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THE BOUNDARY DISTANCE FUNCTION AND THE DIRICHLET-TO-NEUMANN MAP

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Abstract. We outline the proof that two dimensional simple Riemannian manifolds with boundary are boundary distance rigid. In addition we give, in two dimensions, a reconstruction procedure to recover the index of refraction of a bounded medium in Euclidean space from the travel times of sound waves going through the medium.

1. Introduction and statement of the results

Let (M, g) be a compact Riemannian manifold with boundary ∂M. Let $d_g(x, y)$ denote the geodesic distance between x and y. The inverse problem we address in this paper is whether we can determine the Riemannian metric g knowing $d_g(x, y)$ for any $x \in \partial M$, $y \in \partial M$. This problem arose in rigidity questions in Riemannian geometry [M], [C], [Gr]. For the case in which M is a bounded domain of Euclidean space and the metric is conformal to the Euclidean one, this problem is known as the inverse kinematic problem which arose in Geophysics and has a long history (see for instance [R] and the references cited there) and section 5 of this paper.

The metric g cannot be determined from this information alone. We have $d_{\psi^* g} = d_g$ for any diffeomorphism $\psi : M \to M$ that leaves the boundary pointwise fixed, i.e., $\psi|_{\partial M} = Id$, where Id denotes the identity map and $\psi^* g$ is the pull-back of the metric g. The natural question is whether this is the only obstruction to unique identifiability of the metric. It is easy to see that this is not the case. Namely one can construct a metric g and find a point x_0 in M so that $d_g(x_0, \partial M) > \sup_{x,y\in\partial M} d_g(x, y)$. For such a metric, d_g is independent of a change of g in a neighborhood of x_0. The hemisphere of the round sphere is another example.

Therefore it is necessary to impose some a-priori restrictions on the metric. One such restriction is to assume that the Riemannian manifold is **simple**, i.e., given two points there is a unique geodesic joining the points and ∂M is strictly convex. ∂M is strictly convex if the second fundamental form of the boundary is positive definite in every boundary point.

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R. Michel conjectured in [M] that simple manifolds are boundary distance rigid that is d_g determines g uniquely up to an isometry which is the identity on the boundary. This is known for simple subspaces of Euclidean space (see [Gr]), simple subspaces of an open hemisphere in two dimensions (see [M]), simple subspaces of symmetric spaces of constant negative curvature [BCG], simple two dimensional spaces of negative curvature (see [C1] or [O]). We remark that simplicity of a compact manifold with boundary can be determined from the boundary distance function.

In this article we announce the proof of the conjecture in two dimensions. The details will appear in [PU].

Theorem 1.1. *Let $(M, g_i), i = 1, 2,$ be two Riemannian metrics on a compact, simple Riemannian manifold with boundary. Assume*

$$d_{g_1}(x, y) = d_{g_2}(x, y) \quad \forall (x, y) \in \partial M \times \partial M$$

then there exists a diffeomorphism $\psi : M \to M$, $\psi|_{\partial M} = Id$, so that

$$g_2 = \psi^* g_1.$$

As was pointed out in [I] Theorem 1.1 together with the results of [I] implies the following

Theorem 1.2. *Let (M, g_1) be a compact simple Riemannian manifold and g_2 another metric on M such that $d_{g_1}(x, y) \geq d_{g_2}(x, y)$ for all x and y in the boundary. Then Area $(g_1) \geq$ Area (g_2) with equality in area implying the isometry of g_1 and g_2.*

The function d_g measures the first arrival time of geodesics joining points of the boundary. In the case that both g_1 and g_2 are conformal to the Euclidean metric e (i.e., $(g_k)_{ij} = \alpha_k \delta_{ij}$, $k = 1, 2$ with δ_{ij} the Kronecker symbol), as mentioned earlier, the problem we are considering here is known in seismology as the inverse kinematic problem. In this case, it has been proven by Mukhometov in two dimensions [Mu] that if $(M, g_i), i = 1, 2$ is simple and $d_{g_1} = d_{g_2}$, then $g_1 = g_2$.

More generally the same method of proof shows that if $(M, g_i), i = 1, 2,$ are simple compact Riemannian manifolds with boundary and they are in the same conformal class then the metrics are determined by the boundary distance function. More precisely we have:

Theorem 1.3. *Let $(M, g_i), i = 1, 2$ be a two dimensional simple Riemannian compact Riemannian manifold. Assume $g_1 = \rho g_2$ for a positive, smooth function ρ and $d_{g_1} = d_{g_2}$ then $g_1 = g_2$.*

This result and a stability estimate were proven in [Mu1]. We remark that in this case the diffeomorphism ψ must be the identity. For related results and generalizations see [B], [BG], [C], [GN], [MR].

As a consequence of the method of proof of Theorem 1.1 we derive a reconstruction formula for the conformal factor in the two dimensional case. More precisely we have:

Theorem 1.4. *Let (M, g) be a a bounded open set of Euclidean space with smooth boundary and (M, g) simple. Let ρ be a smooth positive function on M so that $(M, \rho g)$ is also simple. Then we develop a reconstruction procedure to recover ρ from $d_{\rho g}(x, y), x, y \in \partial M$.*

The proof of Theorem 1.1 involves a connection between the scattering relation and the Dirichlet-to-Neumann map (DN) associated to the Laplace-Beltrami operator. In section 2 we define the scattering relation. In section 3 we discuss the DN map. In section 4 we outline the proof that from the scattering relation one can determine the DN map in the two dimensional case. Finally in section 5 we give the proof of Theorem 1.4.

We would like to thank Christopher Croke for pointing out to us the reference [I]. We are also very grateful to him for several useful comments on a previous version of this paper.

2. The scattering relation

We mention a closely related inverse problem. Suppose we have a Riemannian metric in Euclidean space which is the Euclidean metric outside a compact set. The inverse scattering problem for metrics is to determine the Riemannian metric by measuring the scattering operator (see [G]). A similar obstruction occurs in this case with ψ equal to the identity outside a compact set. It was proven in [G] that from the wave front set of the scattering operator one can determine, under some non-trapping assumptions on the metric, the **scattering relation** on the boundary of a large ball. We proceed to define in more detail the scattering relation and its relation with the boundary distance function.

Let ν denote the unit-inner normal to ∂M. We denote by $\Omega(M) \to M$ the unit-sphere bundle over M:

$$\Omega(M) = \bigcup_{x \in M} \Omega_x, \quad \Omega_x = \{\xi \in T_x(M) : |\xi|_g = 1\}.$$

$\Omega(M)$ is a $(2 \dim M - 1)$-dimensional compact manifold with boundary, which can be written as the union $\partial\Omega(M) = \partial_+\Omega(M) \cup \partial_-\Omega(M)$

$$\partial_\pm \Omega(M) = \{(x, \xi) \in \partial\Omega(M),\ \pm(\nu(x), \xi) \geq 0\ \}.$$

The manifold of inner vectors $\partial_+\Omega(M)$ and outer vectors $\partial_-\Omega(M)$ intersect at the set of tangent vectors

$$\partial_0\Omega(M) = \{(x, \xi) \in \partial\Omega(M), \quad (\nu(x), \xi) = 0\ \}.$$

Let (M, g) be an n-dimensional compact manifold with boundary. We say that (M, g) is **non-trapping** if each maximal geodesic is finite. Let (M, g) be non-trapping and the boundary ∂M is strictly convex. Denote by $\tau(x, \xi)$ the length of the geodesic $\gamma(x, \xi, t), t \geq 0$, starting at the point x in the direction $\xi \in \Omega_x$. This function is smooth on $\Omega(M) \setminus \partial_0\Omega(M)$. The function $\tau^0 = \tau|_{\partial\Omega(M)}$

is equal zero on $\partial_-\Omega(M)$ and is smooth on $\partial_+\Omega(M)$. Its odd part with respect to ξ

$$\tau^0_-(x,\xi) = \frac{1}{2}\left(\tau^0(x,\xi) - \tau^0(x,-\xi)\right)$$

is a smooth function.

Definition 2.1. *Let (M,g) be non-trapping with strictly convex boundary. The scattering relation $\alpha : \partial\Omega(M) \to \partial\Omega(M)$ is defined by*

$$\alpha(x,\xi) = (\gamma(x,\xi,2\tau^0_-(x,\xi)), \dot\gamma(x,\xi,2\tau^0_-(x,\xi))).$$

The scattering relation is a diffeomorphism $\partial\Omega(M) \to \partial\Omega(M)$. Notice that $\alpha|_{\partial_+\Omega(M)} : \partial_+\Omega(M) \to \partial_-\Omega(M)$, $\alpha|_{\partial_-\Omega(M)} : \partial_-\Omega(M) \to \partial_+\Omega(M)$ are diffeomorphisms as well. Obviously, α is an involution, $\alpha^2 = id$ and $\partial_0\Omega(M)$ is the hypersurface of its fixed points, $\alpha(x,\xi) = (x,\xi)$, $(x,\xi) \in \partial_0\Omega(M)$.

A natural inverse problem is whether the scattering relation determines the metric g up to an isometry which is the identity on the boundary. This information takes into account all the travel times not just the first arrivals.

In the case that (M,g) is a simple manifold, and we know the metric at the boundary, knowing the scattering relation is equivalent to knowing the boundary distance function ([M]). The key to the proof of Theorem 1.1 is to show that if we know the scattering relation we can determine the DN map associated to the Laplace-Beltrami operator of the metric. We proceed to define the DN map.

3. The Dirichlet-to-Neumann Map

Let (M,g) be a compact Riemannian manifold with boundary. The Laplace-Beltrami operator associated to the metric g is given in local coordinates by

$$\Delta_g u = \frac{1}{\sqrt{\det g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(\sqrt{\det g}\, g^{ij} \frac{\partial u}{\partial x_j}\right)$$

where (g^{ij}) is the inverse of the metric g. Let us consider the Dirichlet problem

$$\Delta_g u = 0 \text{ on } M, \quad u\Big|_{\partial M} = f.$$

We define the DN map in this case by

$$\Lambda_g(f) = (\nu, \nabla u|_{\partial M})$$

The inverse problem is to recover g from Λ_g.

A similar obstruction holds for this problem as the one discussed in section 1. Namely

$$\Lambda_{\psi^* g} = \Lambda_g \tag{3.1}$$

where ψ is a C^∞ diffeomorphism of $\overline{M}$ which is the identity on the boundary.

In addition in the two dimensional case the Laplace-Beltrami operator is conformally invariant. More precisely

$$\Delta_{\beta g} = \frac{1}{\beta}\Delta_g$$

for any function β, $\beta \neq 0$. Therefore we have that for $n = 2$

$$\Lambda_{\beta(\psi^* g)} = \Lambda_g$$

for any non-zero β satisfying $\beta|_{\partial M} = 1$.

Therefore the best that one can do in two dimensions is to show that we can determine the conformal class of the metric g up to an isometry which is the identity on the boundary That this is the case is a result proven in [LeU] for simple metrics and for general connected two dimensional Riemannian manifolds with boundary in [LaU].

More precisely we have:

Theorem 3.1. *Let (M, g) be a connected, compact Riemannian surface with boundary. Then $(\Lambda_g, \partial M)$ determines uniquely the conformal class of (M, g).*

As it was shown in [LaU] it is enough to measure the DN map in an open subset of the boundary.

The connection in two dimensions between the DN map and the scattering relation is given by

Theorem 3.2. *Let $(M, g_i), i = 1, 2,$ be compact, simple two dimensional Riemannian manifolds with boundary. Assume that $\alpha_{g_1} = \alpha_{g_2}$. Then $\Lambda_{g_1} = \Lambda_{g_2}$.*

We discuss this result in more detail in the next section.

4. From the scattering relation to the Dirichlet to Neumann map

The proof of Theorem 1.1 is reduced then to the proof of Theorem 3.2. In fact from Theorem 3.2 and Theorem 3.1 we get that we can determine the conformal class of the metric up to an isometry which is the identity on the boundary. Now by Theorem 1.3 we have that the conformal factor must be one proving that the metrics are isometric via a diffeomorphism which is the identity at the boundary. In other words $d_{g_1} = d_{g_2}$ implies that $\alpha_{g_1} = \alpha_{g_2}$. By Theorem 3.2 $\Lambda_{g_1} = \Lambda_{g_2}$. By Theorem 3.1, there exists a diffeomorphism $\psi : M \longrightarrow M$, $\psi|_{\partial M} = \text{Identity}$ and a function $\beta \neq 0, \beta|_{\partial M} = \text{identity}$ such that $g_1 = \beta\psi^* g_2$. By Mukhometov's theorem $\beta = 1$ showing that $g_1 = \psi^* g_2$ proving Theorem 1.1.

Before starting the proof of Theorem 3.2 we recall that Michel [M1] has proven that for two dimensional manifolds Riemannian manifolds with strictly convex boundary one can determine from the boundary distance function, up to the natural obstruction, all the derivatives of the metric at the boundary. This result was generalized to any dimensions in [LSU]. More precisely we have

Theorem 4.1. *Let (M, g) be a connected Riemannian manifold with strictly convex boundary. Then the C^∞-jet of the metric g at the boundary is uniquely determined by the boundary distance function d_g in the following sense. If ∂M is strictly convex with respect to another metric g' on M, then the equality $d_g = d_{g'}$ implies the existence of a diffeomorphism $\varphi : M \to M$ which is the identity on the boundary, $\varphi|_{\partial M} = Id$, and such that the metrics g and $g'' = \varphi^* g'$ satisfy the* *following: In any local coordinate system $(x^1, \dots, x^n)$ defined in a neighborhood of a boundary point, we have $D^\alpha g|_{\partial M} = D^\alpha g''|_{\partial M}$ for every multi-index α.*

The proof of Theorem 3.2 consists in showing that from the scattering relation we can determine the traces at the boundary of conjugate harmonic functions, which is equivalent information to knowing the DN map associated to the Laplace-Beltrami operator. The steps to accomplish this are outlined below. It relies on a connection between the Hilbert transform and geodesic flow.

We embed (M, g) into a compact Riemannian manifold (S, g) with no boundary. Let φ_t be the geodesic flow on $\Omega(S)$ and $\mathcal{H} = \frac{d}{dt}\varphi_t|_{t=0}$ be the geodesic vector field. Introduce the map $\psi : \Omega(M) \to \partial_-\Omega(M)$ defined by

$$\psi(x, \xi) = \varphi_{\tau(x,\xi)}(x, \xi), \quad (x, \xi) \in \Omega(M).$$

The solution of the boundary value problem for the transport equation

$$\mathcal{H}u = 0, \quad u|_{\partial_+\Omega(M)} = w$$

can be written in the form

$$u = w_\psi = w \circ \alpha \circ \psi.$$

Let u^f be the solution of the boundary value problem

$$\mathcal{H}u = -f, \quad u|_{\partial_-\Omega(M)} = 0,$$

which we can write as

$$u^f(x, \xi) = \int_0^{\tau(x,\xi)} f(\varphi_t(x, \xi))dt, \quad (x, \xi) \in \Omega(M).$$

In particular

$$\mathcal{H}\tau = -1.$$

The trace

$$If = u^f|_{\partial_+\Omega(M)} \tag{4.2}$$

is called **the geodesic X-ray transform** of the function f. By the fundamental theorem of calculus we have

$$I\mathcal{H}f = (f \circ \alpha - f)|_{\partial_+\Omega(M)}. \tag{4.3}$$

In what follows we will consider the operator I acting only on functions that do not depend on ξ, unless otherwise indicated. Let $L^2_\mu(\partial_+\Omega(M))$ is the real Hilbert space, with scalar product given by

$$(u, v)_{L^2_\mu(\partial_+\Omega(M))} = \int_{\partial_+\Omega(M)} \mu uv d\Sigma, \quad \mu = (\xi, \nu).$$

Here the measure $d\Sigma = d(\partial M) \wedge d\Omega_x$ where $d(\partial M)$ is the induced volume form on the boundary by the standard measure on M and

$$d\Omega_x = \sum_{k=1}^n (-1)^{k+1} \xi^k d\xi^1 \wedge ... \wedge d\hat{\xi}^k \wedge ... \wedge d\xi^n.$$

As usual the scalar product in $L^2(M)$ is defined by

$$(u,v)=\int_M uv\sqrt{detg}dx.$$

The operator I is a bounded operator from $L^2(M)$ into $L^2_\mu(\partial_+\Omega(M))$. The adjoint $I^*:L^2_\mu(\partial_+\Omega(M))\to L^2(M)$ is given by

$$I^*w(x)=\int_{\Omega_x} w_\psi(x,\xi)d\Omega_x,$$

where $w_\psi=w\circ\alpha\circ\psi$.

We will study the solvability of equation $I^*w=h$ with smooth right hand side. Let $w\in C^\infty(\partial_+\Omega(M))$. Then the function w_ψ will not be smooth on $\Omega(M)$ in general. We have that $w_\psi\in C^\infty(\Omega(M)\setminus\partial_0\Omega(M))$. We give below necessary and sufficient conditions for smoothness of w_ψ on $\Omega(M)$.

We introduce the operators of even and odd continuation with respect to α:

$$A_\pm w(x,\xi)=w(x,\xi),\qquad (x,\xi)\in\partial_+\Omega\left(M\right),$$

$$A_\pm w(x,\xi)=\pm\left(\alpha^*w\right)(x,\xi),\quad (x,\xi)\in\partial_-\Omega\left(M\right).$$

The scattering relation preserves the measure $|(\xi,\nu)|d\Sigma$ and therefore the operators $A_\pm:L^2_\mu(\partial_+\Omega(M))\to L^2_{|\mu|}\left(\partial\Omega\left(M\right)\right)$ are bounded, where $L^2_{|\mu|}\left(\partial\Omega\left(M\right)\right)$ is real Hilbert space with scalar product

$$(u,v)_{L^2_{|\mu|}(\partial\Omega(M))}=\int\limits_{\partial\Omega(M)}|\mu|\,uvd\Sigma,\quad \mu=(\xi,\nu).$$

The adjoint of $A_\pm$ is a bounded operator $A^*_\pm:L^2_{|\mu|}\left(\partial\Omega\left(M\right)\right)\to L^2_\mu(\partial_+\Omega(M))$ given by

$$A^*_\pm u=(u\pm u\circ\alpha)|_{\partial_+\Omega(M)}.$$

Using A^*_- formula (4.3) can be written in the form

$$I\mathcal{H}f=-A^*_-f^0,\quad f^0=f|_{\partial\Omega(M)}. \tag{4.4}$$

The space $C^\infty_\alpha\left(\partial_+\Omega\left(M\right)\right)$ is defined by

$$C^\infty_\alpha\left(\partial_+\Omega\left(M\right)\right)=\{w\in C^\infty\left(\partial_+\Omega\left(M\right)\right):w_\psi\in C^\infty\left(\Omega\left(M\right)\right)\}.$$

We have the following characterization of the space of smooth solutions of the transport equation

Lemma 4.1.

$$C^\infty_\alpha(\partial_+\Omega(M))=\{w\in C^\infty(\partial_+\Omega(M)):A_+w\in C^\infty(\partial\Omega(M))\}.$$

Now we can state the main theorem for solvability for I^*.

Theorem 4.2. *Let (M,g) be a simple, compact two dimensional Riemannian manifold with boundary. Then the operator $I^*:C^\infty_\alpha(\partial_+\Omega(M))\to C^\infty(M)$ is onto.*

The proof of Theorem 4.2 in [PU] uses the fact that I^*I is a pseudodifferential operator or order -1 on any open subset of a simple manifold.

Now we define the Hilbert transform:

$$Hu(x,\xi) = \frac{1}{2\pi}\int_{\Omega_x} \frac{1+(\xi,\eta)}{(\xi_\perp,\eta)} u(x,\eta)d\Omega_x(\eta), \quad \xi \in \Omega_x, \tag{4.5}$$

where the integral is understood as a principal value integral. Here $\perp$ means a 90^o degree rotation. In coordinates $(\xi_\perp)_i = \varepsilon_{ij}\xi^j$, where

$$\varepsilon = \sqrt{\det g}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The Hilbert transform H transforms even (respectively odd) functions with respect to ξ to even (respectively odd) ones. If H_+ (respectively H_-) is the even (respectively odd) part of the operator H:

$$H_+u(x,\xi) = \frac{1}{2\pi}\int_{\Omega_x} \frac{(\xi,\eta)}{(\xi_\perp,\eta)} u(x,\eta)d\Omega_x(\eta),$$

$$H_-u(x,\xi) = \frac{1}{2\pi}\int_{\Omega_x} \frac{1}{(\xi_\perp,\eta)} u(x,\eta)d\Omega_x(\eta)$$

and u_+, u_- are the even and odd parts of the function u, then

$$H_+u = Hu_+, H_-u = Hu_-.$$

We introduce the notation $\mathcal{H}_\perp = (\xi_\perp,\nabla) = -(\xi,\nabla_\perp)$, where $\nabla_\perp = \varepsilon\nabla$ and ∇ is the covariant derivative with respect to the metric g. The following commutator formula for the geodesic vector field and the Hilbert transform is very important in our approach.

Theorem 4.3. *Let (M,g) be a two dimensional Riemannian manifold. For any smooth function u on $\Omega(M)$ we have the identity*

$$[H,\mathcal{H}]u = \mathcal{H}_\perp u_0 + (\mathcal{H}_\perp u)_0 \tag{4.6}$$

where

$$u_0(x) = \frac{1}{2\pi}\int_{\Omega_x} u(x,\xi)d\Omega_x$$

is the average value.

Now we can prove Theorem 3.2.

Separating the odd and even parts with respect to ξ in (4.7) we obtain the identities:

$$H_+\mathcal{H}u - \mathcal{H}H_-u = (\mathcal{H}_\perp u)_0, \quad H_-\mathcal{H}u - \mathcal{H}H_+u = \mathcal{H}_\perp u_0.$$

Let (M,g) be a non-trapping strictly convex manifold. Take $u = w_\psi, w \in C_\alpha^\infty(\partial_+(\Omega))$. Then

$$2\pi\mathcal{H}H_+w_\psi = -\mathcal{H}_\perp I^*w$$

and using the formula (4.4) we conclude

$$2\pi A_-^* H_+ A_+ w = I\mathcal{H}_\perp I^* w, \tag{4.7}$$

since $w_\psi|_{\partial\Omega(M)} = A_+ w$.

Let (h, h_*) be a pair of conjugate harmonic functions on M,

$$\nabla h = \nabla_\perp h_*, \quad \nabla h_* = -\nabla_\perp h.$$

Notice, that $\delta\nabla = \triangle$ is the Laplace-Beltrami operator and $\delta\nabla_\perp = 0$. Let $I^* w = h$. Since $I\mathcal{H}_\perp h = I\mathcal{H} h_* = -A_-^* h_*^0$, where $h_*^0 = h_*|_{\partial M}$, we obtain from (4.7)

$$2\pi A_-^* H_+ A_+ w = -A_-^* h_*^0. \tag{4.8}$$

The following theorem gives the key to obtain the DN map from the scattering relation.

Theorem 4.4. *Let M be a 2-dimensional simple manifold. Let $w \in C_\alpha^\infty(\partial_+\Omega(M))$ and h_* the harmonic continuation of the function h_*^0. Then the equation (4.9) holds iff the functions $h = I^* w$ and h_* are conjugate harmonic functions.*

Proof. The necessity has already been established. Using (4.3) and (4.7) the equality (4.8) can be written in the form

$$I\mathcal{H}_\perp h = I\mathcal{H} q,$$

where q is an arbitrary smooth continuation onto M of the function h_*^0 and $h = I^* w$. Thus, the ray transform of the vector field $\nabla q + \nabla_\perp h$ equal 0.

The next step is to show that this field this field is potential, that is, $\nabla q + \nabla_\perp h = \nabla p$ and $p|_{\partial M} = 0$. This was proven in [An] for simple manifolds. Then functions h and $h_* = q - p$ are conjugate harmonic functions and $h_*|_{\partial M} = h_*^0$. This concludes the proof of Theorem 4.4.

□

In summary we have the following procedure to obtain the DN map from the scattering relation. For an arbitrary given smooth function h_*^0 on ∂M we find a solution $w \in C_\alpha^\infty(\partial_+\Omega(M))$ of the equation (4.8). Then the functions $h^0 = 2\pi(A_+ w)_0$ (notice, that $2\pi(A_+ w)_0 = I^* w|_{\partial M}$) and h_*^0 are the traces of conjugate harmonic functions. It is easy to see that this gives the DN map.

5. Reconstruction of the sound speed and the conformal factor

In this section we prove Theorem 1.4. We first start with the a general simple manifold.

Let (M, g) be a a simple two-dimensional compact Riemannian manifold with boundary. We have that

$$\int_M f I^* w \sqrt{\det g}\, dx = (If, w) \tag{5.9}$$

where in the right hand side the inner product is in $L^2_\mu(\partial_+\Omega(M))$. A conformal Killing vector field X satisfies the equation

$$\frac{1}{2}(\nabla_i X_j + \nabla_j X_i) = g_{ij}\frac{\delta X}{2} \tag{5.10}$$

where ∇ denotes the covariant derivative and δ is the divergence. For such vector field we have

$$\mathcal{H}_g(X,\xi) = \frac{1}{2}\delta X. \tag{5.11}$$

We remark that the right hand side of (5.13) does not depend on ξ. We also note that the local 1-parameter flow generated by the vector field X consists of local conformal isometries of the metric g.

From (4.8) we have that the geodesic X-ray transform $I(\delta X)$ is given by

$$I(\delta X) = -2A^*_-(X^0,\xi), \text{ where } X^0 = X|_{\partial M}. \tag{5.12}$$

Then putting $f = \delta X$ in (5.11) with X solution of (5.12) we get for arbitrary w:

$$\int_M (X, \nabla I^* w)\sqrt{\det g}\,dx = -\frac{2}{\pi}(A^*_-(X,\xi), w) - \int_{\partial M}(X,\nu)(A_+ w)_0 d\Gamma. \tag{5.13}$$

Now we specialize to the case of M a bounded domain of Euclidean space with smooth boundary. We provide M with the Riemannian metric g given by

$$ds^2 = \frac{1}{c^2(x)}dx^2 \tag{5.14}$$

where $c(x)$ is a smooth and positive function on M. In other words the metric g is conformal to the Euclidean metric, $g_{ij} = \delta_{ij}/c^2$. The function $c(x)$ models the sound speed (index of refraction) of the medium M. We denote by $\rho = 1/c^2$.

As mentioned earlier the classical inverse kinematic problem consists in determining $c(x)$ by knowing the lengths of geodesics joining points on the boundary of ∂M which corresponds to the first arrival times of waves going through the domain. The distance function will be denoted by $d_c(x,y); x, y \in \partial M$. This problem arose in Geophysics in order to determine the inner structure of the Earth by measuring the travel times of seismic waves. Herglotz and Wieckert and Zoeppritz considered the case where M is spherically symmetric and the sound speed is smooth and depends only on the radius. Under the condition that $\frac{d}{dr}(\frac{r}{c(r)}) > 0$ then one can reconstruct $c(r)$ from the lengths of geodesics. Notice that this implies that the metric is simple.

We will develop a (linear) method of reconstruction of the sound speed from d_c and then prove Theorem 1.4. In this case the vector field X as in (5.12) is a Cauchy-Riemann vector field, more exactly its contravariant components satisfy the Cauchy-Riemann equations

$$\frac{\partial X^1}{\partial x^1} = \frac{\partial X^2}{\partial x^2}, \quad \frac{\partial X^1}{\partial x^2} + \frac{\partial X^2}{\partial x^1} = 0.$$

Let h be an arbitrary harmonic function. (In our case $\Delta_g = c^2\Delta_e$ and consequently $\Delta_e h = 0$). By Theorem 4.2 we know that we can find find a solution

$w \in C^{\infty}_{\alpha}(\partial_+\Omega(M))$ of equation (4.7), where h^0_* is the trace of the conjugate harmonic function. Then $h = I^*w$. Thus we can calculate the integral in the left hand side of (5.13) for a holomorphic vector field X and harmonic function h. We denote by

$$S_{X,h}[\rho] = \int_M (X, \nabla h)\rho(x)dx \tag{5.15}$$

(since $\sqrt{g} = \rho$) which is known if we know the scattering relation.

The problem of finding ρ is then reduced to find enough holomorphic vector fields u and harmonic functions h so that the product of the gradients is dense in an appropriate space. This is similar to a question considered by Calderón for the linearized inverse conductivity problem at a constant conductivity [Ca]. See [U] for further developments.

We choose

$$X^1 = \zeta_2 e^{<x,\zeta>}, \quad X_2 = \zeta_1 e^{<x,\zeta>}, \quad h = e^{<x,\sigma>} \tag{5.16}$$

with complex vector $\zeta, \sigma \in C^2; \zeta \cdot \zeta = \sigma \cdot \sigma = 0$ with $\sigma \neq -\overline{\zeta}$. Here $<,>$ denotes the standard Euclidean inner product. We remark that we can write for $\zeta \in C^2; \zeta \cdot \zeta = 0$, in the form

$$\zeta = \eta + ik, \text{ with } \eta, k \in \mathbf{R}^2 \text{ satisfying } |k| = |\eta|, < k, \eta >= 0.$$

Substituting (5.18) in (5.17) we obtain:

$$S_{X,h}[\rho] = (\zeta_2\sigma_1 + \zeta_1\sigma_2) \int_M \rho(x) e^{<x,\zeta+\sigma>} dx. \tag{5.17}$$

Therefore we get

$$\frac{S_{X,h}[\rho]}{(\zeta_2\sigma_1 + \zeta_1\sigma_2)} = \int_M \rho(x) e^{<x,\zeta+\sigma>} dx. \tag{5.18}$$

Now by taking the limit

$$lim_{\sigma \to -\overline{\zeta}} \quad \frac{S_{X,h}[\rho]}{(\zeta_2\sigma_1 + \zeta_1\sigma_2)} = \int_M \rho(x) e^{2i<x,k>} dx \tag{5.19}$$

we recover the Fourier transform of ρ.

Thus we have given a recovery procedure to obtain the conformal factor ρ from the scattering relation.

Now let us consider the general case in Theorem 1.4. We can find a conformal diffeomorphism $\phi : (\widetilde{M}, g) \longrightarrow (D, e)$ where D is the unit disk and e denotes the Euclidean metric. Therefore in the argument above we replace $x^i, i = 1, 2$ by $\phi(x^i), i = 1, 2$ and we proceed in a completely analogous fashion. This concludes the proof of Theorem 1.4.

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