denote an algebraically closed field. Let R denote a standard-graded two-dimensional normal K-domain. Let I be a homogeneous R_+-primary ideal and let f denote a homogeneous element. Then $f \in I^\star$ if and only if $e_{HK}(I) = e_{HK}((I,f))$.*

Proof. If the characteristic is positive then this is a standard result from tight closure theory as mentioned in the introduction. So suppose that the characteristic is 0. Let $I = (f_1,\ldots,f_n)$ be generated by homogeneous elements, and set $m = \deg(f)$. The containment in the solid closure, $f \in (f_1,\ldots,f_n)^\star$, is equivalent with the non-affineness of the torsor $\mathbb{P}(\mathcal{S}'^\vee) - \mathbb{P}(\mathcal{S}^\vee)$ [1, Proposition 3.9], where $\mathcal{S} = \mathrm{Syz}(f_1,\ldots,f_n)(m)$ and S' is the extension given by the cohomology class $\delta(f)$. Hence the result follows from Theorem 2.6 and Lemma 3.1. $\square$

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