Mathematical Research Letters **11**, 171–186 (2004)

CONNECTIONS WITH TORSION, PARALLEL SPINORS AND GEOMETRY OF SPIN(7) MANIFOLDS

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Abstract. We show that on every Spin(7)-manifold there always exists a unique linear connection with totally skew-symmetric torsion preserving a nontrivial spinor and the Spin(7) structure. We express its torsion and the Riemannian scalar curvature in terms of the fundamental 4-form. We present an explicit formula for the Riemannian covariant derivative of the fundamental 4-form in terms of its exterior differential. We show the vanishing of the $\hat{A}$-genus and obtain a linear relation between Betti numbers of a compact Spin(7) manifold which is locally but not globally conformally equivalent to a space with closed fundamental 4-form. A general solution to the Killing spinor equations is presented.

1. Introduction

Riemannian manifolds admitting parallel spinors with respect to a metric connection with totally skew-symmetric torsion recently become a subject of interest in theoretical and mathematical physics. One of the main reasons is that the number of preserving supersymmetries in string theory depends essentially on the number of parallel spinors. In 10-dimensional string theory, the Killing spinor equations in the string frame can be written in the following way [43], (see eg [30, 29, 16])

$$\nabla\psi = 0, \tag{1.1}$$

$$(d\Psi - \frac{1}{2}H)\cdot\psi = 0, \tag{1.2}$$

where Ψ is a scalar function called the dilation, H the 3-form field strength, ψ a spinor field and ∇ a metric connection with totally skew-symmetric torsion $T = H$. The number of preserving supersymmetries is determined by the number of solutions of these equations.

The existence of a parallel spinor imposes restrictions on the holonomy group since the spinor holonomy representation has to have a fixed point. In the case of torsion-free metric connections (Levi-Civita connections) the possible Riemannian holonomy groups are known to be SU(n), Sp(n), G_2, Spin(7) [28,

Received July 17, 2002.

Subj. Class.: Special Riemannian manifolds, Spin geometry, String theory *MS* 53C25; 53C27; 53C55; 81T30.

Keywords: Spin(7) structure, torsion, Dirac operator, paralllel spinors, Killing spinor equations.

46]. The Riemannian holonomy condition imposes strong restrictions on the geometry and leads to considerations of Calabi-Yau manifolds, hyper-Kähler manifolds, parallel G_2-manifolds, parallel Spin(7) manifolds. All of them are of great interest in mathematics (see [32] for precise discussions) as well as in high-energy physics, string theory [37].

It just happens that the geometry of these spaces is too restrictive for various questions in string theory [36, 41, 23]. It seems that a 'nice' mathematical generalization of Calabi-Yau manifolds, hyper-Kähler manifolds, parallel G_2-manifolds, parallel Spin(7) manifolds is to consider linear connections with skew-symmetric torsion and holonomy contained in SU(n), Sp(n), G_2, Spin(7).

A remarkable fact is that the existence (in small dimensions) of a parallel spinor with respect to a metric connection with skew-symmetric torsion determines the connection in a unique way in the cases where its holonomy group is a subgroup of SU, Sp, G_2 provided additional differential conditions on the structure are fulfilled [43, 16]. The uniqueness property leads to the idea that it is worth to study the geometry of such a connection with torsion, besides its interest in physics [37, 23], for purely mathematical reasons expecting to get information about the curvature of the metric, Betti numbers, Hodge numbers, $\hat{A}$-genus, etc. In fact, a connection with skew symmetric torsion preserving a given complex structure on a Hermitian manifold was used by Bismut [3] to prove a local index formula for the Dolbeault operator when the manifold is not Kähler. Following this idea, a vanishing theorem for the Dolbeault cohomology on a compact Hermitian non-Kähler manifold was found [1, 29, 30].

In this paper we study the existence of parallel spinors with respect to a metric connection with skew-symmetric torsion in dimension 8 (for dimensions 4,5,6,7 see [43, 12, 29, 16, 17]). The first consequence is that the manifold should be a Spin(7) manifold, i.e. its structure group can be reduced to the group Spin(7). This is because the Euler characteristic $\mathcal{X}(S_\pm)$ of at least one of the (negative S_- or positive S_+) spinor bundles vanishes and therefore the structure group can be reduced to Spin(7) [34]. Surprisingly, we discover that the converse is always true in dimension 8. We show that the existence of a connection with totally skew-symmetric torsion preserving a spinor in dimension 8 is unobstructed, i.e. on every Spin(7) 8-manifold there always exists a unique linear connection with totally skew-symmetric torsion preserving a nontrivial spinor ie with holonomy contained in Spin(7). This phenomena does not occur in the cases of holonomy groups SU, Sp, G_2 (see the end of the paper). We find a formula for the torsion 3-form and for the Riemannian scalar curvature in terms of the fundamental 4-form. Our main result is the following

Theorem 1.1. *Let (M, g, Φ) be an 8-dimensional Spin(7) manifold with fundamental 4-form Φ.*

i). There always exists a unique linear connection ∇ preserving the Spin(7) structure, $\nabla\Phi = \nabla g = 0$, with totally skew-symmetric torsion T given by

$$T = -\delta\Phi - \frac{7}{6} * (\theta \wedge \Phi), \quad \theta = \frac{1}{7} * (\delta\Phi \wedge \Phi). \tag{1.3}$$

On any Spin(7) manifold there exists a ∇-parallel spinor ϕ corresponding to the fundamental form Φ and the Clifford action of the torsion 3-form on it is

$$T \cdot \phi = -\frac{7}{6}\theta \cdot \phi. \tag{1.4}$$

ii). The Riemannian scalar curvature $Scal^g$ and the scalar curvature $Scal$ of the Spin(7) connection ∇ are given in terms of the fundamental 4-form Φ by

$$Scal^g = \frac{49}{18}||\theta||^2 - \frac{1}{12}||T||^2 + \frac{7}{2}\delta\theta, \quad Scal = \frac{49}{18}||\theta||^2 - \frac{1}{3}||T||^2 + \frac{7}{2}\delta\theta. \tag{1.5}$$

The proof relies on our explicit formula expressing the covariant derivative of the fundamental 4-form Φ with respect to the Levi-Civita connection in terms of the exterior derivative of Φ. The existence of such a relation was discovered by R.L.Bryant [7] in his proof that the holonomy group of the Levi-Civita connection is contained in Spin(7) iff $d\Phi = 0$ (see also [39]). We prove ii) using the Schrödinger-Lichnerowicz formula for the connection with torsion established in [16] and the special properties of the Clifford action on the special spinor ϕ.

In the compact case, we use the formula for the Riemannian scalar curvature to show that the Yamabe constant of one of the two classes of Spin(7) manifolds according to Fernandez classification [13] is strictly positive. Applying the Atiyah-Singer index theorem [2], as well as the Lichnerowicz vanishing theorem [35], we find a linear relation between the Betti numbers and show that the Euler characteristic is equal to 3 times the signature.

In the last section we give necessary and sufficient conditions for the existence of a solution to the Killing spinor equations (1.1), (1.2) in an 8-dimensional manifold. We apply our general formula for the torsion of the connection admitting a parallel spinor to the second Killing spinor equation. As a consequence, we obtain a formula for the field strength (torsion) of a solution to both Killing spinor equations in terms of the fundamental 4-form. We discover a relation between a solution to both Killing spinor equations with non-constant dilation and the conformal transformations of the Spin(7) structures. In fact we show that the dilation function arises geometrically (from the Lee form of the structure) and can be interpreted as a conformal factor. Our analysis on the two Killing spinor equations in dimension 8 shows that the physics data (field strength H and the dilation function Ψ) are determined completely by the properties of the parallel spinor or equivalently by the geometry of the corresponding fundamental 4-form.

2. General properties of Spin(7) manifold

We recall some notions of Spin(7) geometry.

Let us consider $\mathcal{R}^8$ endowed with an orientation and its standard inner product $<,>$. Let $\{e_0, ..., e_7\}$ be an oriented orthonormal basis. We shall use the same notation for the dual basis. We denote by e_{ijkl} the monomial $e_i\wedge\epsilon_j\wedge\epsilon_k\wedge\epsilon_l$. Consider the 4-form Φ on $\mathcal{R}^8$ given by

$$\begin{aligned}(2.6)\qquad \Phi &= e_{0123}+e_{0145}+e_{0167}+e_{0246}-e_{0257}-e_{0347}-e_{0356}\\ &+ e_{4567}+e_{2367}+e_{2345}+e_{1357}-e_{1346}-e_{1247}-e_{1256}\quad .\end{aligned}$$

The 4-form Φ is self-dual $*\Phi=\Phi$, where $*$ is the Hodge $*$-operator and the 8-form $\Phi\wedge\Phi$ coincides with the volume form of $\mathcal{R}^8$. The subgroup of $GL(8,R)$ which fixes Φ is isomorphic to the double covering Spin(7) of $SO(7)$ [26]. Moreover, Spin(7) is a compact simply-connected Lie group of dimension 21 [7]. The 4-form Φ corresponds to a real spinor ϕ and therefore, Spin(7) can be identified as the isotropy group of a non-trivial real spinor.

A 3-fold vector cross product P on $\mathcal{R}^8$ can be defined by $<P(x\wedge y\wedge z),t>=\Phi(x,y,z,t)$, for $x,y,z,t\in\mathcal{R}^8$. Then Spin(7) is also characterized by

$$Spin(7)=\{a\in O(8)|P(ax\wedge ay\wedge az)=P(x\wedge y\wedge z),x,y,z\in\mathcal{R}^8\}.$$

The inner product $<,>$ on $\mathcal{R}^8$ can be reconstructed from Φ [13, 25], which corresponds with the fact that Spin(7) is a subgroup of SO(8).

A *Spin(7) structure* on an 8-manifold M is by definition a reduction of the structure group of the tangent bundle to Spin(7); we shall also say that M is a *Spin(7) manifold*. This can be described geometrically by saying that there is a 3-fold vector cross product P defined on M, or equivalently there exists a nowhere vanishing differential 4-form Φ on M which can be locally written as (2.6). The 4-form Φ is called the *fundamental form* of the Spin(7) manifold M [4].

Let (M,g,Φ) be a Spin(7) manifold. The action of Spin(7) on the tangent space gives an action of Spin(7) on $\Lambda^k(M)$ and so the exterior algebra splits orthogonally into components, where Λ^k_l corresponds to an irreducible representation of Spin(7) of dimension l [13, 7]:

$$\Lambda^1(M)=\Lambda^1_8,\quad \Lambda^2(M)=\Lambda^2_7\oplus\Lambda^2_{21},\quad \Lambda^3(M)=\Lambda^3_8\oplus\Lambda^3_{48},$$

$$\Lambda^4(M)=\Lambda^4_+(M)\oplus\Lambda^4_-(M),\quad \Lambda^4_+(M)=\Lambda^4_1\oplus\Lambda^4_7\oplus\Lambda^4_{27},\quad \Lambda^4_-=\Lambda^4_{35};$$

where $\Lambda^4_\pm(M)$ are the $\pm$-eigenspaces of $*$ on $\Lambda^4(M)$ and [4, 7, 39]

$$\Lambda^2_7=\{\alpha\in\Lambda^2(M)|*(\alpha\wedge\Phi)=3\alpha\},\quad \Lambda^2_{21}=\{\alpha\in\Lambda^2(M)|*(\alpha\wedge\Phi)=-\alpha\}$$

$$\Lambda^3_8=\{*(\beta\wedge\Phi)|\beta\in\Lambda^1(M)\},\Lambda^3_{48}=\{\gamma\in\Lambda^3(M)|\gamma\wedge\Phi=0\},\Lambda^4_1=\{f\Phi|f\in\mathcal{F}(M)\}$$

The Hodge star $*$ gives an isometry between Λ^k_l and Λ^{8-k}_l.

If (M,g,Φ) is a Spin(7) manifold, then M is orientable and spin, with preferred spin structure and orientation. If $S=S_+\oplus S_-$ is the spin bundle of M, then there are natural isomorphisms $S_+\equiv\Lambda^0_1\oplus\Lambda^2_7$ and $S_-\equiv\Lambda^1_8$ (see eg [32]).

In general, not every 8-dimensional Riemannian spin manifold M^8 admits a Spin(7) structure. We explain the precise condition [34]. Denote by $p_1(M),p_2(M)$, $\mathcal{X}(M),\mathcal{X}(S_\pm)$ the first and the second Pontrjagin classes, the Euler characteristic of M and the Euler characteristic of the positive and the negative spinor bundles, respectively. It is well known [34] that a spin 8-manifold admits a

Spin(7) structure if and only if $\mathcal{X}(S_+) = 0$ or $\mathcal{X}(S_-) = 0$. The latter conditions are equivalent to [34]

$$p_1^2(M) - 4p_2(M) + 8\mathcal{X}(M) = 0, \tag{2.7}$$

for an appropriate choice of the orientation.

Let us recall that a Spin(7) manifold (M, g, Φ) is said to be parallel (torsion-free [32]) if the holonomy of the metric $Hol(g)$ is a subgroup of Spin(7). This is equivalent to saying that the fundamental form Φ is parallel with respect to the Levi-Civita connection ∇^g of the metric g. Moreover, Hol(g) $\subset$ Spin(7) if and only if $d\Phi = 0$ [7] (see also [39]) and any parallel Spin(7) manifold is Ricci flat [4].

According to the Fernandez classification [13], there are 4-classes of Spin(7) manifolds obtained as irreducible representations of Spin(7) of the space $\nabla^g\Phi$. Following [9] we consider the 1-form θ defined by

$$7\theta = - * (*d\Phi \wedge \Phi) = *(\delta\Phi \wedge \Phi) \tag{2.8}$$

We shall call the 1-form θ *the Lee form* of a given Spin(7) structure.

The 4 classes of Spin(7) manifolds in the Fernandez classification can be described in terms of the Lee form as follows [9]: $W_0 : d\Phi = 0$; $W_1 : \theta = 0$; $W_2 : d\Phi = \theta \wedge \Phi$; $W : W = W_1 \oplus W_2$.

We shall call a Spin(7) structure of the class W_1 (ie Spin(7) structures with zero Lee form) *a balanced* Spin(7) structure.

In [9] Cabrera shows that the Lee form of a Spin(7) structure in the class W_2 is closed and therefore such a manifold is locally conformally equivalent to a parallel Spin(7) manifold and it is called *locally conformally parallel*. If the Lee form is not exact (i.e. the structure is not globally conformally parallel), we shall call it *strict locally conformally parallel*. We shall see later (section 8) that these spaces have very different topology than parallel ones.

Coeffective cohomology and coeffective numbers of Riemannian manifolds with Spin(7) structure are studied in [45].

3. Examples:

Examples of Spin(7) manifolds are constructed relatively recently.

The first known explicit example of complete parallel Spin(7) manifold with Hol(g) = Spin(7) was constructed by Bryant and Salamon [8, 24] on the total space of the spin bundle over the 4-sphere.

The first compact examples of parallel Spin(7) manifolds with Hol(g) = Spin(7) were constructed by Joyce[31, 32] by resolving the singularities of the orbifold T^8/Γ for certain discrete groups Γ.

Most examples of Spin(7) manifolds in the Fernandez classification are constructed by using certain G_2-manifolds. We recall that a G_2-manifold N is a 7-dimensional manifold whose structure group can be reduced to the exceptional group G_2 or equivalently, there exists on N a distinguished associative 3-form γ.

A G_2-manifold is said to be *nearly parallel, cocalibrated of pure type, calibrated* if $d\gamma = const. * \gamma$; $\quad \delta\gamma = 0, d\gamma \wedge \gamma = 0; d\gamma = 0$, respectively [14].

Any 8-manifold of type $M = S^1 \times N$ possesses a Spin(7) structure defined by [39, 45, 10] $\Phi = \eta \wedge \gamma + *\gamma$, where η is a non-zero 1-form on S^1. The induced Spin(7) structure on M is [10]

i) a strict locally conformally parallel if the G_2 structure is nearly parallel;

ii) a balanced one if the G_2-structure is cocalibrated of pure type or calibrated or belongs to the direct sum of these classes.

There are many known examples of compact nearly parallel G_2-manifolds: S^7 [14], SO(5)/SO(3) [8, 39], the Aloff-Wallach spaces $N(g,l)$ =SU(3)/U$(1)_{g,l}$ [11] any Einstein-Sasakian and any 3-Sasakian space in dimension 7 [18, 19], some examples coming from 7-dimensional 3-Sasaki manifolds [19, 20], the 3-Sasakian non-regular spaces $S(p_1, p_2, p_3)$ [5, 6], compact nearly parallel G_2-manifolds with large symmetry groups are classified recently in [19]. The product of each of these spaces by S^1 gives examples of strict locally conformally parallel Spin(7) structures.

Any minimal hypersurface N in R^8 possesses a cocalibrated structure of pure type G_2 [14] and therefore $M = N \times S^1$ has a balanced Spin(7) structure described above.

More general, any principle fibre bundle with one dimensional fibre over a G_2-manifold carries a Spin(7) structure [9]. In this way, a balanced Spin(7) structure arises on a principle circle bundle over a 7-dimensional torus T^7 considered as a G_2-manifold [9].

4. Conformal transformations of Spin(7) structures

We need the next result which is essentially established in [13].

Proposition 4.1. [13] *Let* $\bar{g} = e^{2f} g$, $\quad \bar{\Phi} = e^{4f}\Phi$ *be a conformal change of the given Spin(7) structure* (g, Φ) *and* $\bar{\theta}$, $\quad \theta$ *are the corresponding Lee 1-forms, respectively. Then*

$$\bar{\theta} = \theta + 4df \tag{4.9}$$

Proof. We have [13] $vol._{\bar{g}} = e^{8f} vol._g$, $\quad d\bar{\Phi} = e^{4f}(4df \wedge \Phi + d\Phi)$. We calculate

$$\bar{*} d\bar{\Phi} = e^{4f}(*d\Phi + 4*(df \wedge \Phi)), \quad \bar{*} d\bar{\Phi} \wedge \bar{\Phi} = e^{8f}(*d\Phi \wedge \Phi + 28 * df),$$

where we used the identity $*(\Phi \wedge \gamma) \wedge \Phi = 7 * \gamma$, $\quad \gamma \in \Lambda^1(M)$. We obtain consequently that $\bar{\theta} = -\frac{1}{7}\bar{*}(\bar{*}d\bar{\Phi} \wedge \bar{\Phi}) = -\frac{1}{7} * (*d\Phi \wedge \Phi) - 4 *^2 df = \theta + 4df$. $\square$

More generally, we have

Corollary 4.2. *If the Lee 1-form is closed, then the Spin(7) structure is locally conformal to a balanced Spin(7) structure.*

Proposition 4.1 allows us to find a distinguished Spin(7) structure on a compact 8-dimensional Spin(7) manifold.

Theorem 4.3. *Let (M^8, g, Φ) be a compact 8-dimensional Spin(7) manifold. Then there exists a unique (up to homothety) conformal Spin(7) structure $g_0 = e^{2f} g, \Phi_0 = e^{4f}\Phi$ such that the corresponding Lee 1-form is coclosed, $\delta_0\theta_0 = 0$.*

Proof. We shall use the Gauduchon theorem for the existence of a distinguished metric (Gauduchon metric) on a compact Hermitian or Weyl manifold [21, 22]. We shall use the expression of this theorem in terms of a Weyl structure (see [44], Appendix 1). We consider the Weyl manifold $(M^8, g, \theta, \nabla^W)$ with the Weyl 1-form θ where ∇^W is a torsion-free linear connection on M^8 determined by the condition $\nabla^W g = \theta \otimes g$. Applying the Gauduchon theorem we can find in a unique way a conformal metric g_0 such that the corresponding Weyl 1-form is coclosed with respect to g_0. The key point is that by Proposition 4.1 the Lee 1-form transforms under conformal rescaling according to (4.9) which is exactly the transformation of the Weyl 1-form under conformal rescaling of the metric $\bar{g} = e^{4f} g$. Thus, there exists a unique (up to homothety) conformal Spin(7) structure (g_0, Φ_0) with coclosed Lee 1-form. □

We shall call the Spin(7) structure with coclosed Lee 1-form *the Gauduchon Spin(7) structure.*

Corollary 4.4. *Let (M, g, Φ) be a compact Spin(7) manifold and (g, Φ) be the Gauduchon structure. Then the following formula holds* $*(d\delta\Phi \wedge \Phi) = -||d\Phi||^2$.

Proof. Using (2.8), we calculate that

$$0 = 7\delta\theta = *d(*d\Phi \wedge \Phi) = *(d\delta\Phi \wedge \Phi - *d\Phi \wedge d\Phi) = *\left(d\delta\Phi \wedge \Phi + ||d\Phi||^2.vol\right).$$

□

Corollary 4.5. *On a compact Spin(7) manifold with closed Lee form the first Betti number $b_1 \geq 1$ provided the Gauduchon Spin(7) structure is not balanced. In particular, on any strict locally conformally parallel Spin(7) manifold, $b_1 \geq 1$.*

5. A formula for the covariant derivative of the fundamental form

In [7] R.L. Bryant proved that on a Spin(7) manifold (M, g, Φ) the holonomy group $Hol(g)$ of the metric g is contained in Spin(7) iff the fundamental form Φ is closed i.e. $\nabla^g\Phi = 0$ is equivalent to $d\Phi = 0$. This shows that there is an identification of $\nabla^g\Phi$ and $d\Phi$ (see also [39]). The aim of this section is to give an explicit formula.

Let γ be an 1-form, $\gamma \in \Lambda^1(M)$. We use the same notation for the dual vector field via the metric and denote by i_γ the interior multiplication. The next algebraic fact follows by direct computations

Proposition 5.1. *For any 1-form γ the identity* $*(\Phi \wedge \gamma) = i_\gamma\Phi$ *holds.*

Theorem 5.2. *Let (M,g,Φ) be a Spin(7) manifold with fundamental 4-form Φ, P be the corresponding 3-fold vector cross product and ∇^g be the Levi-Civita connection of g. Then the following formula holds for all vector fields X,Y,Z,V,W:*

$$
\begin{aligned}
(\nabla^g_X\Phi)(Y,Z,V,W) = &\\
&\frac{1}{2}\{\delta\Phi(X,Y,P(Z,V,W)) - \delta\Phi(X,Z,P(Y,V,W))\}\\
&+\frac{1}{2}\{\delta\Phi(X,V,P(Y,Z,W)) - \delta\Phi(X,W,P(Y,Z,V))\}\\
&-\frac{1}{12}\{*(\delta\Phi\wedge\Phi)(P(X,Y,P(Z,V,W)) - *(\delta\Phi\wedge\Phi)(P(X,Z,P(Y,V,W))\}\\
&+\frac{1}{12}\{*(\delta\Phi\wedge\Phi)(P(X,V,P(Y,Z,W)) - *(\delta\Phi\wedge\Phi)(P(X,W,P(Y,Z,V))\}
\end{aligned} \tag{5.10}
$$

Proof. We have the general formulas (see eg [33])

$$
\begin{aligned}
(\nabla^g_X\Phi)(Y,Z,V,W) &= X\Phi(Y,Z,V,W) - \Phi(\nabla^g_XY,Z,V,W)\\
-\Phi(Y,\nabla^g_XZ,V,W) &- \Phi(Y,Z,\nabla^g_XV,W) - \Phi(Y,Z,V,\nabla^g_XW),
\end{aligned} \tag{5.11}
$$

$$
\begin{aligned}
2g(\nabla^g_XY,Z) &= Xg(Y,Z) + Yg(X,Z) - Zg(X,Y)\\
&+ g([X,Y],Z) + g([Z,X],Y) - g([Y,Z],X).
\end{aligned} \tag{5.12}
$$

Let $\{e_0,e_1,...,e_7\}$ be an orthonormal basis and the fundamental form Φ be given by (2.6). We substitute (5.12) into (5.11). Using the expression (2.6) and keeping in mind Proposition 5.1, we check that the right hand side of the obtained equality coincides with the right hand side of (5.10) by long but straightforward calculations evaluating the both sides on the basis $e_0,e_1,...,e_7$. □

6. Proof of Theorem 1.1 part i)

Suppose that a connection ∇ determined by

$$
g(\nabla_XY,Z) = g(\nabla^g_XY,Z) + \frac{1}{2}T(X,Y,Z), \tag{6.13}
$$

where T is a 3-form, satisfies $\nabla\Phi = 0$. Then we have

$$
\begin{aligned}
2(\nabla^g_X\Phi)(Y,Z,V,W) &= \Phi(T(X,Y),Z,V,W) + \Phi(Y,T(X,Z),V,W)\\
&+ \Phi(Y,Z,T(X,V),W) + \Phi(Y,Z,V,T(X,W))
\end{aligned} \tag{6.14}
$$

and consequently

$$
\delta\Phi = -*d*\Phi = \sum_{i,j=0}^{7}\left((i_{e_j}i_{e_i}T)\wedge(i_{e_j}i_{e_i}\Phi)\right) \tag{6.15}
$$

Evaluating (6.15) on the orthonormal basis and using the expression of the fundamental 4-form (2.6) with respect to this basis we arrive to a linear system of maximal rank of 56 linear equations with respect to 56 unknown variables $T(e_i,e_j,e_k), i,j,k = 0,..,7$ since T is a 3-form. By the symmetries of the fundamental 4-form this system is separated into 8 linear systems and each of them consists of 7 linear equations with respect to 7 unknown variables. Solving each of these systems explicitly and using the definition of the Lee form θ we obtain (1.3).

For the converse, we define by (6.13) a connection ∇ with totally skew symmetric torsion T given by (1.3). Clearly $\nabla g = 0$. Substitute (1.3) into (6.14) and using Theorem 5.2 we get $\nabla\Phi = 0$.

Let ϕ be the spinor corresponding to Φ. Clearly ϕ is ∇ parallel. The Clifford action $T \cdot \phi$ depends only on the Λ^3_8-part of T. Using (1.3) and the algebraic formulas $*(\gamma \wedge \Phi) \cdot \phi = i_\gamma(\Phi) \cdot \phi = 7\gamma \cdot \phi$ we obtain (1.4). This proves part i). Part ii) will be proved in the next section. $\Box$

Further, we shall call the connection determined by Theorem 1.1 *the Spin(7)-connection* of a given Spin(7) manifold.

Corollary 6.1. *The Lee 1-form of any Spin(7) structure and the projections* $\pi^3_8(d\Phi)$, $\pi^3_8(T)$ *onto the space* Λ^3_8 *are given by* $\theta = \frac{6}{7} * (\Phi \wedge T)$, $\quad \pi^3_8(d\Phi) = \theta \wedge \Phi$, $\quad \pi^3_8(T) = -\frac{1}{6} * (\theta \wedge \Phi)$.

Keeping in mind Proposition 4.1, we get

Corollary 6.2. *The torsion 3-form* T *of the Spin(7) connection* ∇ *changes by a conformal transformation* $(g_o = e^{2f} g, \Phi_o = e^{4f}\Phi)$ *of the Spin(7) structure* (g, Φ) *by* $T_o = e^{4f}\left(T - \frac{2}{3} * (df \wedge \Phi)\right)$.

7. The Ricci tensor and the scalar curvature

In this section we give formulas for the Ricci tensor and the scalar curvature of the connection ∇ on a Spin(7) manifold and, consequently, formulas for the Ricci tensor and the scalar curvature of the metric g using the special properties of the Clifford action on the ∇-parallel spinor. We apply the Schrödinger-Lichnerowicz formula for the Dirac operator of a metric connection with totally skew-symmetric torsion proved in [16] to the case of the unique Spin(7)-connection ∇ on a Spin(7) manifold (M, g, Φ). Finally, we prove the part ii) of Theorem 1.1.

Let $D, Ric, Scal$ be the Dirac operator, the Ricci tensor and the scalar curvature of the Spin(7) connection defined as usually by

$$D = \sum_{i=0}^{7} e_i.\nabla_{e_i}, \quad Ric(X,Y) = \sum_{i=0}^{7} R(e_i, X, Y, e_i), \quad Scal = \sum_{i=0}^{7} Ric(e_i, e_i).$$

The relations between the Ricci tensor Ric^g and the scalar curvature $Scal^g$ of the metric are (see [29, 16])

$$Ric^g = Ric + \frac{1}{2}\delta T + \frac{1}{4}(i_{(.)}T, i_{(.)}T), \quad Scal^g = Scal + \frac{1}{4}||T||^2, \tag{7.16}$$

where $(,)$ and $||.||^2$ denote the inner product on tensors induced by g and the corresponding norm. In particular, Ric is symmetric iff the torsion 3-form is coclosed, $\delta T = 0$.

Let σ^T be the 4-form defined by $\sigma^T = \frac{1}{2}\sum_{i=0}^{7}(i_{e_i}T) \wedge (i_{e_i}T)$. We take the following result from [16].

Theorem 7.1. [16] *Let Ψ be a parallel spinor with respect to a metric connection ∇ with totally skew- symmetric torsion T on a Riemannian spin manifold M. The following formulas hold*

$$\begin{aligned} &3dT\cdot\Psi - 2\sigma^T\cdot\Psi + Scal\ \Psi = 0, \\ &\frac{1}{2}i_X dT\cdot\Psi + \nabla_X T\cdot\Psi - Ric(X)\cdot\Psi = 0, \\ &D(T\cdot\Psi) = dT\cdot\Psi + \delta T\cdot\Psi - 2\sigma^T\cdot\Psi. \end{aligned} \tag{7.17}$$

If M is compact, then for any spinor field ψ the following formula is true

$$\int_M ||D\psi||^2\, dVol = \int_M \left(||\nabla\psi||^2 + (dT\cdot\psi,\psi) + 2(\sigma^T\cdot\psi,\psi) + Scal||\psi||^2\right)\, dVol. \tag{7.18}$$

In particular, if the eigenvalues of the endomorphism $dT + 2\sigma^T + Scal$ acting on spinors are nonnegative, then every ∇-harmonic spinor is ∇-parallel. If the eigenvalues are positive, then there are no ∇ parallel spinors.

We apply Theorem 7.1 to the ∇-parallel spinor ϕ corresponding to the fundamental 4-form Φ on a Spin(7) manifold to get

Proposition 7.2. *Let (M, g, Φ, ∇) be an 8-dimensional Spin(7) manifold with the Spin(7) connection ∇ of torsion T. The Ricci tensors Ric, Ric^g are given by*

$$Ric(X) = -\frac{1}{2} * (i_X dT\wedge\Phi) - * (\nabla_X T\wedge\Phi), \tag{7.19}$$

$$Ric^g(X,Y) = \frac{1}{2}(i_X dT\wedge\Phi, *Y) + (\nabla_X T\wedge\Phi, *Y) + \frac{1}{2}\delta T(X,Y) + \frac{1}{4}(i_X T, i_Y T). \tag{7.20}$$

7.1. Proof of Theorem 1.1 ii). Let ϕ be the ∇-parallel spinor corresponding to the fundamental 4-form Φ. Then the Riemannian Dirac operator D^g and the Levi-Civita connection ∇^g act on ϕ by the rule

$$\nabla^g_X\phi = -\frac{1}{4}(i_X T)\cdot\phi, \quad D^g\phi = -\frac{3}{4}T\cdot\phi = \frac{7}{8}\theta\cdot\phi, \tag{7.21}$$

where we used (1.4). We are going to apply the well known Schrödinger-Lichnerowicz (S-L) formula [35, 42]

$$(D^g)^2 = \triangle^g + \frac{1}{4}Scal^g, \quad \triangle^g = -\sum\left(\nabla^g_{e_i}\nabla^g_{e_i} - \nabla^g_{\nabla_{e_i}e_i}\right)$$

to the ∇-parallel spinor field ϕ.

Using (7.21) we calculate as a consequence that

$$(D^g)^2\phi = \frac{7}{8}D^g(\theta\phi) = \left(\frac{49}{64}||\theta||^2 + \frac{7}{8}\delta\theta\right)\cdot\phi + \frac{7}{8}d\theta\cdot\phi + \frac{7}{16}(i_\theta T)\cdot\phi, \tag{7.22}$$

where we used the general identity $D^g\theta + \theta D^g = d\theta + \delta\theta - 2\nabla_\theta$.

We compute the Laplacian $\triangle^g$ in the general.

Lemma 7.3. *Let ϕ be a parallel spinor with respect to a metric connection ∇ with skew symmetric torsion T on a Riemannian manifold (M, g). For the Riemannian Laplacian acting on ϕ we have*

$$\triangle^g \phi = -\frac{1}{4}\delta T \cdot \phi - \frac{1}{16}\left(2\sigma^T - \frac{1}{2}||T||^2\right) \cdot \phi. \tag{7.23}$$

Proof of Lemma 7.3. We take a normal coordinate system such that $(\nabla_{e_i} e_i)_p = 0, p \in M$. We use (7.21) to get

$$\triangle^g \phi = \frac{1}{4}\sum_i \left(\nabla_{e_i} i_{e_i} T) \cdot \phi - \frac{1}{16}(i_{e_i} T) \cdot (i_{e_i} T) \cdot \phi\right).$$

Applying the properties of the Clifford multiplication we obtain (7.23) and Lemma 7.3 is proved.

Further, substituting (7.22) and (7.23) into the S-L formula, multiplying the obtained result by ϕ and taking the real part, we arrive at

$$\left(\frac{49}{64}||\theta||^2 + \frac{7}{8}\delta\theta\right)||\phi||^2 = \left(\frac{1}{32}||T||^2 + \frac{1}{4}Scal^g\right)||\phi||^2 - \frac{1}{8}(\sigma^T \cdot \phi, \phi). \tag{7.24}$$

On the other hand, using (1.4), we get

$$D(T \cdot \phi) = -\frac{7}{6}D(\theta \cdot \phi) = -\frac{7}{6}\left(d^\nabla \theta \cdot \phi + \delta\theta \cdot \phi\right),$$

where d^∇ is the exterior derivative with respect to the Spin(7) connection ∇. Now, (7.17) gives
$-\frac{7}{6}\left(d^\nabla \theta \cdot \phi + \delta\theta \cdot \phi\right) = dT \cdot \phi - 2\sigma^T \cdot \phi + \delta T \cdot \phi$. Multiplying the last equality by ϕ and taking the real part, we obtain $-\frac{7}{6}\delta\theta||\phi||^2 = (dT \cdot \phi, \phi) - (2\sigma^T \cdot \phi, \phi)$. Consequently, (7.17) and (7.16) imply

$$\left(-\frac{7}{2}\delta\theta - \frac{1}{4}||T||^2 + Scal^g\right)||\phi||^2 + 4(\sigma^T \cdot \phi, \phi) = 0. \tag{7.25}$$

Finally, we get (1.5) from (7.24) and (7.25). Thus, the proof of Theorem 1.1 is completed. □

Corollary 7.4. *On a balanced Spin(7) manifold the Ricci tensor Ric is symmetric and the Riemannian Ricci tensor and scalar curvatures are given by*

$$Ric(X, Y) = \frac{1}{2}\left(i_X(d\delta\Phi) \wedge \Phi, *Y\right), \quad Scal = -\frac{1}{3}||\delta\Phi||^2;$$

$$Ric^g(X, Y) = \frac{1}{2}\left(i_X(d\delta\Phi) \wedge \Phi, *Y\right) + \frac{1}{4}(i_X T, i_Y T), Scal^g = -\frac{1}{12}||\delta\Phi||^2. \tag{7.26}$$

In particular the Riemannian scalar curvature on a balanced Spin(7)-manifold is non-positive and vanishes identically if and only if the Spin(7)-structure is co-closed, $\delta\Phi = 0$ and therefore parallel.

A balanced Spin(7)-manifold has harmonic fundamental form, $d\delta\Phi = 0$ or equivalently it has closed torsion 3-form, $dT = 0$ if and only if the Spin(7)-structure is co-closed, $\delta\Phi = 0$ and therefore parallel.

Proof. In the case of a balanced structure, the torsion 3-form T satisfies $T = -\delta\Phi$ by Theorem 1.1. Clearly, $\delta T = 0$ and Ric is a symmetric tensor. The Clifford multiplication of a 3-form by the spinor ϕ depends only on its projection in the space Λ^3_8. The 3-form T belongs to Λ^3_{48} by Corollary 6.1 and hence, ∇T, as a 3-form, also belongs to Λ^3_{48} since the Spin(7) connection preserves the fundamental 4-form and therefore it preserves also the splitting Λ^p_l. Hence, the Clifford action of ∇T on the special spinor ϕ is trivial. The rest of the claim follows from Theorem 1.1 and Proposition 7.2. □

8. Topology of compact Spin(7) manifold

In this section we apply our results to obtain information about Betti numbers, $\hat{A}$-genus and the signature of certain classes of Spin(7) manifolds. We use essentially the solution of the Yamabe conjecture [40] as well as the fundamental Atiyah-Singer Index theorem [2] which gives a topological formula for the index of any linear elliptic operator. On a Spin(7) manifold M this reads as *ind* D$= \hat{A}(M) = indD^g$, where $\hat{A}(M)$ is a topological invariant called $\hat{A}$-genus, *ind* D$= dimKerD_+ - dimKerD_-, D_\pm : \Gamma(S_\pm) \to \Gamma(S_\mp)$ are the Dirac operators of a linear connection on M.

First, we notice that the expression of the $\hat{A}$-genus in terms of Betti numbers proved by Joyce [31, 32] for a parallel compact Spin(7) manifold holds for any compact Spin(7) manifold.

Proposition 8.1. *On a compact Spin(7) manifold (M, g, Φ) the $\hat{A}$-genus is given by*

$$24\hat{A}(M) = -1 + b_1 - b_2 + b_3 + (b_4)_+ - 2(b_4)_-, \tag{8.27}$$

where b_i are the Betti numbers of M and $(b_4)_+$ (resp. $(b_4)_-$) is the dimension of the space of harmonic self-dual (resp. anti-self dual) 4-forms.

Proof. The proof goes as in [32] following the reasoning of [38]. We recall the basic identities. The formula for the signature $\tau(M)$ and the $\hat{A}$-genus in terms of Pontrjagin classes are [27]

$$45\left((b_4)_+ - (b_4)_-\right) = 45\tau(M) = 7p_2(M) - p_1^2(M), \tag{8.28}$$

$$45.2^7\hat{A}(M) = 7p_1^2(M) - 4p_2(M).$$

Combining (8.28) with (2.7) gives (8.27). □

We state the main result of this section

Theorem 8.2. *Let M be a compact connected spin 8-manifold with a fixed orientation. If it admits a strict locally conformally parallel Spin(7) structure (g, Φ), then M admits a Riemannian metric g_Y with strictly positive constant scalar curvature, $Scal^{g_Y} > 0$.*

Consequently, the following formulas hold

i). $\hat{A}(M) = 0$;

ii).$\mathcal{X}(M) = 3\tau(M)$;

iii). $b_2 + 2(b_4)_+ - b_3 - (b_4)_- \geq 0$ *with equality iff* $b_1 = 1$.

In particular, M does not admit a metric with holonomy

$$Hol(g) = Spin(7); SU(4); Sp(2); SU(2) \times SU(2).$$

Proof. Let θ be the Lee form of (g, Φ). We need the following algebraic lemma.

Lemma 8.3. *On a Spin(7) manifold the inequality* $||T||^2 \geq \frac{7}{6}||\theta||^2$ *holds. The equality is attained if and only if the Spin(7) structure is locally conformally parallel.*

The proof of Lemma 8.3 follows from Theorem 1.1 and the equality $0 \leq ||T + \frac{1}{6} * (\theta \wedge \Phi)||^2 = ||T||^2 - \frac{7}{6}||\theta||^2$.

Lemma 8.3 gives $||T||^2 = \frac{7}{6}||\theta||^2$ since the structure is locally conformally parallel. Theorem 1.1 leads to the formula

$$Scal^g = \frac{21}{36}||\theta||^2 + \frac{7}{2}\delta\theta \quad . \tag{8.29}$$

According to the solution of the Yamabe conjecture [40] there is a metric $g_Y = e^{2f}g$ in the conformal class of g with constant scalar curvature, $Scal^{g_Y} = const.$ Consider the locally conformally parallel Spin(7) structure $(g_Y = e^{2f}g, \Phi_Y = e^{4f}\Phi)$. Equality (8.29) is true also for the structure (g_Y, Φ_Y). An integration of the last equality over a compact M gives

$$Scal^{g_Y}.Vol_{g_Y} = \frac{21}{36}\int_M ||\theta||^2 \, dVol_{g_Y} > 0,$$

since the structure is strictly locally conformally parallel. Then, by the Lichnerowicz vanishing theorem [35], $indD^{g_Y} = 0$ and $\hat{A}(M) = 0$ by the index theorem. Condition ii) follows exactly as in [38] from (8.28) and (2.7). Statement iii) is a consequence of (8.27) and Corollary 4.5. We derive the last assertion by contradiction with the already proved vanishing of the $\hat{A}$-genus and the result of Joyce [31, 32] claiming non-vanishing of the $\hat{A}$-genus for a Riemannian manifold with Riemannian holonomy groups listed in the condition of the theorem. □

Remark Information for the $\hat{A}$-genus on a compact Spin(7) manifold can be obtained if the eigenvalues of the endomorphism $dT + 2\sigma^T + Scal$ acting on spinors are known according to Theorem 7.1. In particular, if the eigenvalues are non-negative (they cannot be positive since there always exists a parallel spinor), then the holonomy group Hol(∇) will determine the $\hat{A}$ genus in the simply connected case since the index of D is given by the ∇-parallel spinors. For example, if Hol(∇) =Spin(7); SU(4); Sp(2); SU(2)× SU(2), then $\hat{A} = 1; 2; 3; 4$, respectively by pure algebraic arguments, namely by considering the fixed spinors by the action of the holonomy representation of ∇ on spinors.

9. Solutions to the Killing spinor equations in dimension 8

We consider the Killing spinor equations (1.1) and (1.2) in dimension 8. The existence of a non-trivial ∇-parallel spinor is equivalent to the existence of a

Spin(7) structure (g, Φ) [34]. Then the 3-form field strength $H = T$ is given by Theorem 1.1. Involving the second Killing spinor equation (1.2) we have

Theorem 9.1. *In dimension 8 the following conditions are equivalent:*

i) The Killing spinor equations (1.1) and (1.2) admit solution with dilation Ψ;

ii) There exists a Spin(7) structure (g, Φ) with closed Lee form $\theta = -\frac{12}{7}d\Psi$ and therefore it is locally conformal to a balanced Spin(7) structure.

The 3-form field strength $H = T$ and the Riemannian scalar curvature $Scal^g$ are given by

$$T = -\delta\Phi + 2 * (d\Psi \wedge \Phi), \tag{9.30}$$

$$Scal^g = 8||d\Psi||^2 - \frac{1}{12}||T||^2 - 6\triangle\Psi, \tag{9.31}$$

where $\triangle\Psi = \delta d\Psi$ is the Laplacian.

The solution is with constant dilation if and only if the Spin(7) structure is balanced.

Proof. We apply Theorem 1.1. Let ∇ be a connection with torsion 3-form T. Let ϕ be an arbitrary ∇-parallel spinor field such that $(2d\Psi - T)\cdot\phi = 0$. The spinor field ϕ defines a Spin(7) structure Φ which is ∇-parallel. On the other hand, the connection preserving Φ with torsion any 3-form is unique given by Theorem 1.1. Comparing (1.4) with the second Killing spinor equation (1.2) we find $\frac{12}{7}d\Phi = -\theta$. Inserting the last equality into (1.3) and (1.5), we get (9.30) and (9.31) which completes the proof. $\square$

A similar formula as (9.30) was derived in [23] as a necessary condition.

Theorem 9.1 allows us to obtain a lot of compact solutions to the Killing spinor equations. If the dilation is a globally defined function, then any solution is globally conformal equivalent to a balanced Spin(7) structure. For example, any conformal transformation of a compact 8-dimensional manifold with Riemannian holonomy group Spin(7) constructed by Joyce [31, 32] is a solution with a globally defined non-constant dilation.

Summarizing we obtain

Corollary 9.2. *Any solution (M^8, g, Φ) to the Killing spinor equations (1.1), (1.2) in dimension 8 with non-constant globally defined dilation function Ψ comes from a solution with constant dilation by a conformal transformation ie $(g = e^{\frac{6}{7}\Psi}g_0, \Phi = e_0^{\frac{12}{7}\Phi})$, where (g_0, Φ_0) is a balanced Spin(7) structure.*

Note added to the proof. It has been shown in [15] that if an n-ddimensional G-structure with structure group G satisfying certain weak conditions admits a G-connection with totally skew-symmetric torsion then the G-structure has to be a Spin(7)-structure in dimension 8.

Acknowledgements The research was done during the author's visit at the Abdus Salam International Centre for Theoretical Physics, Trieste, Italy, Fall

2001. The author thanks the Abdus Salam ICTP for its support and the excellent facilities. The author is a member of the EDGE, Research Training Network HPRN-CT-2000-00101, supported by the European Human Potential Programme. The research is partially supported by Contract MM 809/1998 with the Ministry of Science and Education of Bulgaria, Contract 586/2002 with the University of Sofia "St. Kl. Ohridski".

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