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PROOF AND INTERPRETATION OF A STRING DUALITY

Jian Zhou

Abstract. We announce the proof of a conjecture of Iqbal on the relationship between multiple covering formula in Gromov-Witten theory with Wess-Zumino-Witten theory. We interpret it in terms of localizations on moduli spaces of stable maps to the projective line, using the recently proved Mariño-Vafa formula for Hodge integrals.

1. Introduction

1.1. The string duality. Duality in physics literature means the equivalence of different quantum field theories. For example, mirror symmetry is a duality in string theory which through intensive researches in the past two decades is more or less mathematically well understood; the physical Seiberg-Witten theory is concerned with the equivalence of the Donaldson theory and the mathematical Seiberg-Witten theory. In recent years, many more dualities have arisen in string theory. See e.g. [32] for an exposition. Most of the string dualities remain very mysterious, lacking mathematical understandings. We now announce an approach to mathematically understand an important special case by some techniques well-known in both mathematics and string theory. More precisely, we will prove a conjecture by Iqbal [11] on a relationship between the Wess-Zumino-Witten (WZW) theory and the multiple covering formula in enumerative geometry of Calabi-Yau threefolds. We also give an interpretation in terms of localization on moduli spaces of stable maps to $\mathbb{P}^1$, using the recently proved Mariño-Vafa formula for Hodge integrals [24, 25]. The details of the proof and its generalization to the proof of Iqbal's conjecture in general local Calabi-Yau toric geometry will appear in [42].

1.2. Multiple covering formula. Denote by $\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, d)$ the moduli space of stable maps to $\mathbb{P}^1$ of degree d. Let $\mathcal{O}(-1)^g_d$ be the bundle on $\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, d)$ whose fiber at a map $f : C \to \mathbb{P}^1$ representing a point in the moduli space is given by

$$H^1(C, f^*\mathcal{O}(-1)).$$

Define

$$K^g_d = \int_{[\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1,d)]^{vir}} e(\mathcal{O}(-1)^g_d \oplus \mathcal{O}(-1)^g_d),$$

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and its generating series:

$$F_{inst}(\lambda) = \sum_{d>0} \sum_{g \geq 0} K_d^g \lambda^{2g-2} t^d.$$

The *multiple covering formula* is the following identity proved in [7]:

$$F_{inst}(\lambda) = \sum_{d>0} \frac{1}{d} \frac{t^d}{(2\sin(d\lambda/2))^2}. \tag{1}$$

Equivalently,

$$K_d^g = \begin{cases} \frac{1}{d^3}, & g = 0, \\ \frac{1}{12d}, & g = 1, \\ \frac{|B_{2g}| \cdot d^{2g-3}}{2g \cdot (2g-2)!}, & g \geq 2. \end{cases}$$

The $g = 0$ case was established in [2, 27, 35, 20], the $g = 1$ case in [3, 10]. In physics $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1$ is regarded as an open Calabi-Yau three-fold, and F_{inst} is the instanton piece of the free energy of the A-model closed string theory on it.

1.3. Physical backgrounds. Relationship of the multiple covering formula with the WZW theory is derived in physics literature by the following ideas: geometric transition, Chern-Simons theory as string theory, 't Hooft large N expansion, relationship between Chern-Simons theory and WZW theory.

1.3.1. *Conifold transition.* There are two procedures to remove a rational double point in a Calabi-Yau three-fold. Under suitable deformations the singular point gets replaced by a copy of S^3 which locally is the same as the zero section in T^*S^3. One can also replace the singular point by a copy of $\mathbb{P}^1$ with normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$. There are two different ways of doing this, and the resulting spaces are related by a flop. Locally, one performs surgery on the $\mathbb{P}^1$ lying at the zero section of $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ by replacing it with a copy of S^3, then the resulting space can be identified with T^*S^3 (cf e.g. [9]). This process is called the conifold transition in the physics literature.

1.3.2. *Relationship with Chern-Simons theory.* In [37] Witten proposed a relationship between the open string theory of T^*M of a three-manifold M and the Chern-Simons theory on M for $SU(N)$, by large N 't Hooft expansion. Gopakumar and Vafa [8] conjectured that under the conifold transition, the large N Chern-Simons theory on S^3 is dual to the A-model closed string theory on $\mathcal{O}(-1) \oplus (-1) \to \mathbb{P}^1$. See also [9]. This conjecture was further tested in [29]. In connection with Witten's work [36] on relationship between Chern-Simons theory and link invariants, this duality leads to the idea that both open and string invariants are related to link invariants. This has been checked for many cases [16, 31, 17, 28, 33]. Closed string invariants for more complicated geometries, such as local toric del Pezzo surfaces, have been calculated also from the Chern-Simons theory using geometric transition [5, 6, 1].

1.3.3. *WZW theory.* Another important idea in the seminal paper [36] is the relationship between Chern-Simons theory and WZW theory. In mathematics WZW theory is the representation theory of affine Kac-Moody algebras. See e.g. [15]. For a fixed integer k, there are only finitely many integrable highest weight representations of level k of an affine Kac-Moody algebra up to equivalences. Denote their characters by $\chi_0(\tau), \dots, \chi_n(\tau)$. Then there are holomorphic functions $S_{ij}(\tau)$, such that

$$\chi_i(-\frac{1}{\tau}) = \sum_j S_{ij}(\tau)\chi_j(\tau).$$

From this one can construction a representation of a double covering of $SL(2, \mathbb{Z})$ (cf. [34]).

1.4. Iqbal's conjecture. In [11] Iqbal proposed a method to compute closed string invariants in local toric Fano geometries by WZW theory by interpreting the associated 5-brane web as a Feynman diagram.

1.4.1. *General remarks on Feynman rules.* In a quantum field theory one is concerned with a certain Feynman integral which gives the correlation function Z of the theory. The Feynman integral usually depends on a parameter λ called the coupling constant. When λ is small, one is in the regime of weak coupling, where perturbative method is applicable. More precisely, one can expand $Z(\lambda)$ as a power series

$$Z(\lambda) = \sum_{g \geq 0} Z_g \lambda^{2g-2},$$

the coefficients of which received contributions from terms indexed by some graphs, called the Feynman diagrams. Here g is the genus of the Feynman diagram. The *Feynman rule* determines the contribution of a Feynman diagram to the Feynman integral as follows: It is a product of factors indexed by the edges and the vertices, divided by the order of the automorphism group of the graph. The factor for an edge is called a *propagator*, and the factor for a vertex is called an *interaction vertex*. One of the most important problems in a quantum field theory is the determination of the Feynman diagrams and the Feynman rule.

1.4.2. *The conjecture.* Recall the 5-brane web is a graph that encodes the relevant toric geometry [18]. Motivated by a lattice model interpretation of the calculations in [1], Iqbal [11] defines the propagators and vertices in terms of WZW theory. He makes many conjectures for the partition functions of various local toric geometries. To state his conjecture for $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1$, we need some notations. For a partition $\eta = (\eta_1 \geq \cdots \geq \eta_l > 0)$, define

$$\kappa_\eta = |\eta| + \sum_{i=1}^{l} \eta_i(\eta_i - 2i),$$

and

$$\mathcal{W}_\eta = q^{\kappa_\eta/4} \prod_{1 \leq i < j \leq l} \frac{[\eta_i - \eta_j + j - i]}{[j-i]} \prod_{i=1}^{l} \prod_{v=1}^{\eta_i} \frac{1}{[v - i + l]},$$

where

$$[x] = q^{\frac{x}{2}} - q^{-\frac{x}{2}} = e^{\sqrt{-1}\lambda/2} - e^{-\sqrt{-1}\lambda/2}.$$

Here

$$q = e^{\sqrt{-1}\lambda}.$$

The series $\mathcal{W}_\eta$ arise in WZW theory as the leading term of certain coefficients of the S matrix. See [1, 11] for details.

For example,

$$\mathcal{W}_{(1)} = \frac{1}{(q^{1/2} - q^{-1/2})^2},$$

$$\mathcal{W}_{(2)} = \frac{q^{\frac{1}{2}}}{(q - 1^{-1})^2 (q^{1/2} - q^{-1/2})^2},$$

$$\mathcal{W}_{(1,1)} = \frac{q^{-\frac{1}{2}}}{(q - 1^{-1})^2 (q^{1/2} - q^{-1/2})^2},$$

In general, by results from an earlier work [40],

$$\mathcal{W}_\eta = \frac{q^{\kappa_\eta/4}}{\prod_{x \in \eta} (q^{h(x)/2} - q^{-h(x)/2})},$$

where x denotes a square in the Young diagram of η, and $h(x)$ denotes its hook length.

Conjecture 1. [11] *The instanton piece of the partition function of $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1$ is given by:*

$$\begin{aligned} -F_{inst} &= \log(1 + \sum_{n=1}^{\infty} I_n) \\ &= I_1 + \{I_2 - \frac{1}{2} I_1^2\} + \{I_3 - I_2 I_1 + \frac{1}{3} I_1^3\} + \cdots, \end{aligned} \tag{2}$$

where

$$I_n = \sum_{|\eta| = n} \mathcal{W}_\eta^2.$$

Using the explicit expressions for $\mathcal{W}_\eta$ Iqbal has verified his conjecture in some low degrees.

1.5. Our results. We announce the following result. The details will appear in [42]:

Theorem 1.1. *Iqbal's conjecture is true. Furthermore, we have*

$$\begin{aligned} F_{inst} &= \log(1 + \sum_{n=1}^{\infty} (-1)^n J_n) \\ &= - J_1 + \{J_2 - \frac{1}{2} J_1^2\} - \{J_3 - J_2 J_1 + \frac{1}{3} J_1^3\} + \cdots , \end{aligned} \tag{3}$$

where

$$J_n = \sum_{|\eta|=n} \mathcal{W}_\eta \mathcal{W}_{\eta'}.$$

Here η' denotes the partition whose Young diagram is the conjugate of that of η.

1.5.1. *Interpretation by localization.* Even though our proof of (2) and (3) can be presented in terms of only standard techniques from algebraic combinatorics of symmetric functions (cf. §6), we arrive at it by localization on $\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, d)$ as in the proof of (1) by Faber and Pandharipande [7]. We use the following key ideas:

(1) the graphs arising in the localization calculations can be interpreted as Feynman diagrams,
(2) localizations give rise to Feynman rules which involves Hodge integrals exactly as in the Mariño-Vafa formula (cf. §7.1);
(3) one can use bosonic Fock space and Wick theorem to produce the localization graphs (cf. Theorem 5.1).

The first two ideas first appear in [41]. The third is standard in string theory. The bosonic Fock space can be realized as the space of symmetric function.

The key ingredients of our proof will be explained in the following sections. The details will appear in a separate paper [42].

2. Combinatorial Preliminaries

We use Macdonald's book [26] as our reference. Most of the result in this section are proved in [40] for the purpose of getting initial values for the cut-and-join equations for Hodge integrals in the Mariño-Vafa formula.

2.1. Partitions. A partition of a positive integer d is a sequence of integers $n_1 \geq n_2 \geq \cdots \geq n_l > 0$ such that $n_1 + \cdots + n_l = d$. We write

$$|\eta| = d, \qquad l(\eta) = l, \qquad \vec{n}(\eta) = (n_1, \ldots, n_l).$$

Denote by $m_j(\eta)$ the number of j's among $n_1, \ldots, n_l$. Each partition η of d corresponds to a conjugacy class C_η of S_d. Denote by $C_{(2)}$ the conjugacy class of transpositions. The number of elements in C_η is

$$|C_\eta| = \frac{d!}{z_\eta},$$

where

$$z_\eta = \prod_j m_j(\eta)! j^{m_j(\eta)}.$$

Denote by $(-1)^g$ the sign of an element in S_d. It is easy to see that

$$(-1)^g = (-1)^{|\eta|-l(\eta)},$$

for $g \in C_\eta$.

Another way of representing a partition is by the Young diagrams: The Young diagram of η has $m_j(\eta)$ rows of boxes of length j. The partition corresponding to the transpose of the Young diagram of η will be denoted by η'. The number of squares in the i-th row of η' will be written as η'_i. For a partition η, define

$$n(\eta) = \sum_i (i-1)\eta_i = \sum_i \binom{\eta'_i}{2}.$$

For any square $e \in \eta$, denote by $h(e)$ its hook length. Then one has (cf. [26]):

$$\sum_{e\in\eta} h(e) = n(\eta) + n(\eta') + |\eta|. \tag{4}$$

2.2. Representations of symmetric groups. Each partition λ corresponds to an irreducible representation R_λ of S_d. For example, $R_{(d)}$ corresponds to the trivial representation, $R_{(1^d)}$ corresponds to the sign representation. The value of the character χ_{R_λ} on the conjugacy class C_η is denoted by $\chi_\lambda(\eta)$. Set

$$c_\eta = \sum_{g\in C_\eta} g.$$

Since c_η lies in the center of the group algebra $\mathbb{C}S_d$, it acts as an scalar $f_\rho(\eta)$ on any irreducible representation R_ρ. Now

$$\dim R\rho \cdot f_\rho(\eta) = \operatorname{tr}|_{R_\rho} c_\eta = \sum_{g\in C_\eta} \operatorname{tr}|_{R_\rho} g = |C_\eta|\chi_\rho(\eta),$$

hence

$$f_\rho(\eta) = |C_\eta| \frac{\chi_\rho(\eta)}{\dim R_\rho} = \frac{d!}{\dim R_\rho} \cdot \frac{\chi_\rho(\eta)}{z_\eta}.$$

See [40] for the following properties of these numbers:

Proposition 2.1. *Each $f_\rho(\eta)$ is an integer. Furthermore,*

$$f_\rho(2) = \frac{1}{2}\kappa_\rho, \tag{5}$$

$$f_{\rho'}(\eta) = (-1)^{|\eta|-l(\eta)} f_\rho(\eta), \tag{6}$$

$$\frac{1}{2}\sum_{e\in\rho} h(e) - n(\rho) = \frac{1}{2}(f_\rho(2) + |\rho|). \tag{7}$$

2.3. A technical result. Denote by Λ the space of symmetric functions. Let p_i be the i-th Newton function. I.e.,

$$p_i(x_1, \dots, x_n, \dots) = x_1^i + \cdots + x_n^i + \cdots ,$$

where $x_1, \dots, x_n, \dots$ are formal variables. For a partition $\eta = (\eta_1 \geq \cdots \geq \eta_l > 0)$, define

$$p_\eta = p_{\eta_1} \cdots p_{\eta_n}.$$

Then $\{p_\eta\}$ form a basis of Λ. The following result proved in [40] for other purposes plays an essential role in our proof of (2) and (3).

Theorem 2.1. *we have the following identity:*

$$\log\left(\sum_{n\geq 0}\sum_{|\rho|=n} \frac{e^{\frac{1}{2}f_\rho(2)\sqrt{-1}\lambda}}{\prod_{e\in\rho} 2\sqrt{-1}\sin(h(e)\lambda/2)} \frac{\chi_\rho(\eta)}{z_\eta} p_\eta\right) = \sum_{d\geq 0} \frac{\sqrt{-1}p_d}{2d\sin(d\lambda/2)}. \tag{8}$$

Let us briefly recall the proof. Denote by s_ρ the Schur function associated to a partition ρ. Recall the following facts about Schur functions [26]:

$$s_\rho(x) = \sum_\eta \frac{\chi_\rho(\eta)}{z_\eta} p_\eta(x), \tag{9}$$

$$s_\rho(1, q, q^2, \dots) = \frac{q^{n(\rho)}}{\prod_{e\in\rho}(1 - q^{h(e)})}, \tag{10}$$

$$\sum_{n\geq 0} t^n \sum_{|\rho|=n} s_\rho(x) s_\rho(y) = \frac{1}{\prod_{i,j}(1 - t x_i y_j)}. \tag{11}$$

Combining these three identities, one gets:

$$\sum_{n\geq 0} t^n \sum_{|\rho|=n} \frac{q^{n(\rho)}}{\prod_{e\in\rho}(1 - q^{h(e)})} \frac{\chi_\rho(\eta)}{z_\rho} p_\eta = \frac{1}{\prod_{i,j}(1 - t x_i q^{j-1})}. \tag{12}$$

When $q = e^{-\sqrt{-1}\lambda}$, and $t = \sqrt{-1}q^{1/2}$, the left-hand side of (12) can be identified with

$$\sum_{n\geq 0}\sum_{|\rho|=n} \frac{e^{\frac{1}{2}f_\rho(2)\sqrt{-1}\lambda}}{\prod_{e\in\rho} 2\sqrt{-1}\sin(h(e)\lambda/2)} \frac{\chi_\rho(\eta)}{z_\rho} p_\eta.$$

On the other hand, take logarithm of the right-hand side of (12), one gets:

$$\log \frac{1}{\prod_{i,j}(1 - t x_i q^{j-1})} = \sum_{d\geq 1} \frac{\sqrt{-1}p_d}{2d\sin(d\lambda/2)}.$$

3. Heisenberg Algebra and Free Boson System

We recall in this section some standard results from bosonic string theory.

3.1. Heisenberg algebra and bosonic Fock space. The space Λ admits an action of the Heisneberg algebra as follows. Define operators $\{\beta_n\}_{n\in\mathbb{Z}}$ on Λ by:

$$\beta_n(f) = \begin{cases} p_{-n}f, & n < 0, \\ 0, & n = 0, \\ n\frac{\partial}{\partial p_n}f, & n > 0. \end{cases}$$

For a partition η of length l define:

$$\beta_\eta = \beta_{\eta_1}\cdots\beta_{\eta_l}, \qquad \beta_{-\eta} = \beta_{-\eta_1}\cdots\beta_{-\eta_l}.$$

Then we have:

$$\begin{aligned} &[\beta_m, \beta_n] = m\delta_{m,-n}, \\ &\beta_n 1 = 0, \quad n \geq 0, \\ &p_\eta = \beta_{-\eta}1. \end{aligned}$$

In other words, Λ is the bosonic Fock space in which 1 is the vacuum vector $|0\rangle$. Define a Hermitian metric on Λ such that

$$\langle p_\mu, p_\nu\rangle = z_\mu\delta_{\mu\nu}.$$

Then in this metric, one has:

$$\beta_n^* = \beta_{-n},$$

for $n \in \mathbb{Z}$.

3.2. Vacuum expectation values and Wick theorem. Following physicists' notations, we will write the inner product of $A|0\rangle$ with $|0\rangle$ as

$$\langle 0|A|0\rangle,$$

where A is a linear operator on Λ. It is called the *vacuum expectation value* (vev) of A, and will simply be denoted as

$$\langle A\rangle.$$

We will be interested in the vev of operators of the form $\beta_\mu\beta_{-\nu}$ which is given by the Wick Theorem:

$$\langle\beta_\mu\beta_{-\nu}\rangle = \delta_{\mu,\nu}z_\mu. \tag{13}$$

As a consequence one has:

$$\left\langle \exp(\sum_{n\geq1}\frac{a_nt^n}{n}\beta_n)\exp(\sum_{n\geq1}\frac{a_{-n}t^n}{n}\beta_{-n})\right\rangle = \exp\left(\sum_{n\geq1}\frac{a_na_{-n}}{n}t^{2n}\right). \tag{14}$$

4. Chemistry of Two-Colored Labelled Graphs

4.1. Two-colored labelled graphs. For a graph Γ, denote by $E(\Gamma)$ the set of edges of Γ, $V(\Gamma)$ the set of vertices of Γ. The *genus* of the graph is given by:

$$g(\Gamma) = 1 - |V(\Gamma)| + |E(\Gamma)|. \tag{15}$$

Recall the valence $val(v)$ of a vertex v is the number of edges incident at v. Since every edge has two vertices, one has the following identity:

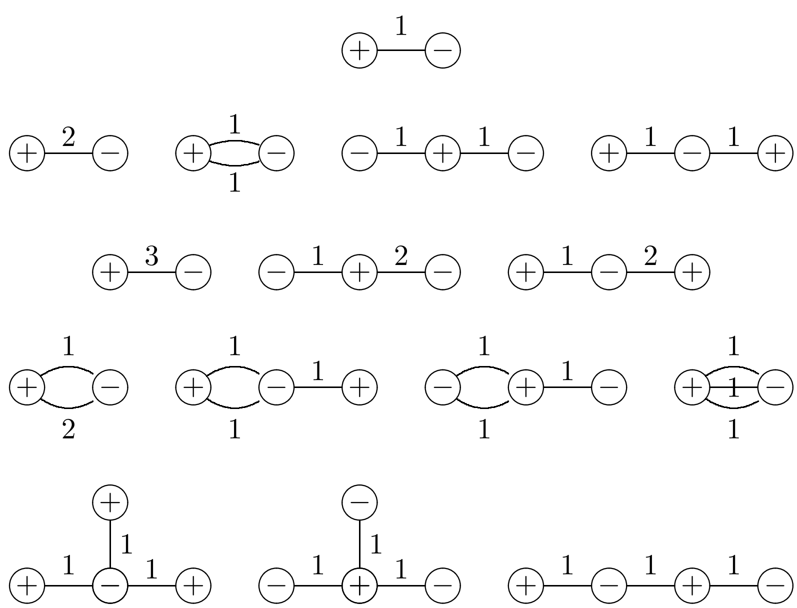
$$\sum_{v \in V(\Gamma)} \mathrm{val}(v) = 2|E(\Gamma)|. \tag{16}$$

We refer to a labelled graph with the following property as a *two-colored labelled graph*. Each vertex v is labelled by $i(v) = \pm$; each edge e is assigned a degree $d_e \in \mathbb{N}$, and its two vertices are marked by $+$ and $-$ respectively. The *degree of the graph* is defined by:

$$d(\Gamma) = \sum_{e \in E(\Gamma)} d_e. \tag{17}$$

Denote by $G_g(\{\pm\}, d)$ the set of not necessarily connected two-colored labelled graphs of genus g and degree d. The set of connected ones will be denoted by $G_g(\{\pm\}, d)^\circ$.

The two-colored labelled graphs will be regarded as Feynman diagrams. The following are some low degree examples of connected two-colored labelled graphs (up to degree 3):



Such graphs appear in the localization on $\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, d)$.

4.2. Chemistry of two-colored labelled graphs. For the discussions below, we introduce some terminologies. We will refer to a vertex together with all the edges incident at it as an *atom*. If $i(v) = \pm$, then we will say the vertex is a $\pm$-atom. The edges will be referred to as the *chemical bonds*. The degree of a bond is the degree of the edge. An $\pm$-atom of type $(d_1, \dots, d_n)$ is a vertex v such that $i(v) = \pm$, and edges incident at it have degrees $d_1, \dots, d_n$. For a partition μ, denote by $n_{\pm\mu}(\Gamma)$ the number of $\pm$-atoms of type μ in Γ. The following result is proved in [42]:

Lemma 4.1. *For any* $\Gamma \in G_g(\{\pm\}, d)$*, we have:*

$$\prod_\mu \beta_{-\mu}^{n_\mu(\Gamma)} = \prod_\mu \beta_{-\mu}^{n_{-\mu}(\Gamma)}, \tag{18}$$

$$d(\Gamma) = \sum_\mu n_+(\mu)(\Gamma)|\mu| = \sum_\mu n_{-\mu}(\Gamma)|\mu|, \tag{19}$$

$$2g(\Gamma) - 2 = \sum_\mu n_+(\mu)(\Gamma)(l_\mu - 2) + \sum_\mu n_{-\mu}(\Gamma)(l_\mu - 2). \tag{20}$$

We write

$$\{n_\mu\}_\mu \sim \{n_{-\mu}\}_\mu$$

if they satisfy (18).

4.3. Automorphism group of two-colored labelled graphs. An automorphism of a two-colored labelled graph Γ consists of two one-to-one correspondences: $f^V : V(\Gamma) \to V(\Gamma)$ and $f^E : E(\Gamma) \to E(\Gamma)$, with the following requirements.

- $i(f^V(v)) = i(v)$, for all $v \in V(\Gamma)$;
- $d_{f^E(e)} = d_e$, for all $e \in E(\Gamma)$;
- if $v_1, v_2 \in V(\Gamma)$ are the two vertices of an edge $e \in E(\Gamma)$, then $f^V(v_1)$ and $f^V(v_2)$ are the two vertices of $f^E(e)$.

The following result is prove in [42] by Wick Theorem:

Lemma 4.2. *Suppose* $\{n_\mu\}_\mu$ *and* $\{n_{-\mu}\}_\mu$ *are two collections of nonnegative integers, labelled by partitions, which contain only finitely many nonzero integers and satisfy (18). Then we have:*

$$\left\langle \prod_\mu \frac{1}{n_\mu!}\left(\frac{\beta_\mu}{z_\mu}\right)^{n_\mu} \cdot \prod_\mu \frac{1}{n_{-\mu}!}\left(\frac{\beta_{-\mu}}{z_\mu}\right)^{n_{-\mu}} \right\rangle = \sum_\Gamma \frac{1}{|\operatorname{Aut}_\Gamma| \cdot \prod_{e\in E(\Gamma)} d_e},$$

where the sum is taken over all graphs $\Gamma \in G_g(\{\pm\}, d)$ *which satisfy:*

$$\{n_{\pm\mu}(\Gamma)\} \sim \{n_{\pm\mu}\}_\mu.$$

5. Generalized Vertex Operators and Feynman Rule

5.1. Generalized vertex operators. Introduce

$$Y_{\pm}(\beta) = \sum_{d>0} \sum_{|\mu|=d} w_{\pm\mu} \frac{\beta_{\pm\mu}}{z_\mu} \lambda^{l(\mu)-2} t^{\frac{d}{2}},$$
$$X_{\pm}(\beta) = e^{Y_{\pm}(\beta)}.$$

In the following, we will see that the exponent of t is the degree, hence t will be referred to as the *degree tracking parameter*. Similarly, the exponent of λ will give $2g-2$, hence λ will be referred to as the *genus tracking parameter*. We will consider the correlation function

$$Z = \langle X_+(\beta) X_-(\beta) \rangle,$$

and the free energy:

$$F = \log Z.$$

5.2. Feynman rule. Now we regard $X_{\pm}(\beta)$ and $Y_{\pm}(\beta)$ as collections of $\pm$-atoms. Taking the vevs

$$\langle Y_+(\beta) Y_-(\beta) \rangle$$

can be regarded as chemical reactions in which $+$-atoms are joined to $-$-atoms by the chemical bonds. This is exactly the context of Wick theorem. The following result is proved in [42] by Wick Theorem. It plays an essential role in our interpretation by localizations.

Theorem 5.1. *The following identities hold:*

$$Z = \sum_{d\geq 0} \sum_{g\geq 0} \lambda^{2g-2} \sum_{\Gamma \in G_g(\{\pm\},d)} \frac{1}{|A_\Gamma|} \prod_{v\in V(\Gamma)} w_v \prod_{e\in E(\Gamma)} w_e, \tag{21}$$

$$F = \sum_{d>0} \sum_{g\geq 0} \lambda^{2g-2} \sum_{\Gamma \in G_g(\{\pm\},d)^\circ} \frac{1}{|A_\Gamma|} \prod_{v\in V(\Gamma)} w_v \prod_{e\in E(\Gamma)} w_e. \tag{22}$$

Here for a vertex v of type η with $i(v) = \pm$,

$$w_v = w_{\pm(\eta_1,\dots,\eta_l)};$$

for an edge e,

$$w_e = t^{d_e}.$$

6. Proof of Iqbal's Conjecture in the Case of Multiple Covering Formula

Now we have all the ingredients to prove (2) and (3). We begin with the former. We compute

$$Z(\lambda, t) = \langle e^{Y_+(\beta)} e^{Y_-(\beta)} \rangle,$$

for

$$Y_{\pm}(\beta) = \sum_{n\geq 1} \frac{(-1)^{n-1}}{n} \prod_{j=1}^{n} \sum_{\nu_j} \sum_{|\eta_j|=|\nu_j|} \frac{\chi_{\eta_j}(C(\nu_j))}{z_{\nu_j}} \cdot \mathcal{W}_{\eta_j} t^{\frac{|\nu_j|}{2}} \beta_{\pm\nu_j}$$

by two different methods. We begin with the fact that:

$$X_{\pm}(\beta) \quad = \quad e^{Y_{\pm}(\beta)} = 1 + \sum_{\nu} \sum_{|\eta|=|\nu|} \frac{\chi_{\eta}(C(\nu))}{z_{\nu}} \cdot \mathcal{W}_{\eta} t^{\frac{|\nu|}{2}} \beta_{\pm\nu}.$$

Now by (8),

$$X_{\pm}(\beta) \quad = \quad \exp\left(\sum_{d\geq 1} \frac{\sqrt{-1} t^{\frac{d}{2}} \beta_{\pm d}}{2d \sin(d\lambda/2)} \right).$$

Hence by (14),

$$\begin{aligned} Z &= \left\langle \exp\left(\sum_{d\geq 1} \frac{\sqrt{-1} t^{\frac{d}{2}} \beta_{d}}{2d \sin(d\lambda/2)} \right) \exp\left(\sum_{d\geq 1} \frac{\sqrt{-1} t^{\frac{d}{2}} \beta_{-d}}{2d \sin(d\lambda/2)} \right) \right\rangle \\ &= \exp\left(-\sum_{d\geq 0} \frac{t^d}{d(2\sin(d\lambda/2))^2} \right). \end{aligned}$$

On the other hand,

$$\begin{aligned} Z &= \langle X_+(\beta) X_-(\beta) \rangle \\ &= \left\langle \left(1 + \sum_{\nu} \sum_{|\eta|=|\nu|} \frac{\chi_{\eta}(C(\nu))}{z_{\nu}} \cdot \mathcal{W}_{\eta} t^{\frac{|\nu|}{2}} \beta_{\nu} \right) \right. \\ &\quad \left. \cdot \left(1 + \sum_{\nu} \sum_{|\eta|=|\nu|} \frac{\chi_{\eta}(C(\nu))}{z_{\nu}} \cdot \mathcal{W}_{\eta} t^{\frac{|\nu|}{2}} \beta_{-\nu} \right) \right\rangle \\ &= 1 + \sum_{\nu} \sum_{|\eta_1|=|\nu|} \sum_{|\eta_2|=|\nu|} \frac{\chi_{\eta_1}(C(\nu))}{z_{\nu}} \mathcal{W}_{\eta_1} \cdot \frac{\chi_{\eta_2}(C(\nu))}{z_{\nu}} \mathcal{W}_{\eta_2} t^{|\nu|} z_{\nu} \\ &= \sum_{|\eta_1|=|\eta_2|=d} \delta_{\eta_1,\eta_2} \mathcal{W}_{\eta_1} \mathcal{W}_2 t^d \\ &= 1 + \sum_{d>0} t^d \sum_{|\eta|=d} \mathcal{W}_{\eta}^2. \end{aligned}$$

Hence we have:

$$-\sum_{d\geq 0} \frac{t^d}{d(2\sin(d\lambda/2))^2} = \log\left(1 + \sum_{d>0} t^d \sum_{|\eta|=d} \mathcal{W}_{\eta}^2 \right). \tag{23}$$

This is exactly (2).

Identity (3) is proved in the same fashion by computing

$$Z(\lambda, t) = \langle e^{Y_+(\beta)} e^{Y_-(\beta)} \rangle, \tag{24}$$

for

$$Y_\pm(\beta) = \sum_{n\geq 1} \frac{(-1)^{n-1}}{n} \prod_{j=1}^{n} \sum_{\nu_j} \sum_{|\eta_j|=|\nu_j|} \frac{\chi_{\eta_j}(C(\nu_j))}{z_{\nu_j}} \cdot \mathcal{W}_{\eta_j} t^{\frac{|\nu_j|}{2}} (\pm 1)^{l(\nu_j)} \beta_{\pm\nu_j} \tag{25}$$

by two different methods.

7. Interpretation by localizations on $\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, d)$

In this section we explain how to interpret (3) in terms of localizations on $\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, d)$.

7.1. Mariño-Vafa formula. A key observation is that by taking suitable choices the localization techniques yield exactly the same Hodge integrals as in the Mariño-Vafa formula. (This idea was first used by the author in [41] to compute BPS numbers in local $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ geometry.) So we will now recall this formula as formulated in [38].

The open string theory for $\mathcal{O}(-1)\oplus\mathcal{O}(-1) \to \mathbb{P}^1$ have been studied in different approaches. In [12] and [19] localization calculations have been carried out. In [28] and many other related works, duality with Chern-Simons theory with Wilson loop observables was used. By comparing with the results of [12], Mariño and Vafa proposed a remarkable formula for Hodge integrals which we come to recall. For a partition η of length l, introduce the following generating series for Hodge integrals:

$$C_\eta(\lambda; \alpha, \beta) = \sum_{g\geq 0} \lambda^{2g} \int_{\overline{\mathcal{M}}_{g,h}} \frac{\Lambda_g^\vee(1)\Lambda_g^\vee(\alpha)\Lambda_g^\vee(\beta)}{\prod_{i=1}^h \frac{1}{n_i}(\frac{1}{n_i} - \psi_i)},$$

where $\Lambda_g^\vee(\alpha) = \sum_{i=0}^g (-1)^i \alpha^{g-i} \lambda_i$. Here the leading term is

$$|\eta|^{h-3} \cdot \prod_{i=1}^{l} \eta_i^2 \tag{26}$$

in all cases. We will often write $C_\eta(\lambda; \alpha, \beta)$ as $C_\eta(\alpha, \beta)$ for simplicity of notations. Then the Marino-Vafa formula is:

$$\begin{aligned} &C_\eta(p, -p-1) \\ =&\lambda^{2-l(\eta)} \cdot \frac{1}{(p(p+1))^{l(\eta)-1}} \cdot \prod_{i=1}^{l(\eta)} \frac{\eta_i \cdot \eta_i!}{\prod_{j=1}^{\eta_i - 1}(j + \eta_i p)} \cdot \frac{\prod_j m_j(\eta)!}{\sqrt{-1}^{l(\eta)}} \\ &\cdot \sum_{n\geq 1} \frac{(-1)^n}{n} \sum_{\cup_{j=1}^n \mu^j = \eta} \prod_{j=1}^{n} \sum_{|\nu^j|=|\mu^j|} \frac{\chi_{\nu^j}(C(\mu^j))}{z_{\mu^j}} \cdot e^{\sqrt{-1}p\kappa_{\nu^j}\lambda/2} \cdot \mathcal{W}_{\nu^j}. \end{aligned} \tag{27}$$

Some low degree cases of this formula has been proved by the author [39] (see [38] for the announcement). In [40] we propose to prove the general case based on the cut-and-join equation. This has been carried in joint work with Chiu-Chu Liu and Kefeng Liu [24, 25]. See [30] for a different approach.

7.2. From Localization on $\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, d)$ to Feynman Rule. The localizations on the moduli space $\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, d)$ involve the description of the fixed point set components on the moduli spaces and the computation of the equivariant Euler class of their normal bundles; furthermore, one has to study the restriction of the equivariant Euler classes of $\mathcal{O}(-1)^g_d$ to the fixed point components. Such calculations have been carried out in various contexts in e.g. [14, 10, 20, 21, 22, 23, 4, 12, 19, 39]. As in [41] we formulate the results as Feynman rules.

We use the following $T(= S^1)$-action on $\mathbb{P}^1$:

$$e^{\sqrt{-1}t} \cdot [z^0 : z^1] = [z_0 : e^{\sqrt{-1}t} z_1].$$

It has two fixed points

$$p_0 = [1 : 0], \qquad\qquad p_1 = [0 : 1].$$

At each fixed point p_i, the induced action on the tangent space is given by:

$$e^{\sqrt{-1}t} \cdot v = e^{\sqrt{-1}(j-i)t} v,$$

where $\{i, j\} = \{0, 1\}$. It is easy to see that

$$j - i = (-1)^i.$$

The above T-action induces T-actions on $\overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, d)$. The fixed point components of $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1, d)^T$ are very easy to describe. They are in one-to-one correspondence with a set $G_g(\mathbb{P}^1, d)$ of decorated graphs described below. Each vertex v of the graph $\Gamma \in G_g(\mathbb{P}^1, d)$ is assigned an index $i(v) \in \{0, 1\}$, and a genus $g(v)$. The valence $val(v)$ of v is the number of edges incident at v. If two vertices u and v are joined by an edge e, then $i(u) \neq i(v)$, and e is assigned a degree d_e. Denote by $E(\Gamma)$ the set of edges of Γ, $V(\Gamma)$ the set of vertices of Γ. The genus of the graph is given by

$$g(\Gamma) = 1 - |V(\Gamma)| + |E(\Gamma)|.$$

The decorations of Γ are required to satisfy the following conditions:

$$\sum_{e \in E(\Gamma)} d_e = d, \qquad\qquad \sum_{v \in V(\Gamma)} g(v) + g(\Gamma) = g.$$

Let $f : C \to \mathbb{P}^1$ represent a fixed point. When $2g(v) - 2 + \mathrm{val}(v) > 0$ the vertex v corresponds to a connected component C_v of genus $g(v)$, with $\mathrm{val}(v)$ nodal points. The component C_v is mapped by f to the fixed point $p_{i(v)}$. When $2g(v) - 2 + \mathrm{val}(v) \leq 0$, C_v is simply a point. There are only two cases: $g(v) = 0$ and $\mathrm{val}(v) = 1$, $g(v) = 0$ and $\mathrm{val}(v) = 2$. They will be referred to as the type I and type II unstable vertices respectively. Each edge e corresponds to a component of C, isomorphic to $\mathbb{P}^1$. Each C_e is mapped to $\mathbb{P}^1$ with degree d_e, which in suitable coordinates is given by:

$$[u_0 : u_1] \mapsto [u_0^{d_e} : u_1^{d_e}].$$

Define

$$\overline{\mathcal{M}}_\Gamma = \prod_{v \in V(\Gamma)} \overline{\mathcal{M}}_{g(v),val(v)}.$$

In this product, $\overline{\mathcal{M}}_{0,1}$ and $\overline{\mathcal{M}}_{0,2}$ are interpreted as points. There are natural morphisms

$$\tau_\Gamma : \overline{\mathcal{M}}_\Gamma \to \overline{\mathcal{M}}_{g,0}(\mathbb{P}^1, d).$$

Its image is $\overline{\mathcal{M}}_\Gamma / A_\Gamma$, where for A_Γ we have an exact sequence:

$$0 \to \prod_{e \in E(\Gamma)} \mathbb{Z}_{d_e} \to A_\Gamma \to \mathrm{Aut}(\Gamma) \to 1.$$

Given a graph Γ in $G_g(\mathbb{P}^1, d)$, we call the labelled graph obtained from Γ by ignoring the markings of $g(v)$ of the vertices the type of Γ. Denote by $G(\mathbb{P}^1, d)$ the set of types of graphs in $G_g(\mathbb{P}^1, d)$. Clearly $G_g(\mathbb{P}^1, d)$ is exactly $G_g(\{\pm\}, d)$ discussed above.

For $k \in \mathbb{Z}$, a lifting of the T-action to $L_k = \mathcal{O}_{\mathbb{P}^1}(k)$ is determined by the weights a_0 of $L_k|_{[1:0]}$ and a_1 of $L_k|_{[0:1]}$. It is easy to see that $a_0 - a_1 = k$. Following [10], we say L_k is given the weights $[a_0, a_1]$. For example, the induced action on $T\mathbb{P}^1$ has weights $[1, -1]$, the cotangent bundle has weights $[-1, 1]$.

For the two copies of $\mathcal{O}(-1)$, we use weights $[p, -p-1]$ and $[-p-1, p]$ respectively to carry out the localization. Then we take the limit $p \to 0$. This makes sense because of the following observation. By localization we get an expression of F_{inst} as formal power series in t and λ whose coefficients are polynomials in p. But F_{inst} itself is independent of p, hence even though we obtain the expression originally for integral p, it actually holds for all complex numbers of p, hence we can take the limit. (It is interesting to compare with [7] where the localization was done directly for $p = 0$ to prove (1).) As a result, we get the following:

Theorem 7.1. [42] *We have the following Feynman rule:*

$$F_{inst}(\lambda, t) = \sum_{g \geq 0} \sum_{d > 0} \lambda^{2g-2} t^d \sum_{\Gamma \in G_g(\{\pm\}, d)} \frac{\lambda^{2g(\Gamma)-2}}{|A_\Gamma|} \prod_{e \in E(\Gamma)} W_e \prod_{v \in V(\Gamma)} W_v, \tag{28}$$

where

$$\begin{aligned} W_e &= t^{d_e/2}, \\ W_v &= (-1)^{i(v)\,\mathrm{val}(v)} \lambda^{2-\mathrm{val}(v)} \cdot z_{\mu(v)} \\ &\quad \cdot \sum_{n \geq 1} \frac{(-1)^{n-1}}{n} \sum_{\cup_{j=1}^n \nu_j = \mu(v)} \prod_{j=1}^n \sum_{|\eta_j| = |\nu_j|} \frac{\chi_{\eta_j}(C(\nu_j))}{z_{\nu_j}} \cdot \mathcal{W}_{\eta_j}. \end{aligned}$$

By Theorem 5.1, this exactly corresponds to (24) and (25).

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Department of Mathematical Sciences, Tsinghua University, Beijing, 100084, China
E-mail address: jzhou@math.tsinghua.edu.cn