Mathematical Research Letters **11**, 327–344 (2004)

ON THE GEOMETRY OF WEIL-PETERSSON COMPLETION OF TEICHMÜLLER SPACES

Sumio Yamada

Abstract. Given a surface of higher genus, we will look at the Weil-Petersson completion of the Teichmüller space of the surface, and will study the geometry induced by the Weil-Petersson distance functional. Although the completion is no longer a Riemannian manifold, it has characteristics similar to those of Cartan-Hadamard manifolds.

1. Introduction

It is well known [20] that the Weil-Petersson metric is not complete on the Teichmüller space over a closed surface of higher genus. When a Weil-Petersson geodesic cannot be further extended, a non-trivial closed geodesic shrinks in length (with respect to the hyperbolic metric) to zero, thus developing a node. Take the Weil-Petersson completion $\overline{\mathcal{T}}$ of the Teichmüller space $\mathcal{T}$. It was shown by Masur [14] that the Weil-Petersson metric extends to $\overline{\mathcal{T}}$. In this paper, we show that the space $(\overline{\mathcal{T}}, d)$ is an NPC (or CAT(0)) space in the sense of Toponogov [12], even though the distance function d induced by the Weil-Petersson metric is no longer smooth (with respect to geometric quantities such as the hyperbolic length of closed geodesics.) By construction, the mapping class group (Teichmüller modular group) acts isometrically on the Teichmüller space $\mathcal{T}$. One can extend the isometric action of the mapping class group to the completion $\overline{\mathcal{T}}$. It will be noted that the geometry of $\overline{\mathcal{T}}$ is closely related to the isometric actions of various subgroups of the mapping class group. Although $\overline{\mathcal{T}}$ is no longer a manifold, it still has many geometric characteristics shared with the so called Cartan-Hadamard manifolds; complete simply-connected manifolds with non-positive sectional curvature. The aim of this paper and its sequel is to rewrite the paper [3] of Lipman Bers' where he characterizes, after Thurston [16], the elements of mapping class group in terms of their translation distances with respect to the Teichmüller metric, only to replace the Teichmüller metric by the Weil-Petersson metric.

In Section **2** we will define and characterize the Weil-Petersson completion of the Teichmüller space for a closed surface of genus g. We will show that the space is NPC/CAT(0). Next in Section **3** we investigate the singular behavior of the Weil-Petersson metric tensor as the surface develops nodal singularities

Received September 18, 2003.
Supported in part by NSF DMS0222387.

by sharpening the results of H. Masur [14]. In the last section, the behaviors of Weil-Petersson geodesics are studied. In particular we observe that the copies of frontier Teichmüller spaces are imbedded totally geodesically inside $\overline{\mathcal{T}}$. Moreover it is noted that those sets are left invariant under the action of Dehn twists when the twist occurs around the nodes, regarded as degenerated simple closed curves.

This paper is motivated to provide a geometric approach to the subject of super/strong rigidity where lattices of Lie groups are represented in the mapping class group of a surface. As in the papers of Corlette [5], Gromov-Schoen [11], the rigidity questions can be transcribed into the study of equivariant harmonic maps into the NPC space on which the isometry group acts. In this approach, the negative curvature condition is crucial to controlling analytic properties of the harmonic maps. In the case of strong rigidity, the representation arises as the monodromy of some fibration where the fiber is the Riemann surfaces of varying conformal structures. The monodromy is created by existence of singular surfaces/fibers, or equivalently vanishing cycles. It should be noted that the super rigidity of lattices of rank two and higher in mapping class groups have been studied recently by Farb and Masur [9] via a group theoretic approach.

Also it should be pointed out that there has been much work done on so-called augmented Teichmüller space, and its mapping class group action on it (see [2] for example). One should note that the Weil-Petersson completion of a Teichmüller space can be identified with the augmented Teichmüller space set-theoretically.

The author wishes to thank G. Tian for originally suggesting to look at the geometry of moduli space behind the monodromy of Lefschitz fibration, which motivated this investigation. He also wishes to thank H. Masur, M. Wolf and S. Wolpert who have offered numerous suggestions and comments in the course of completing the paper.

The author would like to record that the set of results presented here have been available since 2000 in several versions, and they were as presented in April 2001 at a joint meeting of Pacific Northwest Geometry Seminar and Wasatch Topology hosted at University of Utah. It should be also remarked that there appeared since two preprints [6] and [22] which contain results partly motivated on the material presented in the previous versions of this paper.

2. Weil-Petersson completion of a Teichmüller space

Let Σ^2 be a closed (compact and without boundary) surface of genus g with $g > 1$. Denote the set of all smooth Riemannian metrics on Σ by $\mathcal{M}$. Denote the set of all hyperbolic metrics on Σ by $\mathcal{M}_{-1}$. Note that by the uniformization theorem, $\mathcal{M}_{-1}$ can be identified with the set of all conformal structures on Σ^2. Let $\mathcal{D}$ be the group of smooth orientation-preserving diffeomorphisms of Σ , and $\mathcal{D}_0$ the subgroup of diffeomorphisms homotopic to the identity map from a fixed Riemann surface $\tilde{\Sigma}$ (this gives markings to all the points in $\mathcal{M}_{-1}$.)

Define the Teichmüller space $\mathcal{T}_g$ of Σ to be

$$\mathcal{T}_g = \mathcal{M}_{-1}/\mathcal{D}_0.$$

Define the moduli space M_g of Σ to be

$$M_g = \mathcal{M}_{-1}/\mathcal{D}.$$

The discrete group $\mathcal{D}/\mathcal{D}_0$ is called the mapping class group, or the Teichmüller modular group. which we will denote by $\mathrm{Map}(\Sigma)$.

The space $\mathcal{M}$ of all Riemannian metrics has a natural L^2-metric defined by

$$< h, k >_{L^2(G)} = \int_N < h(x), k(x) >_{G(x)} d\mu_G(x)$$

where h and k are symmetric $(0,2)$-tensors, which belong to $T_G\mathcal{M}$. Knowing that $\mathcal{M}_{-1}$ is smoothly imbedded in $\mathcal{M}$ with the induced L^2-metric, and also that $\mathcal{M}_{-1} \to \mathcal{M}_{-1}/\mathcal{D}_0$ is a Riemannian submersion (see [10]), it makes sense to restrict the L^2-metric defined on $\mathcal{M}$ to $\mathcal{M}_{-1}/\mathcal{D}_0$. Thus the Teichmüller space has a L^2-inner product structure, and it is called *Weil-Petersson* metric. It should be noted that the Weil-Petersson cometric was introduced (Ahlfors [1]) as an L^2 pairing of two cotangent vectors, or equivalently two holomorphic quadratic differentials on the surface. It was then identified with the L^2 metric defined as above by Fischer and Tromba [10]. Recall the standard geometric fact [8] that any Weil-Petersson geodesic in $\mathcal{T}$ can be lifted horizontally once the initial point of the lift is specified, and the lift is then itself a geodesic in $\mathcal{M}_{-1}$ with respect to the L^2 metric. In what follows, we will not distinguish a Weil-Petersson geodesic in $\mathcal{T}$ and its horizontal lift in $\mathcal{M}_{-1}$ unless it is necessary. By construction each element of the mapping class group acts as a Weil-Petersson isometry. In [15] it was shown that the mapping class group is the full Weil-Petersson isometry group of the Teichmüller space.

With respect to this metric, the Teichmüller space $\mathcal{T}$ has non-positive sectional curvature (see Tromba [17] or Wolpert [19]) and though the metric is incomplete (Wolpert [20]) —not every Weil-Petersson geodesic can be extended indefinitely— $\mathcal{T}$ is still geodesically convex, that is, every pair of points can be joined by a unique length minimizing geodesic (Wolpert [18].) It is also known that the space is simply connected, diffeomorphic to the $6g-6$ dimensional Euclidean ball, where $g > 1$ is the genus of the surface Σ (see [17] for references.)

We will first show that the incompleteness is always caused by pinching of (at least) one neck of the Riemann surface. Since the proof (as presented in [17]) is short and elementary, we will include it here.

Proposition 1. *Suppose that $\sigma : [0,T) \to \mathcal{T}$, where $T < +\infty$ is a Weil-Petersson geodesic, which cannot be extended beyond T. Then for any sequence $\{t_n\}$ with $\lim t_n = T$, the hyperbolic length of the shortest closed geodesic(s) on $(\Sigma, \sigma(t_n))$ converges to zero.*

Proof. Suppose the contrary. Then there is some lower bound ε for the length of all closed geodesics in Σ on $\sigma([0,T))$. Then the compactness theorem of

Mumford and Mahler says that there exists a subsequence of $\{t_n\}$, which we denote by $\{t_n\}$ again, and a sequence of diffeomorphisms $\{\phi_n\}$ of Σ such that $\phi_n^*\sigma(t_n)$ converges to a hyperbolic metric G. Note $\phi_n^*\sigma$ is a horizontal lift of a Weil-Petersson geodesic defined on $(0, T]$ for each n. (Here we are using the fact that $\mathcal{M}_{-1} \to \mathcal{T}$ is a Riemannian submersion.)

In the meantime, the existence theorem of solutions to ordinary differential equation says that given G in the space $\mathcal{M}_{-1}$ of hyperbolic metrics, there exist an open neighborhood U of G and $\delta > 0$ such that any geodesic with an initial point G' in U is defined on $(-\delta, \delta)$.

Choose n sufficiently large so that $\phi_n^*\sigma(t_n)$ is in U, and $T - t_n < \delta/2$. Then the geodesic $\phi_n^*\sigma(t)$ can be extended to the interval $(t_n - \delta, t_n + \delta)$, which is a contradiction since $T < t_n + \delta$. □

Definition 1. *Let $\overline{\mathcal{T}}$ be the Weil-Petersson completion of the Teichmüller space of a Riemann surface of genus greater than one. Denote by $\partial\mathcal{T}$ the frontier set $\overline{\mathcal{T}}\backslash\mathcal{T}$.*

The preceding proposition states that every point in $\partial\mathcal{T}$ represent a nodal surface, that is, a surface with a node or equivalently a pinched neck. H. Masur has shown in [14] that $\partial\mathcal{T}$ consists of a union of Teichmüller spaces of topologically reduced Riemann surfaces, created by neck pinching as the conformal structure degenerates toward the frontier points. Masur also showed that the Weil-Petersson metric tensor of $\mathcal{T}$ restricted to the directions tangent to the frontier set $\partial\mathcal{T}$, spanned by the holomorphic quadratic differentials developing poles over the pinching neck, converges to the Weil-Petersson metric tensor of the Teichmüller space of the topologically reduced Riemann surface. In this sense the Weil-Petersson metric extends to $\overline{\mathcal{T}}$. The Weil-Petersson metric tensor evaluated in the directions spanned by holomorphic quadratic differentials with order two poles over the pinching neck, blows up at various rates (also in [14]), which we will carefully analyze in the following section.

Set-theoretically there is a natural stratification of the Weil-Petersson completion $\overline{\mathcal{T}}$ studied in the name of *augmented Teichmüller space* (See [2]). Let $\mathcal{S}$ be the equivalent classes of homotopically nontrivial simple closed curves on the Riemann surface Σ, two curves equivalent when there is an isotopic diffeomorphism sending one to the other. Denote by $\mathcal{T}_C$ the Teichmüller space of (or a product of Teichmüller spaces of) punctured Riemann surface(s) obtained by pinching a collection of *mutually disjoint* simple closed geodesics $C = \{c_i\}$ with $0 \leq i \leq 3g - 3$. Note that $3g - 3$ is the upper bound of the number of mutually disjoint simple closed geodesics on Σ of genus g. Then we have

$$\overline{\mathcal{T}} = \cup_{c \in \mathcal{S}} \overline{\mathcal{T}_c}$$

where $\mathcal{T}$ is denoted as $\mathcal{T}_\emptyset$. It should be noted that $\overline{\mathcal{T}}$ can be also seen as

$$\overline{\mathcal{T}} = \cup_{C \subset \mathcal{S}} \mathcal{T}_C$$

since we have the following set theoretic relation

$$\mathcal{T}_{C_1 \cup C_2} \subset \overline{\mathcal{T}_{C_1}} \cap \overline{\mathcal{T}_{C_2}}$$

provided $C_1 \cup C_2$ is a subset of $\mathcal{S}$ representing a collection of mutually disjoint simple closed geodesics.

Lastly in this section we prove the following theorem.

Theorem 1. *The Weil-Petersson completed Teichmüller space $\overline{\mathcal{T}}$ is an NPC space (or equivalently a CAT(0) space.)*

Remark NPC stands for "non-positively curved" as defined in [12]. It is a length space (X, d), in which any pair of points p and q can be connected by a rectifiable curve whose length realizes the distance $d(p, q)$, and in which any triangle satisfies the length comparison in the sense of Toponogov with a comparison triangle in $\mathbf{R}^2$.

Proof. The result (Corollary 3.11) cited in [4] says that the metric completion of an NPC space is an NPC space. The Teichmüller space equipped with the Weil-Petersson metric is an NPC space, since it is simply connected, non-positively curved, geodesically convex, open manifold as described above. Hence it follows that its Weil-Petersson metric completion $\overline{\mathcal{T}}$ is an NPC space. □

Remark When the statement of the theorem was first proved, the author was unaware of the general fact as it appears in [4]. The direct relevance of the fact in this context was first pointed out by B. Farb.

3. Singular Behavior of Weil-Petersson Metric

Consider the case where P in $\partial\mathcal{T}$ represents a Riemann surface Σ_0 with $p \leq 3g-3$ nodes. Suppose that this Σ_0 is obtained by pinching mutually disjoint closed geodesics c_i of a non-singular hyperbolic surface Σ (i.e. without nodes) of genus g to a point. It belongs to a copy of a Teichmüller space $\mathcal{T}_{\cup c_i}$ of a topological surface with p nodes (or equivalently a surface with p pairs of punctures a_i and b_i.) Now introduce a complex coordinate system, as demonstrated in [14], $t = (t_1, ..., t_p, t_{p+1}, ..., t_{3g-3})$ where the origin $0 \in \mathbf{C}^{3g-3}$ represents Σ_0, where $t_{p+1}, ...t_{3g-3}$ parametrize the Teichmüller space $\mathcal{T}_{\cup c_i}$ while $t_i,\ 1 \leq i \leq p$ is defined by local coordinates on the surface Σ_0 near the node N_i as follows.

At the node N_i with $1 \leq i \leq p$, Σ_0 has a neighborhood V_i isomorphic to $\{|z_i| < c, |w_i| < c, z_i w_i = 0\}$ in $\mathbf{C}^2$ for a sufficiently small $c < 1$. The isomorphism is given by local coordinate functions $F_i : U_i \to \mathbf{C}$ and $G_i : W_i \to \mathbf{C}$ where U_i and W_i are disjoint neighborhoods around the pair of points a_i and b_i respectively identified with the node N_i such that for p in U_i and q in W_i, we have $z_i = F_i(p), w_i = G_i(q)$ with $F_i(a_i) = G_i(b_i) = 0$, $\{|z_i| < c\} \subset F_i(U_i)$ and $\{|w_i| < c\} \subset G_i(W_i)$.

Recall the plumbing construction of a nodal surface([21].) Remove p pairs of discs $\{z_i : 0 < |z_i| < c^2 < 1\}$ and $\{w_i : 0 < |w_i| < c^2 < 1\}$ from Σ_0, and denote the surface thus obtained by $(\Sigma)^*_{c^2}$. Let t_i be a complex number so that $|t_i| < c^4$. Consider a model for a hyperboloid parametrized by t_i as follows,

$$\mathcal{V}_{i,c} = \{(z_i, w_i, t_i) : z_i w_i = t_i, |z_i|, |w_i| < c \text{ and } |t_i| < c^4\}$$

For a given t_i we glue $(\Sigma)^*_{c^2}$ to $\mathcal{V}_{i,c}$ by the maps $\hat{F}_i : F_i^{-1}\{c > |z_i| > c^2\} \to \mathcal{V}_{i,c}$ defined by $\hat{F}_i(p) = (F_i(p), t_i/F_i(p), t_i)$ and $\hat{G}_i : G_i^{-1}\{c > |w_i| > c^2\} \to \mathcal{V}_{i,c}$ defined by $\hat{G}_i(q) = (G_i(q), t_i/F_i(q), t_i)$. We denote by Σ_t the Riemann surface obtained by plumbing the p necks with $t = (t_1, ..., t_p)$. Now each node N_i have been replaced by a neck of size $|t_i|$. Given the complex structure of Σ_t, we will assume that Σ_t is uniformized, that is, equipped with the hyperbolic metric ds_t^2. As $|t_i| \to 0$, the surface Σ develops a node N_i, or equivalently a hyperbolic cusp.

Observe that by a pinching a closed geodesic c to a point, one can have two topologically distinct pictures depending on whether $[c]$ is homologically trivial or not. One is when the resulting surface Σ_0 has one path-connected component, with genus $g-1$ and with two punctures. The other is that the surface Σ_0 consists of two disconnected surfaces, of genus g_1 and g_2 with $g_1+g_2 = g$ and each surface has one puncture.

In the first case, the frontier component $\mathcal{T}_{c_1}$ is the Teichmüller space of surfaces of genus $g-1$ with two punctures. The complex dimension of $\mathcal{T}_{c_1}$ then is $3[(g-1)-1]+2 = 3g-3-1$, where the extra two complex dimensions is due to the freedom to choose the positioning of the two punctures.

In the second case, $\mathcal{T}_{c_1}$ is a product space of two Teichmüller spaces $\mathcal{T}^1_{c_1}$ and $\mathcal{T}^2_{c_1}$, where $\mathcal{T}^i_{c_1}$ represents the set of Riemann surfaces of genus g_i with one puncture. Then the dimension of the product space is

$$[3(g_1-1)+1]+[3(g_2-1)+1] = 3(g_1+g_2-1)-3+2 = 3g-3-1.$$

Hence in either case the dimension of the frontier Teichmüller space $\mathcal{T}_1$ is of complex codimension one. Similarly when Σ_0 has p nodes, the frontier component that parametrized the nodal surfaces is of complex codimension p. A neighborhood of Σ_0 in this frontier component is parametrized by linear combinations by a set of $3g-3-p$ Beltrami differentials ν_j with $p+1 \le j \le 3g-3$ where each ν_i is supported away from the neighborhoods of the nodes, denoted above by U_j, W_j with $1 \le j \le p$. In other words, $\{\nu_i\}$ are supported on $(\Sigma)^*_c$ where c is chosen above. In particular for $\nu(s) = \sum_{j=p+1}^{3g-3} s_j \partial/\partial t_j$ with s_j are sufficiently small, $\zeta^{\nu(s)} : \Sigma_0 \to \Sigma_{(0,s)}$ is a quasiconformal diffeomorphism satisfying the Beltrami equation $\overline{\partial}\zeta^{\nu(s)} = \nu(s)\partial\zeta^{\nu(s)}$. The fact that $\{\partial/\partial\tau_j\}$ with $p+1 \le j \le 3g-3$ are supported away from the nodes implies that the same set can be used to span a subspace of the tangent space at the point in $\overline{\mathcal{T}}$ representing the surface Σ_t with $t = (t_1, ..., t_p, 0, ..., 0)$

On the other hand the Beltrami differential $\partial/\partial t_i$ with $1 \le i \le p$ are given by Beltrami differentials, each of which is defined by the following one complex-parameter family of quasiconformal diffeomorphisms $\zeta^{|t'_i|} : A_{|t_i|} \to A_{|t'_i|}$ where

$\zeta^{|t_i'|}(z_i)$ is given by

$$\zeta^{|t_i'|}(z_i) = z_i|z_i|^{\alpha(|z_i|,t_i')}$$

where for every t_i', $\alpha(c, t_i') = 0$, and for each t_i' $\zeta^{|t_i'|}(t_i) = t_i'$. $\zeta^{|t_i'|}$ pushes in/out the inner circle $\{|z_i| = |t_i|\}$ of the annulus A_{t_i} onto $\{|z_i| = |t_i'|\}$ while keeping the outer circle $\{|z_i| = c\}$ intact. Differentiate $\zeta^{|t_i'|}(z_i)$ with respect to t_i' and evaluate $t_i' = t_i$, and then further differentiate by $\overline{z_i}$ to obtain the Beltrami differential

$$\frac{\partial}{\partial t_i}(z_i) = \frac{z_i}{2\overline{z_i}} \frac{\partial}{\partial \log |z_i|}\Big(\frac{\partial \alpha}{\partial t_1'}(|z_i|, t_1) \log |z_i|\Big) \frac{d\overline{z_i}}{dz_i}.$$

Note that $(t_1, ..., t_p, t_{p+1}, ..., t_{3g-3})$ provides a complex coordinate system for a *local manifold cover* of the compactified moduli space $\overline{M}_g$ as explained in [14] (p.625) and [21] (2.4C). For $1 \leq i \leq p$, t_i determines the surface up to the Dehn twists around c_i, and hence Σ_t is determined up to a product of Dehn twists about the curves $\{c_j\}_{j=1}^p$. It is important to recall the Weil-Petersson metric is invariant under the action of the mapping class group. In particular the action of the Dehn twists is isometric. Hence the coordinate system $t \in \mathbf{C}^{3p-3}$ may be used to fully describe the Weil-Petersson metric on $\overline{\mathcal{T}}$.

With respect to the coordinate system $(t_1, ..., t_p, t_{p+1}, ..., t_{3g-3})$ defined as above, H. Masur [14] showed that the Weil-Petersson Hermitian metric tensor blows up as $|t_i| \to 0$. In particular, it was shown that for $1 \leq i \leq p$

$$0 < \liminf_{t \to 0} |t_i|^2(-\log |t_i|)^3 G_{i\bar{i}} < \limsup_{t \to 0} |t_i|^2(-\log |t_i|)^3 G_{i\bar{i}} < C$$

where $t = 0 \in \mathbf{C}^{3g-3}$ represent the surface with the nodes $\{N_i\}_{i=1}^p$. On the other hand restricted to the directions represented by deformations supported away from the nodes, the metric converges to that of the frontier Teichmüller space;

$$\lim_{(t_1,...,t_p) \to 0} G_{i\bar{j}}(t) = G_{i\bar{j}}(0, ..., 0, t_{p+1}, ..., t_{3g-3})$$

We will refine Masur's result and show the following.

Proposition 2. *As $|t_i| \to 0$, that is, as the p nodes develop, one has the following description of the blowing up of the Weil-Petersson metric component.*

$$|G_{i\bar{i}}(t)| = \pi^3 \Big(1 + O(\sum_{j=1}^p (-\log |t_j|)^{-2})\Big) |t_i|^{-2}(-\log |t_i|)^{-3}.$$

Remark The exact value π^3 of the constant above is due to S. Wolpert [22].

Proposition 3. *The Weil-Petersson metric evaluated in the directions* $\{\partial/\partial t_i\}_{i=p+1}^{3g-3}$ *have the following convergence.*

$$G_{i\bar{j}}(t_1,...,t_p,t_{p+1},...,t_{3g-3}) = G_{i\bar{j}}(0,...,0,t_{p+1},...,t_{3g-3})\{1+O(\sum_{i=1}^{p}(-\log^{-2}|t_i|))\}$$

These results complement the following expansions in [14];

$$|G_{i\bar{j}}(t)| = O(|t_i|^{-1}|t_j|^{-1}(-\log^{-3}|t_i|)(-\log^{-3}|t_j|)) \text{ for } 1\le i,j\le p,\ i\ne j$$
$$|G_{i\bar{j}}(t)| = O(|t_i|^{-1}(-\log^{-3}|t_i|)) \text{ for } 1\le i\le p \text{ and } j>p$$

Proof of Proposition 2. We will first show that the i-th diagonal component of the Weil-Petersson cometric satisfy

$$G^{i\bar{i}}(t) = \frac{1}{\pi^3}(-\log^3|t_i|)|t_i|^2\Big(1+O(\sum_{k=1}^{p}(-\log|t_k|)^{-2})\Big).$$

for $1\le i\le p$.

We start by recalling Masur's construction [14] of regular 2-differentials as the dual basis for the set of Beltrami differentials introduced above. There is a natural pairing between Beltrami differentials and the holomorphic quadratic differentials over the i-th annulus, and one can check to see that for $z_i^\alpha dz_i^2$,

$$\int_{A_{|t_i|}}\Big(\frac{\partial}{\partial t_i}\Big)\Big(z_i^\alpha dz_i^2\Big) = \begin{cases} -\frac{\pi}{t_i} & \text{if } \alpha=-2 \\ 0 & \text{otherwise}\end{cases}$$

This suggests that the dual element of $\frac{\partial}{\partial t_i}$ in the cotangent space $T^*\mathcal{T}$ be of the form $\phi_i(z_i) = -\frac{t_i}{\pi}f(z_i)\Big(\frac{dz_i}{z_i}\Big)^2$, where $f(z_i)$ is holomorphic with $f(0)=1$. The Proposition 7.1 of [14] says that indeed this can be done so that there are 2-differentials $\phi_1(z,t),...,\phi_{3g-3}(z,t)$ on Σ_t which we identify with $dt_1,...,dt_{3g-3}$ using the fact that the pairings between the Beltrami differentials $\partial/\partial t_i$ and the regular 2-forms as described in 5.4 and 5.5 of [14] are by construction parametrized holomorphically in $t=(t_1,...,t_{3g-3})$. In particular, near each neck for $0<|t_i|<c$, $1\le i\le p$ we have the following descriptions of the differentials ϕ_i on Σ_t. For $1\le i,j\le p$

$$\phi_i(z_j,t) = -\frac{t_i}{\pi}\Big\{\delta_{ij}+f_1(z_j,t)+f_2(w_j,t)\Big\}\Big(\frac{dz_j}{z_j}\Big)^2$$

and for $k>p$

$$\phi_k(z_i,t) = \Big\{f_1(z_i,t)+f_2(w_i,t)\Big\}\Big(\frac{dz_i}{z_i}\Big)^2$$

where f_1 is holomorphic in z_i on $\{|z_i|<c\}$ and $f_1(0)=0$, and f_2 is holomorphic in w_i on $\{|w_i|<c\}$ and $f_2(0)=0$.

We are ready to calculate the Weil-Petersson cometric tensor with respect to the dual basis $\{dt_i\}_{i=1}^{3g-3}$ at a point in $\overline{\mathcal{T}}$ representing the surface Σ_t. First introduce a one-parameter family of approximately hyperbolic surfaces which models the development of the node as $|t_i|$ goes to zero.

We denote by $(\Sigma)^*_c$ a surface obtained by removing from Σ_0 p pairs of disjoint discs of radius c centered at a_i and b_i for each pair of punctures $\{a_i, b_i\}$. $(\Sigma)^*_c$ has a hyperbolic metric which is the restriction of the hyperbolic metric on Σ_0. By construction, the complement of $(\Sigma)^*_c$ in Σ_t is a union of annuli $A_{|t_i|} = \{z : |t_i|/c < |z_i| < c\}$ where $1 \leq i \leq p$, each of which we uniformize by the hyperbolic metric

$$d\omega^2_{|t_i|} = \Big(\frac{\pi}{\log|t_i|} \csc \frac{\pi \log|z_i|}{\log|t_i|} \Big|\frac{dz_i}{z_i}\Big|\Big)^2.$$

Hence we have a hyperbolic metric $d\omega^2_t$ on the disjoint union of $(\Sigma)^*_c$ and $A_{|t_i|}$. As the neck pinches ($|t_i| \to 0$), the hyperbolic metric on the annulus converges pointwise to the hyperbolic metric on two copies of the punctured disc $\{0 < |z| < c\}$;

$$ds^2_0 = \Big(\frac{|dz|}{|z|\log|z|}\Big)^2,$$

which models the standard hyperbolic cusp.

Now for the Riemann surface Σ_t, we have two conformally equivalent metrics: the hyperbolic metric ds^2_t uniformizing Σ_t and the approximate metric dw^2_t where the latter is possibly discontinuous across $\{|z_i| = c\}$. Note that when $t = 0$, the approximate metric $d\omega^2_0$ coincides with ds^2_0. In [21] (Expansion 4.2), Wolpert studied the explicit dependence of the hyperbolic metric $ds^2_{t_i}$ on t_i. In particular, it follows that

$$\Big|\frac{d\omega^2_t}{ds^2_t} - 1\Big| = O(\Sigma^p_{i=1}(-\log|t_i|)^{-2})$$

over Σ_t. Using the complex coordinate z_i over the neck, the hyperbolic metrics $d\omega^2_{|t_i|}$ and $ds^2_{t_i}$ are related by

$$ds^2_t = \rho^2_t(z_i)dz_i \otimes \overline{z}_i \text{ and } d\omega^2_{|t_i|} = \lambda^2_t(z_i)dz_i \otimes d\overline{z}_i.$$

Then the estimate above says that

$$\lambda_t/\rho_t = 1 + O(\sum_j^{3g-3}(-\log|t_j|)^{-2}).$$

The contribution of the Weil-Petersson pairing of $\phi_i = dt_i$ with itself over the i-th neck can be now written down in terms of λ_t;

$$\begin{aligned}
&\int_{A_{|t_i|}} \tfrac{|\phi_i|^2(z)}{\rho_t^2(z_i)} dx_i dy_i \\
&= (1+O(\textstyle\sum_{j=1}^p (-\log|t_j|)^{-2}) \int_{A_{|t_i|}} \tfrac{|\phi_i|^2(z)}{\lambda_t^2(z_i)} dx_i dy_i \\
&= \{1+O(\textstyle\sum_{j=1}^p (-\log|t_j|)^{-2})\}\pi^{-2}|t_i|^2 \int_{A_{|t_1|}} \tfrac{1+O(r_i^2)}{r_i^4} \sin^2\left(\tfrac{\pi(-\log r_i)}{(-\log|t_i|)}\right) r_i dr_i d\theta_i \\
&= |t_i|^2 \tfrac{2}{\pi^3}\{1+O(\textstyle\sum_{j=1}^p (-\log|t_j|)^{-2})\}(-\log^3|t_i|) \int_{\log(|t_i|/c)/\log|t_i|}^{\log c/\log|t_i|} \sin^2\pi\mu d\mu \\
&= |t_i|^2\left(\tfrac{1}{\pi^3}\{1+O(\textstyle\sum_{j=1}^p (-\log|t_j|)^{-2})\}(-\log^3|t_i|) + O(1)\right) \\
&= |t_i|^2 \tfrac{1}{\pi^3}\{1+O(\textstyle\sum_{j=1}^p (-\log|t_j|)^{-2})\}(-\log^3|t_i|)
\end{aligned}$$

where $z_i = x_i + iy_i, r_i = |z_i|$ and $\mu = \log r_i/\log|t_i|$ and the term $O(1)$ on the second line from the last depends on the value of c.

The pairing of ϕ_i with itself over the j-th neck for $i \neq j$ is given by the integral

$$\begin{aligned}
&\int_{A_{|t_j|}} \tfrac{|\phi_i|^2(z)}{\rho_t^2(z_j)} dx_j dy_j \\
&= (1+O(\textstyle\sum_{l=1}^p (-\log|t_l|)^{-2}) \int_{A_{|t_j|}} \tfrac{|\phi_i|^2(z)}{\lambda_t^2(z_j)} dx_j dy_j \\
&= \{1+O(\textstyle\sum_{l=1}^p (-\log|t_l|)^{-2})\}\pi^{-2}|t_i|^2 \int_{A_{|t_j|}} \tfrac{1+O(r_j^2)}{r_j^2} \sin^2\left(\tfrac{\pi(-\log r_j)}{(-\log|t_j|)}\right) r_j dr_j d\theta_j \\
&= O(|t_i|^2)
\end{aligned}$$

As described in [14], one checks that on any compact set K in $(\Sigma)_c^*$, we have

$$\int_K \frac{|\phi_i|^2(z)}{\rho_t^2(z)} dxdy = O(\sum_{i=1}^p |t_i|^2).$$

Therefore the Weil-Petersson pairing of ϕ_i with itself over the entire surface Σ_t is dominated by the contribution from the i-th neck, and we have

$$\begin{aligned}
G^{i\bar{i}}(t) &= \int_{\Sigma_t} \tfrac{|\phi_i|^2(z)}{\rho^2(z)} dxdy \\
&= \tfrac{1}{\pi^3}|t_i|^2(-\log|t_i|)^3\{1+O(\textstyle\sum_{j=1}^p (-\log|t_j|)^{-2})\}
\end{aligned}$$

for $1 \leq i \leq p$.

The statement of the proposition follows by inverting the matrix $G^{i\bar{j}}$ as in the argument given by Masur [14]. Note that the diagonal terms $G^{i\bar{i}}$ for $1 \leq i \leq p$ vanish at faster rates than the off diagonal terms $G^{i\bar{k}}$ with $k \neq i$, while the square block $G^{m\bar{n}}$ with $m, n > p$ is a uniformly positive definite matrix. With those two observations in mind, one calculates the determinant and the cofactors to invert the matrix, to obtain the statement of the proposition.

□

Proof of Proposition 3. We will show that the components $G^{m\bar{n}}$ of the cometric for $m, n > p$ satisfy

$$|G^{m\overline{n}}(t_1, ..., t_p, t_{p+1}, ..., t_{3g-3}) - G^{m\overline{n}}(0, ..., 0, t_{p+1}, ..., t_{3g-3})| = O(\sum_{j=1}^{p}\{-\log|t_j|\}^{-2}).$$

The statement of the proposition then follows by inverting the matrix as above, this time with this refined description of $G^{m\overline{n}}(t)$.

Now $G^{m\overline{n}}$ is given by the Weil-Petersson pairing

$$\int_{\Sigma_t} \frac{\phi_m\overline{\phi_n}}{\rho_t^2} dxdy$$

where ϕ_m, and ϕ_n are two regular differentials with possible poles of order at most one over the shrinking neck as $|t| \to 0$.

As before, we consider the region containing the pinching neck and the rest separately. Let $A_{|t_i|}$ be the annulus $\{z : |t_i| < |z| < c\}$ in Σ_t and Let $(\Sigma_t)_c^*$ denote the complement of the p punctured discs of radius c in the nodal surface represented by $(0, ..., 0, t_{p+1}, ..., t_{3g-3})$. Then using the previously quoted estimate of Wolpert's,

$$\lambda_t/\rho_t = 1 + O(\sum_{j=1}^{3g-3}(-\log|t_j|)^{-2}).$$

Over the k-th neck we have

$$\begin{aligned} & \int_{A_{|t_k|}} \frac{\phi_m\overline{\phi_n}}{\rho_t^2} dx_k dy_k - \int_{A_{|t_k|}} \frac{\phi_m\overline{\phi_n}}{\lambda_t^2} dx_k dy_k \\ = & \int_{A_{|t_k|}} \phi_m\overline{\phi_n}\Big[\frac{1}{[1+O(\Sigma_{j=1}^{p}(-\log|t_j|)^{-2})]\lambda_t^2(z)} - \frac{1}{\lambda_t^2(z)}\Big] dx_k dy_k \\ = & O(\textstyle\sum_{j=1}^{p}(-\log|t_j|)^{-2}) \int_{A_{|t_k|}} \frac{\phi_m\overline{\phi_n}}{\lambda_t^2(z)} dx_k dy_k \\ = & O(\textstyle\sum_{j=1}^{p}(-\log|t_j|)^{-2}) \end{aligned}$$

The last equality follows from the fact that the part of the integrand $\phi_i\overline{\phi_j}$ is a term which, as $z_k \to 0$, blows up no faster than the rate of $1/|z_k|^2$, which in turn implies that the integral $\int_{A_{|t_k|}} \frac{\phi_m\overline{\phi_m}}{\rho_0^2(z)} dx_k dy_k$ is a term $O(1)$ as $(t_1, ..., t_p)$ goes to zero.

On any compact set K in $(\Sigma_t)_c^*$, that is away from the p necks, we have

$$\begin{aligned} \int_K \frac{\phi_m\overline{\phi_n}}{\rho_t^2} dxdy - \int_K \frac{\phi_m\overline{\phi_n}}{\lambda_t^2} dxdy &= O(\textstyle\sum_{j_1}^{p}\{-\log|t_j|\}^{-2}) \int_K \frac{\phi_m\overline{\phi_n}}{\rho_t^2(z)} dxdy \\ &= O(\textstyle\sum_{j=1}^{p}(-\log|t_j|)^{-2}) \end{aligned}$$

The last equality follows from the fact that the integrand of the previous line is continuous in z over K.

Combining those estimates, we see that the difference between

$$G^{m\overline{n}}(t_1, t_2, ...t_p, t_{p+1}, ...t_{3g-3})$$

and

$$G^{m\overline{n}}(0, ..., 0, t_{p+1}, ..., t_{3g-3})$$

is a term of $O(\sum_{j+1}^{p}(-\log|t_j|)^{-2})$.

$\square$

Now we perform the following change of variables;

$$t_j = |t_j|e^{i\theta_j}, \ \ \theta_j = \arg t_j \quad \text{and} \quad u_j = (-\log|t_j|)^{-1/2}$$

$$dt_j = e^{i\theta_j}d|t_j| + it_j d\theta_j, \quad d\overline{t_j} = e^{-i\theta_j}d|t_j| - i\overline{t_j}d\theta_j$$

$$\begin{aligned}\Re\Big[\tfrac{\pi^3}{|t_j|^2(-\log|t_j|)^3}dt_j\otimes d\overline{t_j}\Big] &= \tfrac{\pi^3}{|t_j|^2(-\log|t_j|)^3}\Big[(d|t_j|)^2 + |t_j|^2(d\theta_j)^2\Big]\\ &= \tfrac{\pi^3}{(-\log|t_j|)^3}\Big(\tfrac{d|t_j|}{|t_j|}\Big)^2 + \tfrac{\pi^3}{(-\log|t_j|)^3}(d\theta_j)^2\\ &= 4\pi^3\Big((du_j)^2 + \tfrac{1}{4}(u_j)^6(d\theta_j)^2\Big)\end{aligned}$$

where $\Re$ denotes the real part of the complex-valued tensor.

Then the Weil-Petersson Riemannian metric near the frontier point is written down as

$$\begin{aligned} & ds^2\\ = \ & 4\pi^3\Big(1+O((u_i)^4)\Big)\textstyle\sum_{i=1}^{p}\Big[du_i^2 + \tfrac{1}{4}(u_i)^6 d\theta_i^2\Big]\\ + \ & \textstyle\sum_{p<j\leq 3g-3}\Big(1+O((u_j)^4\Big)|dt_j|^2\\ + \ & \textstyle\sum_{1\leq i,\ j\leq p}\Big(\tilde{C_{ij}} + O((u_i)^4) + O((u_j)^4)\Big)(u_i)^3(u_j)^3\times\Big[\text{cross terms } du_i,\ du_j\Big]\\ + \ & \textstyle\sum_{k\leq p,\ l>p}\Big(\hat{C_{kl}} + O((u_k)^4)\Big)(u_k)^3\times\Big[\text{cross terms } du_k,\ dt_l \ (\text{or } d\overline{t_l})\ \Big]\\ + \ & \textstyle\sum_{1\leq i\leq p,\ p<j\leq 3g-3}\Big(\overline{C}_{ij} + O((u_i)^4)\Big)(u_i)^6\times\Big[\ \text{cross terms } d\theta_i, dt_j \ (\text{or } d\overline{t_j})\Big]\\ + \ & \textstyle\sum_{1\leq i,j\leq p}O((u_i)^6(u_j)^6)d\theta_i\otimes d\theta_j.\end{aligned}$$

One can see the almost product structure of the Weil-Petersson metric $G(t)$ near the nodal surface Σ_0 by rewriting the description above as

$$\begin{aligned}(1)\quad & G(t_1,...,t_p,t_{p+1},...,t_{3g-3})\\ & = G(\mathbf{0},t_{p+1},...,t_{3g-3}) + 4\pi^3(1+O(\|u\|^3))\Big[\sum_{j=1}^{p}du_j^2 + \frac{1}{4}(u^j)^6 d\theta_j^2\Big].\end{aligned}$$

where $\|u\| = (\sum_{j=1}^{p}u_j^2)^{1/2}$.

In a sequel to this paper, we will study the differentiability of the metric near the boundary, and hence obtain the singular behavior of the Levi-Civita connection there.

4. Geometry of the Frontier Set $\partial\mathcal{T}$

We start this section with a theorem which describes how each boundary component is embedded in $\partial\mathcal{T}$.

Theorem 2. *Each component of the boundary Teichmüller spaces is totally geodesic; that is, given any pair of points p and q in a Teichmüller space $\mathcal{T}_C$ representing a collection of nodal surfaces Σ_C obtained by pinching a collection C of mutually disjoint simple closed geodesics c_i of the nonsingular surface Σ, a length minimizing geodesic connecting p and q are totally contained in $\mathcal{T}_C$ and it is unique.*

Proof. Suppose $C = \cup_{i=1}^{|C|} c_i$. Let $l_{c_i}(x)$ be the hyperbolic length of the simple closed geodesic c_i with respect to the hyperbolic metric x on Σ. The domain of the functional $l_{c_i} : \mathcal{T} \to \mathbf{R}_{\geq 0} \cup \{\infty\}$ with its values in the extended positive real line can be continuously extended to $\overline{\mathcal{T}}$ from $\mathcal{T}$ by defining $l_{c_i}|_{\mathcal{T}_A} \equiv 0$ on $\mathcal{T}_A$ with $c_i \in A$ and $l_{c_i}|_{\mathcal{T}_A} \equiv \infty$ if A contains a simple closed curve a which intersects with c_i both represented as a simple closed geodesic on a nonsingular hyperbolic surface Σ.

Define a new functional $\mathcal{L}_C : \overline{\mathcal{T}} \to \mathbf{R}_{\geq 0} \cup \infty$ by

$$\mathcal{L}_C(x) = \sum_{i=1}^{|C|} l_{c_i}(x).$$

Note that $\mathcal{L}_C|_{\overline{\mathcal{T}}_C} \equiv 0$, hence that $\mathcal{L}_C(p) = \mathcal{L}_C(q) = 0$.

We now construct a length minimizing geodesic connecting p and q. Let $\{p_i\}$ and $\{q_i\}$ be Cauchy sequences in $\mathcal{T}$ converging to p and q respectively. Let $\sigma_i(t)$ be the unique length minimizing Weil-Petersson geodesic connecting $p_i = \sigma_i(0)$ and $q_i = \sigma_i(1)$. Note that σ_i lies entirely in $\mathcal{T}$ due to the geodesic convexity of $\mathcal{T}$ [18], and it realizes the Weil-Petersson distance between p_i and q_i. Then by the strictly negative sectional curvature of the Weil-Petersson metric on $\mathcal{T}$, we know that

$$d(\sigma_i(t), \sigma_j(t)) \leq \max\Big(d(p_i, p_j), d(q_i, q_j)\Big).$$

The right hand side of the inequality converges to zero, and hence it follows that $\sigma_i(t)$ converges to a point in $\overline{\mathcal{T}}$, which we call $\sigma(t)$.

We now claim that $\sigma(t)$ is a Weil-Petersson geodesic in $\overline{\mathcal{T}}$, that is, it minimizes the length among all the rectifiable curves connecting p and q.

Let us recall that for a harmonic map from a Riemannian domain into an NPC space as defined in [12] by introducing the energy functional as the norm of a functional/measure on the domain obtained from the pull-back distance functional. From now on we will replace the Weil-Petersson length $L(\sigma)$ of a path by the energy $E(\sigma)$ of the path, since in one dimension we have $L = E^{1/2}$ provided that the path is parametrized by the unit interval.

One of the properties of the energy thus defined is the lower semicontinuity. In particular, given the sequence of maps $\sigma_i : [0,1] \to \overline{\mathcal{T}}$ which converges to σ in L^2, that is

$$\lim_{i\to\infty} \int_0^1 d^2(\sigma_i(t), \sigma(t))dt = 0,$$

then we have the inequality

$$E(\sigma) \le \liminf_{i\to\infty} E(\sigma_i).$$

Recall $E(\sigma_i)^{1/2} = L(\sigma_i) = d(p_i, q_i)$. While the continuity of d on $\overline{\mathcal{T}}$ implies that $\lim_i d(p_i, q_i) = d(p, q)$. Putting together, we have

$$L(\sigma) = E(\sigma)^{1/2} \le d(p, q).$$

By the definition of the distance, we have the opposite inequality. Therefore we have shown that

$$L(\sigma) = d(p, q)$$

and hence that σ is a Weil-Petersson geodesic. Note by the NPC condition, it is also unique.

It is known that the length functional l_{c_i} is convex with respect to the Weil-Petersson metric on $\mathcal{T}$ (a result of S. Wolpert [18], see [23] for generalizations.) In particular the function $\mathcal{L}_C(\sigma_i(t))$ is convex in t. Recall the general fact that the supremum of a family of convex functions is itself convex. We apply this to the family $\{\mathcal{L}_C(\sigma_i(t))\}$. On the other hand $\mathcal{L}_C$ is a continuous functional defined on $\overline{\mathcal{T}}$ with its values in the extended real line and $\sigma_i(t)$ converges to $\sigma(t)$ pointwise. Combined together we have

$$\limsup_{i\to\infty} \mathcal{L}_C(\sigma_i(t)) = \lim_{i\to\infty} \mathcal{L}_C(\sigma_i(t)) = \mathcal{L}_C(\sigma(t))$$

is a convex function in $t \in [0, 1]$.

Consider the function $f(t) = \mathcal{L}_C(\sigma(t))$. Suppose that $M = \max f(t) > 0$. It follows from the convexity that $f(t) \equiv M > 0$, which contradicts with $f(0) = f(1) = 0$.

We have so far shown that $f(t) \equiv 0$, which then implies that σ lies in $\overline{\mathcal{T}}_C$ To see σ lies in $\mathcal{T}_C$, note that there exists a Weil-Petersson length realizing geodesic σ' connecting p and q lying entirely in the Teichmüller space $\mathcal{T}_C$ due to the Weil-Petersson geodesic convexity of the $\mathcal{T}_C$. Since given two points in an NPC/CAT(0) space X a length-realizing geodesic is unique (here we take X to be $\mathcal{T}_C$), we know that σ' coincides with σ.

□

Remark Note that in the preceding proof, when the points p and q are chosen to be in $\overline{\mathcal{T}}_C$ instead of $\mathcal{T}_C$ by using the same argument one can deduce the conclusion that *$\overline{\mathcal{T}}_C$ is totally geodesic in $\overline{\mathcal{T}}$.* We make a point here to state that both $\mathcal{T}_C$ and $\overline{\mathcal{T}}_C$ are totally geodesic in $\overline{\mathcal{T}}$, but only the latter is geodesically complete with respect to the induced Weil-Petersson distance function.

The next theorem had been essentially known in the context of geometry of Teichmüller space with respect to the Teichmüller distance function. In particular it is a consequence of a statement (Theorem 6) which appears in [2], due to the fact the Teichmüller distance dominates the Weil-Petersson distance. The proof is based on the fact that the Dehn twist can be arbitrarily localized in the presence of a pinching neck. The proof is presented here for the sake of completeness and also to make this idea of localizing the Dehn twist explicit utilizing the expansion in Section **3** of the Weil-Petersson metric tensor near the frontier sets.

It is of particular interest when one studies a local monodromy around a singular fiber (a nodal surface Σ_0.) (See for example papers of Matsumoto-Montesinos-Amilibia [13], Earle-Sipe [7])

Theorem 3. *Suppose that γ is a Dehn twist around a simple closed geodesic c in Σ. Let $\mathcal{T}_c$ be the Teichmüller space of the surface Σ_0 obtained by pinching c of non-singular surface Σ to a node. Then $\overline{\mathcal{T}}_c$ in $\overline{\mathcal{T}}$ is fixed by the action of γ.*

Remark Suppose that C is a collection of mutually nonintersecting simple closed geodesics c_i on Σ, and let γ_i is the Dehn twist along c_i. Then the group generated by γ_i is a free abelian subgroup G_C of the mapping class group. Note that each γ_i fixes $\overline{\mathcal{T}}_{c_i}$, and hence that the set $\cap_i \overline{\mathcal{T}}_{c_i} = \overline{\mathcal{T}}_C \subset \overline{\mathcal{T}}$ is fixed by the action of the subgroup G_C.

Proof. Suppose that γ is a Dehn twist around a closed geodesic c on Σ. Suppose Σ_0 is a Riemann surface with at least one node N which is obtained by pinching the closed geodesic c.

Let t_1 be the plumbing parameter for the curve c. Then according to the almost-product structure (1) of the Weil-Petersson metric tensor near the frontier sets, we have

$$G(t_1, ..., t_p, t_{p+1}, ..., t_{3g-3}) = 4\pi^3(1 + O(\|u\|^3))(du_1^2 + \frac{1}{4}(u^1)^6 d\theta_1^2)$$

$$+ \Big\{G(\mathbf{0}, t_{p+1}, ..., t_{3g-3}) + 4\pi^3(1 + O(\|u\|^3))\Big[\sum_{j=2}^{p} du_j^2 + \frac{1}{4}(u^j)^6 d\theta_j^2\Big]\Big\}, \tag{2}$$

where we use the following coordinates as before,

$$t_j = |t_j|e^{i\theta_j}, \;\; \theta_j = \arg t_j, \;\; u_j = (-\log|t_j|)^{-1/2} \text{ and } \|u\| = (\sum_{j=1}^{p} u_j^2)^{1/2}.$$

As was explained in a paragraph preceding **Proposition 2**, the coordinates $(t_1, ..., t_p)$ parametrized the surface Σ_t only up to a product of Dehn twists about the curves c_i. In particular the cyclic subgroup generated by the Dehn twist γ around $c = c_1$ acts as covering transformations on the universal covering space of the complement of $\{t_1 = 0\}$ by

$$\gamma : (|t_1| + i\theta_1, ..., t_{3g-3}) \mapsto (|t_1| + i(\theta_1 + 2\pi), ..., t_{3g-3}).$$

Given a point $x = (t_1, ..., t_{3g-3})$ near a point y in the frontier set $\overline{\mathcal{T}}_c$, consider a path $\sigma(s)$ connecting x and γx defined by

$$\sigma(s) = (|t_1| + i(\theta_1 + s), ..., t_{3g-3}).$$

for $0 \leq s \leq 2\pi$. Then using the almost-product structure (2) of the Weil-Petersson metric tensor, the Weil-Petersson length of the path σ is computed to be $O(|t_1|^3)$, which in turn gives an upper bound on the Weil-Petersson distance between x and γx. Therefore for the y in $\overline{\mathcal{T}}_c$ with $|t_1|(y) = 0$, we have

$$d(y, \gamma y) = \lim_{x \to y} d(x, \gamma x) = 0.$$

□

The previous two theorems says that $\overline{\mathcal{T}}_C$ is a convex subset of the NPC space $\overline{\mathcal{T}}$ which is fixed by products of the Dehn twists γ_{c_i} with $c_i \in C$. Recall that a convex subset S, which is complete with respect to the induced metric within a Cartan-Hadamard manifold has a globally defined projection map π_S where $\pi_S(x)$ is defined to be the nearest point to x in S. This can be generalized in the NPC setting, as in [4] (Proposition 2.4.) In particular, given a collection C of mutually disjoint simple closed homotopically nontrivial curves c_i, $|C| \leq 3g-3$, the subset $\overline{\mathcal{T}}_C$ is a convex subset of the NPC space $\overline{\mathcal{T}}$, which is geodesically complete in the induced Weil-Petersson distance function. Hence it enjoys the following properties [4];

1. For every $x \in \overline{\mathcal{T}}$, there exists a unique point $\pi_C(x) \in \overline{\mathcal{T}}$ such that $d(x, \pi_C(x)) = d(x, \overline{\mathcal{T}}_C) := \inf_{y \in \overline{\mathcal{T}}_C} d(x, y)$.
2. If x' belongs to the geodesic segment connecting x and $\pi_C(x)$, then $\pi_C(x') = \pi_C(x)$.
3. Given $x \notin \overline{\mathcal{T}}_C$ and $y \in \overline{\mathcal{T}}_C$, if $y \neq \pi_C(x)$ then the Alexandrov angle (as defined in [4]) $\angle_{\pi_C(x)}(x, y) \geq \pi/2$.
4. The map π_C is a retraction of $\overline{\mathcal{T}}$ onto $\overline{\mathcal{T}}_C$ which does not increase distances, that is $d(x, y) \geq d(\pi_C(x), \pi_C(y))$ for any x and y. Furthermore, the map $H : \overline{\mathcal{T}} \times [0, 1] \to \overline{\mathcal{T}}$ associating to (x, t) the point a distance $td(x, \pi_C(x)$ from x on the geodesic $[x, \pi_C(x)]$ is a continuous homotopy from the identity map of $\overline{\mathcal{T}}$ to π.

Lastly we point out the following observation. It has come to the author's attention thanks to a conversation with M. Bestvina in April of 2001.

Given a surface of genus $g > 1$, by pinching certain nodes it becomes a disjoint union of $\lfloor g/2 \rfloor - 1$ four-times punctures spheres and $\lfloor g \rfloor$ once-punctured tori. The Teichmüller space of a four-times punctured sphere, as well as that of the once-punctured torus are complex one dimensional. Any further pinching of a neck of those two hyperbolic surfaces would produce a product of three punctured spheres, whose Teichmüller space is trivial. Hence $g + (\lfloor g/2 \rfloor - 1)$ can be regarded as the maximal number of copies of nontrivial Teichmüller spaces appearing as a factor of the product space $\mathcal{T}_C$.

Suppose C is a collection of homotopically nontrivial simple closed curves represented by mutually disjoint simple closed geodesics on a uniformized surface Σ of genus g, such that $\mathcal{T}_C$ is a direct product of $\lfloor g/2 \rfloor - 1$ copies of the Teichmüller space for four-times punctures sphere and $\lfloor g \rfloor$ copies of Teichmüller space for once-punctured torus;

$$\mathcal{T}_C = \mathcal{T}_1 \times \cdots \times \mathcal{T}_{g+\lfloor g/2 \rfloor -1}.$$

Let $\sigma_i : [0,1] \to \mathcal{T}_i$ be a nontrivial Weil-Petersson geodesic for each i, $1 \leq i \leq g + \lfloor g/2 \rfloor - 1$. Then the map

$$I : \prod_i^{g+\lfloor g/2 \rfloor -1} [0,1] \to \mathcal{T}_C$$

defined by

$$I(t_1, ..., t_{g+\lfloor g/2 \rfloor -1}) = \Big(\sigma_1(t_1), \cdots, \sigma_{g+\lfloor g/2 \rfloor -1}(t_{g+\lfloor g/2 \rfloor -1})\Big)$$

is clearly an isometric imbedding of the $g + \lfloor g/2 \rfloor - 1$ dimensional locally Euclidean space.

Therefore we conclude;

Theorem 4. *There exists a locally Euclidean isometric embedding of dimension* $g + \lfloor g/2 \rfloor - 1$ *in* $\overline{\mathcal{T}}$.

We remark here that for any Teichmüller space $\mathcal{T}$ of complex dimension at least one, the sectional curvature of Weil-Petersson metric is bounded above by zero, hence there is no flat in $\mathcal{T}$ apart from the geodesics.

References

[1] L. Ahlfors, *Some remarks on Teichmüller's space of Riemann surfaces.* Ann. of Math. **74** 1961, 171–191.

[2] W. Abikoff, *Degenerating families of Riemann surfaces.* Ann. of Math. **105** 1977, 29–44.

[3] L. Bers, *An extremal problem for quasiconformal mappings and a theorem by Thurston.* Acta Math. **141** 1978, 73–98.

[4] M. Bridson, A. Haefliger, *Metric spaces of non-positive curvature* Springer, 1999.

[5] K. Corlette, *Archimedian superrigidity and hyperbolic geometry.* Ann. of Math. **135** 1992, 165–182

[6] G. Daskalopoulos, R. Wentworth, *Classification of Weil-Petersson isometires.* Preprint 2001

[7] C. Earle, P. Sipe, *Families of Riemann surfaces over the punctures disk.* Pacific J. of Math. **150** (1991) 79–96.

[8] S. Gallot, D. Hulin, J. Lafontaine, *Riemannian Geometry.* Springer-Verlag, 1993.

[9] B. Farb and H. Masur, *Superrigidity and mapping class groups.* Topology, **37** 1998, 1169–1176.

[10] A. Fischer, A. Tromba, *On the Weil-Petersson Metric on the Teichmüller Space.* Trans. AMS **42** (1975), 319–335.

[11] M. Gromov and R. Schoen, *Harmonic mappings into singular spaces and p-adic superrigidity for lattices in groups of rank one.* Publ. IHES **76** (1992) 165–246.

[12] N. Korevaar and R. Schoen, *Sobolev spaces and harmonic maps for metric target spaces,* Comm. Anal. Geom. **1** (1993), 561–659.

[13] Y. Matsumoto and J. Montesinos-Amilibia, *Pseudo-periodic homeomorphisms and degeneration of Riemann surfaces.* Bull. Amer. Math. Soc. **30** 1994, 70–75.

[14] H. Masur, *The extension of the Weil-Petersson metric to the boundary of the Teichmüller space.* Duke Math. J. **43** (1976), 623–635.

[15] H. Masur, M. Wolf, *The Weil-Petersson isometry group*. Geom. Dedicata, **93** (2002) 177–190.

[16] W. Thurston, *On the geometry and dynamics of diffeomorphisms of surfaces.* Bull. A.M.S. **19** (1988) 417–431.

[17] A. Tromba, *Teichmüller Theory in Riemannian Geometry*. Birkhäuser, 1992.

[18] S. Wolpert, *Geodesic Length Functions and the Nielsen Problem.* J. Diff. Geom. **25** (1987), 275–295.

[19] ———, *Chern forms and the Riemann tensor for the moduli space of curves,* Invent. math. **85** (1986), 119–145.

[20] ———, *Noncompleteness of the Weil-Petersson Metric for Teichmüller Space.* Pacific J. Math. **61** (1975), 573–577.

[21] ———, *The hyperbolic metric and the geometry of the universal curve.* J. Differential Geometry **31** (1990) 417–472.

[22] ———, *Geometry of the Weil-Petersson completion of Teichmüller space.* Preprint 2002.

[23] S. Yamada, *Weil-Petersson convexity of the energy functional on classical and universal Teichmüller spaces.* J. Differential Geom. **51** (1999) 35–96.

E-mail address: yamada@math.uab.edu