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A NOTE ON STICKELBERGER ELEMENTS FOR CYCLIC P-EXTENSIONS OVER GLOBAL FUNCTION FIELDS OF CHARACTERISTIC P

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Abstract. We prove a special case of Tate's refinement of a conjecture of Gross concerning the Stickelberger element associated to a cyclic extension over a global function field of characteristic p.

1. Introduction

In this note we make some initial progress toward Tate's refinement [8] of a conjecture of Gross [3] concerning the Stickelberger element associated to a cyclic extension over a global function field of characteristic p.

Recall that Gross's conjecture concerns the form of the Stickleberger element which is an element in the integral group ring of the Galois group for an abelian extension of a global field. In this note, we only consider the function field case. To be precise, let us fix a global function field k of characteristic p, and let $S = \{v_0, \dots, v_n\}$ and T be fixed non-empty finite sets of places in k such that S and T are disjoint.

Let L/k be an abelian extension unramified outside S and let $G = \mathrm{Gal}(L/k)$. For $v \notin S$, let Fr_v denote the Frobenius element in G. As modified by Gross, the associated *Stickelberger element* θ_G [3, 7] is the unique element in the group ring $\mathbb{Z}[G]$ such that for every non-trivial ring homomorphism $\chi : \mathbb{Z}[G] \longrightarrow \mathbb{C}$,

$$\chi(\theta_G) = L(\chi, 0), \tag{1}$$

where, $L(\chi, s)$, $s \in \mathbb{C}$ is the modified L-function defined by

$$L(\chi, s) = \prod_{v \notin S} (1 - \chi(Fr_v)N(v)^{-s})^{-1} \times \prod_{v \in T} (1 - \chi(Fr_v)N(v)^{1-s}), \quad (\mathrm{Re}(s) > 1). \tag{2}$$

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In [3] Gross conjectures that if I is the augmentation ideal of $\mathbb{Z}[G]$, then θ_G is in I^n and is congruent to $h \cdot \mathcal{R}_G$ modulo I^{n+1}. Here $h = h_{S,T}$ is the T-modified class number of the S-integers of k and $\mathcal{R}_G$ is the *refined regulator* . This conjecture is the analogue of the class number formula (interpreted as a statement about the order of vanishing of the zeta function at "$s = 0$" and the first non-zero term in its Taylor expansion).

In [3], the refined regulator is defined as an element in I^n/I^{n+1}. We instead choose to adopt Tate's definition [8] and define the refined regulator as an element in the group ring.

Here we describe Tate's definition of the refined regulator. First, for each i, let

$$\deg_i : k_{v_i}^* \longrightarrow \mathbb{Z}$$

be the local valuation map which sends the local parameters to 1. Recall that $U_{S,T}$, the group of S-units which are congruent to 1 modulo T, is torsion free. Let $\{u_j\}_{1 \le j \le n}$ be a $\mathbb{Z}$-basis of it. Then the classical regulator equals

$$\pm \mathrm{M}_S \cdot \det_{1 \le i,j \le n} (\deg_i(u_j)),$$

where the sign is determined by the ordering of the basis and M_S is the index of $U_{S,T}$ as a subgroup of the free part of the S-units. We choose the ordering to have the positive sign. For $0 \le i \le n$, let G_i be the decomposition subgroup of $v_i \in S$, and let

$$f_i : k_{v_i}^* \longrightarrow G_i$$

be the local reciprocity law homomorphism. These local homomorphisms are viewed as analogue of the local valuation maps, and the refined regulator is defined.

Definition 1.1. *We define*

$$\mathcal{R}_G = \det_{1 \le i,j \le n} (f_i(u_j) - 1) \in \mathbb{Z}[G]. \tag{3}$$

The definition of $\mathcal{R}_G$ depends on the choice of v_0 in S.

For a subgroup H of G, let I_H be the kernel of the homomorphism $\mathbb{Z}[G] \longrightarrow \mathbb{Z}[G/H]$ induced from the natural quotient homomorphism. The following lemma is obvious.

Lemma 1.1. ([8]) We have

$$\mathcal{R}_G \in \prod_{1 \le i \le n} I_{G_i}. \tag{4}$$

From now on, we assume that G is cyclic of order p^m and

$$m_0 \le m_1 \le \cdots \le m_n \le m,$$

where

$$p^{m_i} = [G : G_i], \quad i = 0, \ldots n.$$

Let

$$N = p^{m_0} + \cdots + p^{m_{n-1}}. \tag{5}$$

In [8], Tate, using the valuation criterion (see Lemma 2.1), proves that if $m_n = m - 1$, then θ_G and $h \cdot \mathcal{R}_G$ are both in I^N where N is defined in (5), and conjectures that, in the case where $m_0 = 0$, they are congruent modulo I^{N+1}. Since $N \geq n$, this is a refinement of the conjecture of Gross. Note that in the case where $m_0 > 0$ the congruence might not hold ([6]) and a more subtle formulation is needed. For more discussions on Tate's refinement, see [6, 2, 1].

Our main result (proved in Section 2) is that this congruence indeed holds for the "$m_0 = 0$" case.

Theorem 1.1. *If $m_n = m - 1$ and $m_0 = 0$, then $\theta_G, h \cdot \mathcal{R}_G \in I^N$, and*

$$\theta_G \equiv h \cdot \mathcal{R}_G \pmod{I^{N+1}}. \tag{6}$$

Theorem 1.1 is proved through making direct use of the fact that G is a p-group. This allows us to work on $\mathbb{Z}_p[G]$ which is just $\mathbb{Z}[G] \otimes_{\mathbb{Z}} \mathbb{Z}_p$. In particular, if we let

$$I_{H,p} = I_H \otimes_{\mathbb{Z}} \mathbb{Z}_p,$$

then Theorem 1.1 is proved in Section 2 by using the *valuation criterion* (Lemma 2.1) in combination with the following result whose proof is postponed until Section 3.

Theorem 1.2. *If G is a cyclic p-group, then*

$$\theta_G \in \prod_{1 \leq i \leq n} I_{G_i,p}, \tag{7}$$

and

$$\theta_G \equiv h \cdot \mathcal{R}_G \pmod{I_{G,p} \cdot \prod_{1 \leq i \leq n} I_{G_i,p}}. \tag{8}$$

2. The valuation criterion

We follow the notation in [8]. Assume that G is cyclic of order p^m and G_n is of order p^{m-1}. Let σ generate G and put $\rho = \sigma^{p^{m-1}}$. Let χ be a character of G of order p^m so that $\zeta := \chi(\sigma)$ is a primitive p^mth root of unity. Put $\lambda = \zeta - 1$. The proof of the following lemma can be found in [6].

Lemma 2.1. (Tate,[8]) For $j \geq 1$, the character χ induces an isomorphism

$$\begin{array}{rcl} \chi : I^j \cap (\rho - 1)\mathbb{Z}[G]/I^{j+1} \cap (\rho - 1)\mathbb{Z}[G] & \xrightarrow{\sim} & (\lambda)^{j+p^{m-1}-1}/(\lambda)^{j+p^{m-1}} \\ (\sigma - 1)^{j-1}(\rho - 1) & \mapsto & \lambda^{j+p^{m-1}-1} \end{array}$$

Assuming Theorem 1.2, we now easily prove Theorem 1.1.

Proof. (of Theorem 1.1) From the assumption $m_0 = 0$ and $m_n = m-1$, we have

$$\chi(\prod_{1\le i\le n} I_{G_i,p}) \subseteq (\lambda)^{p^{m_1}+\cdots+p^{m_n}} = (\lambda)^{N+p^{m-1}-1},$$

and

$$\chi(I_{G,p}\cdot \prod_{1\le i\le n} I_{G_i,p}) \subseteq (\lambda)^{1+p^{m_1}+\cdots+p^{m_n}} = (\lambda)^{N+p^{m-1}}.$$

□

3. The method of using $\mathbb{Z}_p$-extensions

In this section, we will frequently use the following version of Local Leopoldt Theorem (cf. [5, 9]).

Lemma 3.1. *Let v be a place of k and $\alpha \in k$. If $\alpha^{1/p} \in k_v$, the $\alpha^{1/p} \in k$.*

Proof. If $\alpha^{1/p} \notin k$, then $k(\alpha^{1/p}) = k^{1/p}$. But since k is dense in k_v, we can not have $k^{1/p} \subset k_v$. □

Let Γ_S be the Galois group of the maximal pro-p abelian extension which is unramified outside S. In [5], it is proved that Γ_S is a product of countably many copies of $\mathbb{Z}_p$ (see also [9]). Therefore, we can extend L/k to an extension E/k unramified outside S such that $\mathrm{Gal}(E/k) \simeq \mathbb{Z}_p$.

For each $i = 0, \ldots, n$, let k_i and E_i denote the completions at v_i of k and E respectively. Put $\mathcal{N}_i = \mathrm{Norm}_{E_i/k_i}(E_i^*)$. Then $k_i^*/\mathcal{N}_i$ is either $\mathbb{Z}_p$, $\mathbb{Z}$, or $\{0\}$.

Let $\tilde{\Gamma}$ be the p-completion of the group $k^*\backslash\mathbb{A}_k^*/\prod_{i=1}^n \mathcal{N}_i \times \prod_{v\notin S} \mathcal{O}_v^*$. Here, as usual, $\mathbb{A}_k^*$ denotes the group of ideles and $\mathcal{O}_v^*$ denotes the group of local units at v. Then $\tilde{\Gamma}$ is the Galois group of the maximal abelian pro-p-extension of k over which the decomposition group at each v_i, $i = 1, \ldots, n$, is exactly the p-completion of $k_i^*/\mathcal{N}_i$. Note that here we impose no condition at v_0. We have the natural homomorphism Ψ which maps the p-completion of $\prod_{i=1}^n k_i^*/\mathcal{N}_i$ to $\tilde{\Gamma}$.

Lemma 3.2. *With the Ψ as above, the following are true.*

1. *The homomorphism Ψ is injective and the $\mathbb{Z}_p$-module $\tilde{\Gamma}/\mathrm{Im}(\Psi)$ is torsion free.*
2. *There exists an abelian pro-p-extension F/k which satisfies the following.*
   - (a) *The field F contains E.*
   - (b) *The Galois group $\Gamma := \mathrm{Gal}(F/k)$ is isomorphic to $\mathbb{Z}_p^d$ for some d.*
   - (c) *The decomposition group Γ_i at each v_i, for $i = 1, \ldots, n$, is exactly the p-completion of $k_i^*/\mathcal{N}_i$.*
   - (d) *The natural homomorphism $\prod_{i=1}^n \Gamma_i \longrightarrow \Gamma$ is injective and the cokernel is torsion free.*

Proof. We first prove that $\tilde{\Gamma}$ is p-torsion free. Then, by the fact that it is a quotient of Γ_S, it is also a product of countably many copies of $\mathbb{Z}_p$. Now, suppose that $\gamma \in \tilde{\Gamma}$ and $\gamma^p = 1$. If γ is represented by an idele $z = (z_v)$, then there is an $\alpha \in k^*$ and a $u \in \prod_{i=1}^n \mathcal{N}_i \times \prod_{v\notin S} \mathcal{O}_v^*$ such that $z^p = \alpha \cdot u$. In particular, in k_0^* we have $z_{v_0}^p = \alpha$. By Lemma 3.1, we have $\alpha = \beta^p$ for some $\beta \in k^*$. Since k_0^* and $k_i^*/\mathcal{N}_i$, $i = 1, ..., n$ are p-torsion free, we have $z_{v_i}\beta^{-1} = 1$, for $i = 0, ..., n$. Also, we have $z_v \beta^{-1} \in \mathcal{O}_v^*$, for $v \notin S$. This implies that $\gamma = 1$.

For $i = 1, \ldots, n$, let $x_i \in k_i^*$, and let $\bar{x}$ be the element determined by $(x_1, \ldots, x_n)$ in the p-completion of $\prod_{i=1}^n k_i^*/\mathcal{N}_i$. If $\bar{x}$ is in the kernel of Ψ, then for each $j = 1, 2, \ldots$, there exists $\alpha_j \in K^*$, $u(j) \in \prod_{i=1}^n \mathcal{N}_i \times \prod_{v \notin S} \mathcal{O}_v^*$ and $y(j) \in \mathbb{A}_k^*$ such that in $\mathbb{A}_k^*$,

$$(x_1, \ldots, x_n) = \alpha_j \cdot u(j) \cdot y(j)^{p^j}.$$

In particular, we have

$$\alpha_j \in (k_0^*)^{p^j},$$

and by Lemma 3.1, we have

$$\alpha_j \in (k^*)^{p^j}.$$

Consequently, the element $\bar{x}$ is p-divisible in the p-completion of $\prod_{i=1}^n k_i^*/\mathcal{N}_i$, hence is trivial. This shows that Ψ is injective. If there is a $\gamma \in \tilde{\Gamma}$ obtained from an idele $z = (z_v)$ such that $\gamma^p = \Psi(\bar{x})$, then for each $j = 1, 2, \ldots$, there are $\alpha_j \in K^*$, $u(j) \in \prod_{i=1}^n \mathcal{N}_i \times \prod_{v \notin S} \mathcal{O}_v^*$ and $y(j) \in \mathbb{A}_k^*$ such that

$$(x_1, \ldots, x_n) = z^p \cdot \alpha_l \cdot u(j) \cdot y(j)^{p^j}.$$

Again, by Lemma 3.1, we have

$$\alpha_j \in (k^*)^p,$$

and consequently, there is a $\bar{w}$ in the p-completion of $\prod_{i=1}^n k_i^*/\mathcal{N}_i$ such that $\bar{x} = \bar{w}^p$. Therefore, $\gamma \cdot \Psi(\bar{w})^{-1}$ is in the p-torsion of $\tilde{\Gamma}$. Since $\tilde{\Gamma}$ is p-torsion free, we have $\gamma = \Psi(\bar{w})$. This completes the proof of (1). Also, since $\tilde{\Gamma}$ is a product of countably many copies of $\mathbb{Z}_p$, then by (1), we can find an abelian pro-p-extension F_0/k which satisfies conditions (b), (c), (d). Then we put $F = F_0 E$. □

Now we pay off our last debt and prove Theorem 1.2.

Proof. (of Theorem 1.2) Let F/k satisfy conditions (a), (b), (c), (d), in Lemma 3.2, and let $\Gamma = \mathrm{Gal}(F/k)$. Since θ_G is functorial with respect to G, through the projective limit, we can define the Stickelberger element $\theta_\Gamma \in \mathbb{Z}[[\Gamma]]$ ([9]). Also, the refined regulator $\mathcal{R}_\Gamma \in \mathbb{Z}[[\Gamma]]$ is defined. If there is an $i \in \{1, \ldots, n\}$ such that $k_i^*/\mathcal{N}_i = \{0\}$, then v_i splits completely and (see [3])

$$\theta_\Gamma = 0 = \mathcal{R}_\Gamma,$$

and there is nothing to prove. We assume that $k_i^*/\mathcal{N}_i \neq \{0\}$, for every $i = 1, \ldots, n$. Then $\mathbb{Z}_p\Gamma_i \simeq \mathbb{Z}_p$. Choose a $\mathbb{Z}_p$-basis $\gamma_1, \ldots, \gamma_d$ of Γ, such that $\mathbb{Z}_p\Gamma_i = \mathbb{Z}_p\gamma_i$, for $i = 1, \ldots, n$, and put $t_i = \gamma_i - 1$ for $i = 1, \ldots, d$. Then in this case, the ring $\mathbb{Z}_p[[\Gamma]]$ is just the formal power series ring $\mathbb{Z}_p[[t_1, \ldots, t_d]]$ and the ideals $\bar{I}_{\Gamma,p} := \ker(\mathbb{Z}_p[[\Gamma]] \longrightarrow \mathbb{Z}_p)$ and $\bar{I}_{\Gamma_i,p} := \ker(\mathbb{Z}_p[[\Gamma]] \longrightarrow \mathbb{Z}_p[[\Gamma/\Gamma_i]])$, $i = 1, \ldots, d$, of $\mathbb{Z}_p[[\Gamma]]$ are just $(t_1, \ldots, t_d)$ and (t_i). Here, as before, the natural ring homomorphisms are induced from the quotient homomorphisms

$$\Gamma \longrightarrow \{0\},$$

and

$$\Gamma \longrightarrow \Gamma/\Gamma_i.$$

For each i, let F_i be the fixed field of Γ_i. Since v_i splits completely in F_i, the natural homomorphism $\mathbb{Z}_p[[\Gamma]] \longrightarrow \mathbb{Z}_p[[\Gamma/\Gamma_i]]$, maps both θ_Γ and $h \cdot \mathcal{R}_\Gamma$ to zero. This shows that as power series, both θ_Γ and $h \cdot \mathcal{R}_\Gamma$ are divisible by t_i for every $i \in \{1, \ldots, n\}$. Therefore, we have

$$\theta_\Gamma \in t_1 t_2 \ldots t_n \cdot \mathbb{Z}_p[[t_1, \ldots, t_d]] = \prod_{i=1}^{n} I_{\Gamma_i, p}.$$

Applying the quotient map $\Gamma \longrightarrow G$, we get the inclusion (7).

In this case, Gross's Conjecture has been proved [9], so that $\theta_\Gamma - h \cdot \mathcal{R}_\Gamma$ is in $I_{\Gamma,p}^{n+1}$. Since it is also divisible by $t_1 \ldots t_n$, we have

$$\theta_\Gamma - h \cdot \mathcal{R}_\Gamma \in I_{\Gamma,p} \cdot \prod_{i=1}^{n} I_{\Gamma_i, p}\,.$$

Congruence (8) then can be obtained by applying the quotient map $\Gamma \longrightarrow G$.

□

References

[1] Aoki, N. *On Tate's refinement for a conjecture of Gross and its generalization.* to appear in Journal de Theorie des Nombres Bordeaux, 2003.

[2] Burns, D. *On relations between derivatives of abelian L-functions at $s = 0$.* Preprint, 2002

[3] Gross, B. *On the values of abelian L-functions at $s = 0$.* J. Fac. Sci. Univ. Tokyo Sect. IA, Math. **35** (1988), 177–197.

[4] Hayes, D. *The refined p-adic abelian Stark conjecture in function fields.* Invent. Math. **94** (1988), 505–527.

[5] Kisilevsky, H. *Multiplicative independence in function fields.* J. Number Theory **44** (1993), 352–355.

[6] Lee, J. *Stickelberger elements for cyclic extensions and the order of vanishing of abelian L-functions at $s = 0$. to appear in Comp. Math.*

[7] Tate, J. *Les Conjectures de Stark sur les Fonctions L d'Artin en $s = 0$,* Birkhauser, 1984

[8] Tate, J. *a letter to Joongul Lee,* 1997

[9] Tan, K.-S. *On the Special Values of Abelian L-Function.* J. Math. Sci., Univ. Tokyo **1** (1995), 305–319.

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