Mathematical Research Letters **11**, 435–441 (2004)

PRESCRIBING RICCI CURVATURE ON COMPLEXIFIED SYMMETRIC SPACES

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The aim of this note is to prove the existence of invariant Ricci-flat Kähler metrics on complexifications of symmetric spaces of compact type. Before stating the result, let us fix the notation.

Let (M, g) be a Riemannian symmetric space of compact type and p a point in M. Let G be the identity component of the isometry group of (M, g) and let K be the stabiliser of p in G. Then $M \simeq G/K$ and the complexification of M is TM with the adapted complex structure [7] that can be identified with $G^{\mathbb{C}}/K^{\mathbb{C}}$[1]. We are going to prove

Theorem 1. *Let (M, g) be an irreducible symmetric space of compact type. Let G and K be as above and suppose that K is connected. Let ρ be a real exact G-invariant $(1,1)$-form on the complexification $TM \simeq G^{\mathbb{C}}/K^{\mathbb{C}}$. Then there exists a G-invariant Kähler metric on TM whose Ricci form is ρ.*

Remark. The Kähler form obtained in Theorem 1 is exact.

The above result has been proved in [9] for symmetric spaces of rank 1 and in [2] for compact groups, i.e. for the case when $G = K \times K$ and K acts diagonally. For hermitian symmetric spaces and $\rho = 0$, Theorem 1 has also been known [4].

The proof given here is quite different from that given for group manifolds in [2]. We show that the complex Monge-Ampère equation on $G^{\mathbb{C}}/K^{\mathbb{C}}$ reduces, for G-invariant functions, to a real Monge-Ampère equation on the dual symmetric space G^*/K. We also show that the Monge-Ampère operator on non-compact symmetric spaces has a radial part, i.e. it is equal, for K-invariant functions, to another Monge-Ampère operator on the maximal abelian subspace of G^*/K. These facts, together with the theorem on K-invariant real Monge-Ampère equations proved in [3], yield Theorem 1.

1. Riemannian symmetric spaces of non-compact type

Here we recall some facts about the geometry of Riemannian symmetric spaces. The standard reference for this section is [6].

Let $M = G/K$ be a symmetric space of compact type with K connected, and let G^*/K be its dual. If $\mathfrak{g}$, $\mathfrak{g}^*$ and $\mathfrak{k}$ denote the Lie algebras of G, G^* and

Received February 25, 2004.

[1] The complexification of a compact connected Lie group G is the connected group G^{c} whose Lie algebra is the complexification of the Lie algebra of G and which satisfies $\pi_1(G^{c}) = \pi_1(G)$.

K, then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, $\mathfrak{g}^* = \mathfrak{k} \oplus i\mathfrak{p}$, where $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$ and $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$. The restriction of the Killing form to $i\mathfrak{p}$ is positive definite and induces the Riemannian metric of G^*/K. Moreover, the Riemannian exponential mapping provides a diffeomorphism between $\mathfrak{p}$ and G^*/K. This can be viewed as the map:

$$p \mapsto e^{ip}K, \tag{1.1}$$

where $p \in \mathfrak{p}$ and e is the group-theoretic exponential map for G^*. Thus we have two K-invariant Riemannian metrics on $\mathfrak{p} \simeq \mathbb{R}^n$: the Euclidean one given by the Killing form, and the negatively curved one given by the diffeomorphism (1.1).

Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$ and $\mathfrak{l}$ its centraliser in $\mathfrak{k}$. Let Σ be the set of restricted roots and Σ^+ the set of restricted positive roots. For each $\alpha \in \Sigma$, let $\mathfrak{p}_\alpha$ (resp. $\mathfrak{k}_\alpha$) denote the subspace of $\mathfrak{p}$ (resp. of $\mathfrak{k}$) where each $(\operatorname{ad} H)^2$, $H \in \mathfrak{a}$, acts with eigenvalue $\alpha(H)^2$. We have the direct decompositions

$$\mathfrak{p} = \mathfrak{a} + \sum_{\alpha \in \Sigma^+} \mathfrak{p}_\alpha, \qquad \mathfrak{k} = \mathfrak{l} + \sum_{\alpha \in \Sigma^+} \mathfrak{k}_\alpha. \tag{1.2}$$

Let $\mathfrak{a}^+$ be an open Weyl chamber and let $\mathfrak{p}'$ be the union of K-orbits of points in $\mathfrak{a}^+$. Any K orbit in $\mathfrak{p}'$ is isomorphic to K/L where the Lie algebra of L is $\mathfrak{l}$. Moreover, we have the diffeomorphism:

$$\mathfrak{a}^+ \times K/L \to \mathfrak{p}', \qquad (h, k) \mapsto \operatorname{Ad}(k)h. \tag{1.3}$$

We now wish to write the two K-invariant metrics on $\mathfrak{p}$ in coordinates given by this diffeomorphism. Let $\sum dr_i^2$ be the Killing metric on $\mathfrak{a}^+$ (the r_i can be viewed as K-invariant functions on $\mathfrak{p}'$). For each $\mathfrak{k}_\alpha$, choose a basis $X_{\alpha,m}$ (m runs from 1 to twice the multiplicity of α) of vectors orthonormal for the Killing form and denote by $\theta_{\alpha,m}$ the corresponding basis of invariant 1-forms on K/L. We have

Proposition 1.1. *Let g_0 be the Euclidean metric on $\mathfrak{p}$, given by the restriction of the Killing form, and let g be the negatively curved symmetric metric on $\mathfrak{p}$ given by the diffeomorphism* (1.1). *Then, under the diffeomorphism* (1.3) *the metrics g_0 and g can be written in the form*

$$\sum_i dr_i^2 + \sum_{(\alpha,m)} F(\alpha(r))\theta^2_{(\alpha,m)}, \tag{1.4}$$

where $F(z) = z^2$ for g_0, and $F(z) = \sinh^2(z)$ for g.

Proof. Since all these metrics are K-invariant, it is enough to compute them at points of $\mathfrak{a}^+$. Let H be such a point and let (h, ρ), $h \in \mathfrak{a}$, $\rho \in T_{[1]}K/L$, be a tangent vector to $\mathfrak{a}^+ \times K/L$ at $(H, [1])$. The vector ρ can be identified with an element of $\sum \mathfrak{k}_\alpha \subset \mathfrak{k}$. The corresponding (under (1.3)) tangent vector at $H \in \mathfrak{p}'$ is $h + [\rho, H]$. Computing the Killing form of this vector yields the formula (1.4) with $F(z) = z^2$ for g_0. The formula for g follows from a similar computation, using the expression for the differential of the map (1.1) given in [6], Theorem IV.4.1. $\square$

2. Monge-Ampère equation on symmetric spaces

Let (M, g) be a Riemannian manifold and $u : M \to \mathbb{R}$ a smooth function. Then the Hessian of u is the symmetric $(0, 2)$-tensor Ddu where D is the Levi-Civita connection of g. In local coordinates x_i, Ddu is represented by the matrix

$$H_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j} - \sum_k \Gamma_{ij}^k \frac{\partial u}{\partial x_k}. \tag{2.1}$$

We say that the function u is g-convex (resp. strictly g-convex), if Ddu is non-negative (resp. positive) definite. The Monge-Ampère equation on the manifold (M, g) is then

$$\mathbf{M}_g(u) := (\det g)^{-1} \det Ddu = f \tag{2.2}$$

where f is a given function.

Let $(G^*/K, g)$ be a symmetric space of non-compact type given by a Cartan decomposition $\mathfrak{g}^* = \mathfrak{k} + i\mathfrak{p}$. As in the previous section, we identify $M = G^*/K$ with $\mathfrak{p}$ and denote by g_0 the (flat) metric given by restricting the Killing form to $\mathfrak{p}$. We have:

Theorem 2.1. *Let $M \simeq \mathfrak{p}$ be a symmetric space of noncompact type and let u be a K-invariant (smooth) function on M. Then*

(1) *u is g-convex if and only if u is g_0-convex (i.e. convex in the usual sense on $\mathfrak{p}$).*

(2) *The following equality of Monge-Ampère operators holds:*

$$\mathbf{M}_g(u) = F \cdot \mathbf{M}_{g_0}(u),$$

where $F : M \to \mathbb{R}$ is a positive K-invariant smooth function depending only on M.

We have proved in [3] a theorem on the existence and regularity of K-invariant solutions to Monge-Ampère equations on $\mathbb{R}^n$. From this we immediately obtain

Corollary 2.2. *Let $(G^*/K, g)$ be an irreducible symmetric space of noncompact type and let f be a positive smooth K-invariant function on G^*/K. Then the Monge-Ampère equation* (2.2) *has a global smooth K-invariant strictly g-convex solution.* □

We shall now prove Theorem 2.1. In fact we shall prove it in the following, more general situation. Suppose that we are given a K-invariant metric on $\mathfrak{p}$ whose pullback under (1.3) can be written as (cf. (1.4)):

$$\sum_i dr_i^2 + \sum_{(\alpha,m)} F_{(\alpha,m)}(\alpha(r))\theta_{(\alpha,m)}^2, \tag{2.3}$$

where $F_{(\alpha,m)} : \mathbb{R} \to \mathbb{R}$ are smooth functions vanishing at the origin such that $z^{-1}\frac{dF(\alpha,m)}{dz}$ is smooth and positive everywhere. Proposition 1.1 implies that the symmetric metric on G^*/K is of this form. We claim that Theorem 2.1 holds for any metric g of the form (2.3).

In order to simplify the notation, let us write j for the index (α, m) and α_j for α if $j = (\alpha, m)$. The metric g can be now written as

$$\sum_i dr_i^2 + \sum_j F_j(\alpha_j(r))\theta_j^2.$$

We recall the following formula:

$$2Ddu = L_{\nabla u} g,$$

where L is the Lie derivative and ∇u is the gradient of u with respect to the metric g. On the other hand, for any $(0, 2)$-tensor g and vector fields X, Y, Z, we have:

$$(L_X g)(Y, Z) = X.g(Y, Z) - g([X, Y], Z) - g(Y, [X, Z]).$$

We now compute $L_{\nabla u} g$ on $\mathfrak{p}'$ with respect to the basis vector fields $\partial/\partial r_i$, X_j, where X_j are dual to θ_j. Here u is a K-invariant function. The gradient of u is just $\sum \frac{\partial u}{\partial r_i} \frac{\partial}{\partial r_i}$, in particular it is independent of the functions F_j. It follows immediately that $(L_{\nabla u} g)(\partial/\partial r_i, X_j) = 0$ and that the matrix $(L_{\nabla u} g)(X_j, X_k)$ is equal to $\nabla u.g(X_j, X_k)$ and hence it is diagonal with the (jj)-entry equal to

$$\nabla u \left(F_j(\alpha_j(r))\right) = \frac{dF_j}{dz}_{|_{z=\alpha_j(r)}} \alpha_j(\nabla_0 \bar{u}).$$

Here $\nabla_0 \bar{u} = \sum \frac{\partial u}{\partial r_i} \frac{\partial}{\partial r_i}$ is the gradient of u restricted to the Euclidean space $\mathfrak{a} = \mathbb{R}^n$ in coordinates r_i, and viewed as a map from $\mathbb{R}^n$ to itself.

Theorem 2.1 with the more general metric (2.3) follows easily with the function F given explicitly by

$$F = \frac{\prod \alpha_j(r)}{\prod F_j(\alpha_j(r))} \prod \left(\frac{1}{2} \frac{dF_j}{dz} \right)_{z=\alpha_j(r)}.$$

Observe that the assumptions on the F_j guarantee that F extends to a smooth positive function on $\mathfrak{p}$.

3. Proof of the Main Theorem

Let (M, g) be a Riemannian symmetric space of compact type, G its isometry group, and $K \subset G$ the stabiliser group of a point. There is a canonical isomorphism between $G^{\mathbb{C}}/K^{\mathbb{C}}$ and $G \times_K \mathfrak{p}$ (i.e. the tangent bundle of G/K) given by the map:

$$G \times \mathfrak{p} \to G^{\mathbb{C}} \to G^{\mathbb{C}}/K^{\mathbb{C}}, \ \ (g, p) \mapsto g e^{ip}. \tag{3.1}$$

This isomorphism can be viewed in many ways: as an example of Mostow fibration [8], as given via Kähler reduction of $G^{\mathbb{C}} \simeq G \times \mathfrak{g}$ by the group K [5], or as given by the adapted complex structure construction [7] which provides a canonical diffeomorphism between the tangent bundle of G/K and a complexification of G/K. In any case it provides a fibration

$$\pi : G^{\mathbb{C}}/K^{\mathbb{C}} \to G/K. \tag{3.2}$$

The fibers of this projection can be identified with $\mathfrak{p}$ via the map (3.1). In particular, the fiber over [1] is given by the $K^{\mathbb{C}}$-orbits of elements $e^{ip}, p \in \mathfrak{p}$. We shall relate G-invariant plurisubharmonic functions on $G^{\mathbb{C}}/K^{\mathbb{C}}$ to convex functions on this fiber (see [1] for a different approach to this).

For a function w on a complex manifold one defines its Levy form Lw to be the Hermitian $(0,2)$ tensor given in local coordinates as

$$\frac{\partial^2 w}{\partial z_k \partial \bar{z}_l} dz_k \otimes d\bar{z}_l. \tag{3.3}$$

This form does not depend on the choice of local coordinates. We shall compute this form for a G-invariant function w on $G^{\mathbb{C}}/K^{\mathbb{C}}$. It is enough to compute it at points e^{ip}, $p \in \mathfrak{p}$. First of all, we choose local holomorphic coordinates at such a point:

Lemma 3.1. *In a neighbourhood of a point e^{ip}, $p \in \mathfrak{p}$, complex coordinates are provided by the map $\mathfrak{p}^{\mathbb{C}} \to G^{\mathbb{C}} \to G^{\mathbb{C}}/K^{\mathbb{C}}$, $(a+ib) \mapsto e^{a+ib}e^{ip}$.*

Proof. We have to show that the map $(a+ib) \mapsto e^{a+ib}e^{ip}K^{\mathbb{C}}$ has a non-singular differential at 0. This is equivalent to $\left(\operatorname{ad} e^{-ip}\right) u \notin \mathfrak{k}^{\mathbb{C}}$ for $u \in \mathfrak{p}^{\mathbb{C}}$. We have

$$\left(\operatorname{ad} e^{-ip}\right) u = e^{\operatorname{ad}(-ip)} u = \cosh\big(\operatorname{ad}(-ip)\big)u + \sinh\big(\operatorname{ad}(-ip)\big)u, \tag{3.4}$$

where the first term of the sum lies in $\mathfrak{p}^{\mathbb{C}}$ and the second one in $\mathfrak{k}^{\mathbb{C}}$. To show that the first term does not vanish recall that $(\operatorname{ad}(-ip))^2$ has all eigenvalues nonnegative. □

We now have:

Lemma 3.2. *In the complex coordinates $z = a+ib$ given by the previous lemma, the Levy form* (3.3) *of a G-invariant function w satisfies the equation:*

$$\left(\frac{\partial^2 w}{\partial z_k \partial \bar{z}_l}\right)_{\substack{a=0\\ b=0}} = \frac{1}{4}\frac{\partial^2}{\partial b_k \partial b_l} w\big(e^{ib}e^{ip}\big)_{b=0}. \tag{3.5}$$

Proof. The polar decomposition of $G^{\mathbb{C}}$ implies that e^{a+ib} can be uniquely written as ge^{iy}, where $g \in G$ and $y \in \mathfrak{g}$. Any G-invariant function on $G^{\mathbb{C}}/K^{\mathbb{C}}$ in a neighbourhood of e^{ip} is a function of y only. On the other hand, as $e^{2iy} = \left(e^x e^{iy}\right)^*\left(e^x e^{iy}\right) = e^{-a+ib}e^{a+ib}$, it follows from the Campbell-Hausdorff formula that $y = b + [b,a]/2 + \textit{higher order terms}$. Hence the matrix of second derivatives in (3.3) at e^{ip} (i.e. at $a=0, b=0$) is the same as the matrix of second derivatives of

$$(a,b) \mapsto e^{(ib+\frac{i}{2}[b,a])}e^{ip} \tag{3.6}$$

at $a=0, b=0$. We shall now show that for a G-invariant function w on $G^{\mathbb{C}}/K^{\mathbb{C}}$, this matrix of second derivatives is equal to the right-hand side of (3.5).

The Campbell-Hausdorff formula implies that up to order 2 in a, b, we have $e^{(ib+\frac{i}{2}[b,a])} = e^{ib}e^{\frac{i}{2}[b,a])}$. Set $c = [b,a]/2$, which is a point in $\mathfrak{k}$. We are going to show that modulo terms of order 2 in c (hence of order 4 in a, b), $e^{ic}e^{ip}$ is equal to $e^{\rho}e^{ip}e^{iq}$, where $\rho \in \mathfrak{g}$ and $q \in \mathfrak{k}$ are both linearly dependent on c. We note that this proves the lemma, as

$$e^{ib}e^{\rho}e^{ip}e^{iq} = e^{\rho}e^{ib+O(3)}e^{ip}e^{iq} = e^{\rho}e^{ib+O(3)}e^{ip}$$

in $G^{\mathbb{C}}/K^{\mathbb{C}}$, where $O(3)$ denotes terms of order 3 and higher in a, b.

We find q from the equation $\cosh \operatorname{ad}(ip)(q) = c$, which can be solved uniquely as $\cosh \operatorname{ad}(ip)$ is symmetric and positive-definite on $\mathfrak{k} \subset \mathfrak{g}$. We then put $\rho = -i \sinh \operatorname{ad}(ip)(q)$. We observe that $\rho \in \mathfrak{g}$ and $e^{\rho - ic} = e^{ip}e^{-iq}e^{-ip}$, thanks to (3.4). Moreover, modulo terms quadratic in c, $e^{\rho} = e^{ic}e^{\rho-ic}$ and, consequently:

$$e^{\rho}e^{ip}e^{iq} = e^{ic}e^{\rho-ic}e^{ip}e^{iq} = e^{ic}\big(e^{ip}e^{-iq}e^{-ip}\big)e^{ip}e^{iq} = e^{ic}e^{ip},$$

again modulo terms quadratic in c. This finishes the proof of the lemma. □

According to this lemma, we have to compute $\frac{\partial^2}{\partial b_k \partial b_l} w\big(e^{ib}e^{ip}\big)_{b=0}$. Now, since $e^{ib}e^{ip} \in G^*$, $e^{ib}e^{ip} = ke^{iz}$, where $z = z(b) \in \mathfrak{p}$ and $k \in K$. As w is G-invariant, $w\big(e^{ib}e^{ip}\big) = w(e^{iz})$ and therefore

$$\frac{\partial^2}{\partial b_k \partial b_l} w\big(e^{ib}e^{ip}\big)_{b=0} = \frac{\partial^2}{\partial b_k \partial b_l} w\big(e^{iz(b)}\big)_{b=0}.$$

Thus we compute the matrix of second derivatives of a function defined on $\exp(i\mathfrak{p})$ in the coordinates given by $b \mapsto e^{ib}e^{ip} \mapsto e^{iz(b)}$. These, however, are the geodesic coordinates at the point e^{ip} in the symmetric space dual to M (being translations of geodesics at [1]) and hence the matrix of second derivatives in these coordinates is equal to the Riemannian Hessian (2.1) for the symmetric metric on the dual space. If we assume that K is connected, then this dual space is G^*/K, and we obtain

Theorem 3.3. *Suppose that K is connected. Let w be a smooth G-invariant function on $X = G^{\mathbb{C}}/K^{\mathbb{C}}$ and let $\bar{w}$ be its restriction to the fiber $S = \exp(i\mathfrak{p})$ of* (3.2) *over* [1]. *Let g denote the symmetric metric on $S \simeq G^*/K$. Then w is (strictly) plurisubharmonic if and only if $\bar{w}$ is (strictly) g-convex. Moreover, the following equality holds:*

$$\partial\bar{\partial} \log \det Lw = \partial\bar{\partial} \log \widehat{\mathbf{M}_g(\bar{w})},$$

where $\hat{u} : X \to \mathbb{R}$ is a G-invariant function such that $\bar{\hat{u}}$ is a given K-invariant function u on S. □

We are now ready to prove Theorem 1. Recall that $X = G^{\mathbb{C}}/K^{\mathbb{C}}$ is a Stein manifold and so if ρ is an exact $(1,1)$ form on X, then $\rho = -i\partial\bar{\partial}h$ for some function h. If ρ is G-invariant, then we can assume that h is G-invariant. We can restrict h to the fiber S defined in the last theorem and thanks to Corollary 2.2 we can find a strictly g-convex K-invariant smooth solution $\bar{u}$ to the equation (2.2) with $f = e^h$, where the metric g is the symmetric metric on $S \simeq G^*/K$. We can extend this solution via G-action to a G-invariant function u on X. Theorem 3.3 implies now that u is strictly plurisubharmonic and that the Ricci form of the Kähler metric with potential u is ρ.

Acknowledgement

This work has been supported by an Advanced Research Fellowship from the Engineering and Physical Sciences Research Council of Great Britain. I thank Hassan Azad and the anonymous referee for helpful comments which resulted in an improved presentation.

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