Mathematical Research Letters **11**, 85–1 (2004)

ENDPOINT L^p-L^q ESTIMATES FOR SOME CLASSES OF DEGENERATE RADON TRANSFORMS IN $\mathbb{R}^2$

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Abstract. We study endpoint L^p-L^q estimates for the degenerate Radon transforms in $\mathbb{R}^2$ given by

$$Rf(t,x) = \int_{\mathbb{R}} f(t+S(x,y),y)\psi(x,y)dy$$

where ψ is a smooth function supported in a small neighborhood of the origin. Under the assumption S is a smooth function satisfying the left and right finite type conditions ($\partial_x\partial_y^n S(0,0) \neq 0$ and $\partial_x^m\partial_y S(0,0) \neq 0$ for some n, $m \geq 1$), we obtain complete L^p-L^q estimates for R.

1. Introduction and statement of results

The generalized Radon transform $\mathcal{R}$ in the plane is defined by

$$\mathcal{R}f(P) = \int_{\mathcal{M}_P} f(Q)\chi(P,Q)d\sigma_P(Q) \tag{1.1}$$

where $\mathcal{M}_P$ are curves in $\mathbb{R}^2$, and $d\sigma_P$ is a smooth density on $\mathcal{M}_P$ varying smoothly along P and χ is a compactly supported smooth function. The regularity properties of $\mathcal{R}$ have been studied in terms of L^p-Sobolev and L^p-L^q estimates ([PS1], [PS2], [PS3]). Up to ϵ-loss, sharp bounds for both problems were obtained by Seeger ([Se1], [Se2]). However, most endpoint estimates remain unsettled.

In this note we study endpoint L^p-L^q estimates for the degenerate Radon transform given by

$$Rf(t,x) = \int_{\mathbb{R}} f(t+S(x,y),y)\psi(x,y)dy \tag{1.2}$$

where S is a smooth function and ψ is a smooth cutoff function supported in a small neighborhood of the origin. It was introduced by Phong and Stein [PS2] as a *semi-translation invariant* model of the generalized Radon transform. These models are given in (1.1) by the curves for $P=(x,t)$

$$\mathcal{M}_{x,t} = \{(s,y) : s = t + S(x,y)\}.$$

Received February 19, 2003.

2000 *Mathematics Subject Classification.* Primary 44A12; Secondary 35S30.

Key words and phrases. degenerate Radon transform, endpoint estimate.

Research of the author is supported in part by The Interdisciplinary Research Program R01-1999-00005 (PI: K.T. Kim) of The Korea Science and Engineering Foundation.

For real analytic S the best possible L^2-regularity results for R were obtained by Phong and Stein [PS3], and later the sharp L^p-Sobolev estimates were obtained by Yang [Y] except some endpoints. For smooth S, Rychkov [Ry] obtained L^2-regularity only slightly weaker than that of [PS3].

The results for L^p-L^q estimates for R as well as L^p-Sobolev estimates can be described in terms of the *Newton polygon*(see [Se2] and [Y]). Let S be a smooth function given in a formal power series at the origin by $S(x,y) = \sum c_{\alpha,\beta} x^\alpha y^\beta$. We recall that the *Newton polygon* $\Gamma(S)$ of S at the origin is the convex hull of

$$\bigcup_{(\alpha,\beta), c_{\alpha,\beta}\neq 0} ((\alpha,\beta) + \mathbb{R}_+ \times \mathbb{R}_+)$$

where $\mathbb{R}_+ = [0,\infty)$. The *reduced Newton polyon* $\widetilde{\Gamma}(S)$ of S at the origin is defined by $\widetilde{\Gamma}(S) = \Gamma(S''_{xy}) + (1,1)$. Let $\mathcal{E}(S)$ be the collection of extreme points of $\widetilde{\Gamma}(S)$ and let $\mathcal{H}(S) \subset [0,1]\times[0,1]$ be the convex hull of the set

$$\{(1,1),(0,0)\} \cup \left\{ \left(\frac{m+1}{m+n+1}, \frac{m}{m+n+1}\right) : (m,n) \in \mathcal{E}(S) \right\}.$$

As shown in [Se2], R is unbounded from $L^p(\mathbb{R}^2)$ to $L^q(\mathbb{R}^2)$ if $(1/p, 1/q)$ lies below $\mathcal{H}(S)$ and $\psi(0,0) \neq 0$; analogous statements involving suitable finite type conditions also hold in the general case (1.1) (see [Se2]).

Under the assumption that S is *real analytic*, it was shown by the author [L2] that the necessary condition above is sufficient (for earlier partial results see [B], [L1], [PS2]). Let n, m be integers ≥ 2. Assuming left and right finite type conditions of degree n and m, the critical $L^{\frac{n+1}{2}}$-L^{n+1} and $L^{\frac{m+1}{m}}$-$L^{\frac{m+1}{m-1}}$ estimates were obtained by Bak, Oberlin and Seeger [BOS] in the general case (1.1). For the semi-translation invariant model case, the left finite type condition of degree n is read as

$$\partial_x \partial_y^{n-1} S(0,0) \neq 0 \text{ and } \partial_x \partial_y^j S(0,0) = 0 \quad for \quad 1 \leq j \leq n-2 \tag{1.3}$$

and the right finite type condition of degree m is equivalent to the condition

$$\partial_x^{m-1} \partial_y S(0,0) \neq 0 \text{ and } \partial_x^i \partial_y S(0,0) = 0 \quad for \quad 1 \leq i \leq m-2. \tag{1.4}$$

The result in [BOS] implies that if (1.3), (1.4) are satisfied, then the same $L^{\frac{n+1}{2}}$-L^{n+1} and L^{m+1}-$L^{\frac{m+1}{m-1}}$ estimates for R hold for smooth S. However, the endpoint estimates corresponding to other vertices remain open. The following is our main result.

Theorem 1.1. *Suppose S is a smooth function satisfying both the left and right finite type conditions (1.3), (1.4) with $n = N+1, m = M+1$, respectively, for some $N, M \geq 1$. Then if ψ is supported in a sufficiently small neighborhood of the origin, there is a constant C, independent of (p,q), such that $\|Rf\|_{L^q(\mathbb{R}^2)} \leq C\|f\|_{L^p(\mathbb{R}^2)}$ if $(1/p, 1/q) \in \mathcal{H}(S)$.*

For the proof of theorem we make use of Puiseux decomposition of C^∞ functions due to Rychkov [Ry], and combine the argument in [L2] based on multilinear interpolation with the technique of Yang [Y]. The method in this note can also be used to simplify the proof of the result in [L2]. In section 2 we study some damped estimates for R, which will be used in the proof of theorem. Also, it seems to be interesting in its own right. In section 3, the proof of Theorem 1.1 is given.

2. Damping estimates

The $L^{\frac{n+1}{2}}$-L^{n+1}, $L^{\frac{m+1}{m}}$-$L^{\frac{m+1}{m-1}}$ estimates for R are possible only when the left finite type of degree n and right finite type of degree m, or the better type conditions are satisfied. Even though these are not satisfied, the estimates corresponding to left and right finite type conditions can be obtained using suitable damping factors which compensate some bad behavior near the degeneracy. In view of the left finite type of degree n and the right finite type of degree m, the degeneracies can be thought to be located in the sets

$$\{(x,y) : \partial_x \partial_y^{n-1} S(x,y) = 0\}, \qquad \{(x,y) : \partial_x^{m-1} \partial_y S(x,y) = 0\},$$

respectively. So, it seems natural to use $|\partial_x \partial_y^n S|^a$, $|\partial_x^m \partial_y S|^b$ with suitable $a, b > 0$ as damping factors to recover the estimates associated to the left and right finite type conditions.

Let I_1, I_2 be finite open intervals. For $n \geq 1$, let us define $\mathcal{R}^n$ by

$$\mathcal{R}^n f(t,x) = \int_{\mathbb{R}} f(t + S(x,y), y) \chi_{I_1 \times I_2}(x,y) |\partial_x \partial_y^n S(x,y)|^{\frac{1}{n+2}} dy \tag{2.1}$$

which has damping factor related to the left finite type of degree $n+1$. By the standard argument involving characteristic functions, taking $S(x,y) = (x-y)^l$, one can easily see that the exponent $\frac{1}{n+2}$ over $|\partial_x \partial_y^n S|$ is the best possible one for $L^{\frac{n+2}{2}}$-L^{n+2} estimate. Interchanging the roles of x, y, the damped estimates for the right finite type follow from duality.

When S is real analytic, the $L^{3/2}$-L^3 estimate for $\mathcal{R}^1$ can be deduced from the damped oscillatory integral estimate in [PS4] and the argument in [PS2]. In the case S is a polynomial, an improvement of this was obtained by Bak, Oberlin and Seeger [BOS] in terms of Lorentz-spaces. Let $L^{p,q}$ be the Lorentz space with the (quasi-)norm denoted by $\|\cdot\|_{p,q}$ or $\|\cdot\|_{L^{p,q}}$.

Proposition 2.1. *Let $S(x,y)$ be a smooth function on $I_1 \times I_2$. Suppose that $\partial_x^M \partial_y S(x,y)$ is either positive or negative on $I_1 \times I_2$ for some positive integer M, and suppose there are a constant $B > 0$ such that for $N \geq 1$, the inequality*

$$0 < \sup_{y \in I_2} |\partial_x \partial_y^N S(x,y)| \leq B \inf_{y \in I_2} |\partial_x \partial_y^N S(x,y)| \tag{2.2}$$

holds for all $x \in I_1$. Then for $1 \leq n \leq N$, there is a constant $C = C(B,M,N,n)$ such that for all $f \in L^{\frac{n+2}{2}, n+2}(\mathbb{R}^2)$,

$$\|\mathcal{R}^n f\|_{L^{n+2}(\mathbb{R}^2)} \leq C \|f\|_{L^{\frac{n+2}{2}, n+2}(\mathbb{R}^2)}.$$

Proposition 2.1 can be used to obtain estimate for R defined by some infinitely degenerating S, for example, $S(x,y) = e^{-1/x}y^l$ with $I_1, I_2 = (0,\epsilon)$, $0 < \epsilon \ll 1$. When S is a polynomial in x, y, the condition (2.2) is trivially satisfied for some N. Moreover, if S is a nontrivial analytic function defined on a neighborhood of the origin, it can be shown that there is an interval $(-\delta, \delta)$ such that for all $x \in (-\delta,\delta)$ (2.2) holds with $I_2 = (-\delta,\delta)$ with sufficiently small δ. One can see it from the following.

Lemma 2.2. *If F is an analytic function defined on a neighborhood of the origin with $\partial_x\partial_y F \not\equiv 0$, then there are constants C, $\delta > 0$ and nonnegative integers $M, N \geq 1, c, d$ such that*

$$C^{-1}|x|^c \leq |\partial_y^N \partial_x F(x,y)| \leq C|x|^c,\ C^{-1}|y|^d \leq |\partial_x^M \partial_y F(x,y)| \leq C|y|^d \tag{2.3}$$

if $(x,y) \in (-\delta,\delta)\times(-\delta,\delta)$.

Dividing $(-\delta,\delta)\times(-\delta,\delta)$ into $(0,\delta)\times(0,\delta)$, $(0,\delta)\times(-\delta,0)$, $(-\delta,0)\times(0,\delta)$, $(-\delta,0)\times(-\delta,0)$, and using Lemma 2.2, we see that the conditions in Proposition 2.1 are satisfied on each of the decomposed. Hence, from Proposition 2.1 it follows that $\|\mathcal{R}^1 f\|_{3,3} \leq C\|f\|_{\frac{3}{2},3}$ if S is a (nontrivial) real analytic function and $I_1, I_2 = (-\delta,\delta)$. Interchanging the roles of x, y, we see the same estimate also holds for the adjoint of $\mathcal{R}^1$. From duality we get $\|\mathcal{R}^1 f\|_{3,\frac{3}{2}} \leq C\|f\|_{\frac{3}{2},\frac{3}{2}}$. Interpolation between these two estimates for $\mathcal{R}^1$ gives the following which is an extension of Theorem 1.2 in [BOS].

Corollary 2.3. *Let $S(x,y)$ be an analytic function defined on a neighborhood of the origin and let $I_1, I_2 = (-\delta,\delta)$. If $\widetilde{\Gamma}(S)$ is not empty and δ is sufficiently small, then there is a constant C such that*

$$\|\mathcal{R}^1 f\|_{L^{3,r}(\mathbb{R}^2)} \leq C\|f\|_{L^{\frac{3}{2},r}(\mathbb{R}^2)} \tag{2.4}$$

for all $f \in L^{\frac{3}{2},r}(\mathbb{R}^2)$ and $\frac{3}{2} \leq r \leq 3$.

Proof of Lemma 2.2. Using the Weierstrass preparation theorem (see [H], p.195-196), the real-analytic function $\partial_y F$ can be written as

$$\partial_y F(x,y) = U(x,y)x^a y^b (x^n + c_{n-1}(y)x^{n-1} + \cdots + c_0(y)) \tag{2.5}$$

where U, c_i are real-analytic functions on a neighborhood of the origin with $U(0,0) \neq 0$ and $c_i(0) = 0$ for $i = n-1, \ldots, 0$. Since the reduced Newton polygon of S at the origin is nonempty, $a + n \geq 1$. Note that

$$\partial_x^{n+a}\partial_y F(x,y) = (n+a)!U(x,y)y^b + R(x,y)$$

where $R(x,y) = \sum_{i=0}^{n-1} c_i(y)y^b\partial_x^{n+a}(x^{a+i}U(x,y))$. Since $c_i(0) = 0$ for $i = 0, \ldots, n-1$, $R(x,y) = O(|y^b x| + |y^{b+1}|)$. Therefore, if δ is sufficiently small, then on the set $(0,\delta)\times(0,\delta)$

$$C_1|y^b| \geq |\partial_x^{n+a}\partial_y F(x,y)| \geq C_2|y^b|$$

for some $C_1, C_2 > 0$ because $U(0,0) \neq 0$. The same argument holds if we interchange the roles of x, y. This completes the proof. $\square$

2.1. Polynomial like functions on $\mathbb{R}$. Now we consider a class of functions introduced in [PS3]. Let I be a finite interval. We say $F \in C^\infty(I)$ is polynomial type of degree N on I with constant B if the inequality

$$0 < \sup_{x\in I} |F^{(N)}(x)| \leq B \inf_{x\in I} |F^{(N)}(x)| \tag{2.6}$$

holds. We denote by $\mathcal{P}_N^B(I)$ this class of functions.

Lemma 2.4. *Let $P \in \mathcal{P}_N^B(I)$ and n be an integer satisfying $1 \leq n \leq N$. Then for any $(a,b) \subset I$, there is a constant $C = C(N,B,n)$, independent of particular function P and a,b, such that*

$$|a-b|^n \sup_{x\in(a,b)} |P^{(n)}(x)| \leq C \sup_{x\in(a,b)} |P(x)|. \tag{2.7}$$

Proof of Lemma 2.4. The following is an adaptation of the argument in [PS3]. By normalization we may assume that for all $x \in I$,

$$1 \leq P^{(N)}(x) \leq B. \tag{2.8}$$

Let $\mu = \sup_{x\in(a,b)} |P(x)|$. Then trivially $(a,b) \in \{x \in I : |P(x)| \leq \mu\}$.

First we consider the case when $N = 1$. Since $P' \geq 1$ on the interval I, we see that for some constant C, independent of P, I,

$$|a-b| \leq C\mu = C \sup_{x\in(a,b)} |P(x)|$$

using the following well-known sublevel set estimates for smooth functions on $\mathbb{R}$.

Lemma 2.5 (Sublevel set estimates, [C1])**.** *Let v be a smooth function on $\mathbb{R}$ and k be a positive integer. If $|v^{(k)}| \geq L$ on an interval I, then there is a constant C, independent of v, L and I, such that*

$$|\{x \in I : |v(t)| \leq \lambda\}| \leq C\,(\lambda/L)^{\frac{1}{k}}\,.$$

We now proceed by induction on N. Assume that Lemma 2.4 holds with N replaced by $N-1$. By Taylor's expansion we see that for $x, x+t \in (a,b)$,

$$P(x+t) = \sum_{k=1}^{N-1} P^{(k)}(x)\frac{t^k}{k!} + O(t^N) \sup_{x\in(a,b)} |P^N(x)|.$$

For any $x \in (a,b)$, there is an h, $|h| = \frac{b-a}{2}$, so that $x + sh \in (a,b)$ for all $0 \leq s \leq 1$. Integrating with $t = hs$, we see

$$\int_0^1 P(x+sh)\psi(s)ds = \sum_{k=0}^{N-1} P^{(k)}(x)\frac{h^k}{k!}\int_0^1 s^k\psi(s)ds + O(|b-a|^N).$$

Choose ψ so that $\int_0^1 s\psi(s)ds = 1$ and $\int_0^1 s^k\psi(s)ds = 0$ for $k = 0, 2, 3, \ldots, N-1$. Since $\left|\int_0^1 P(x+sh)\psi(s)ds\right| \leq C\sup_{x\in(a,b)} |P(x)| = C\mu$, it follows that

$$|(b-a)P'(x)| \leq C(\mu + |a-b|^N) \leq C\mu = C \sup_{x\in(a,b)} |P(x)|$$

because $|a-b|^N \leq C\mu$ by Lemma 2.5 and the fact that $1 \leq P^{(N)}(x)$. Since $P' \in \mathcal{P}^B_{N-1}(I)$, Now we can apply induction hypothesis to P' to get that for $1 \leq n \leq N$,

$$\begin{aligned}|a-b|^n \sup_{x\in(a,b)} |P^n(x)| &= |a-b|^n \sup_{x\in(a,b)} |(P')^{(n-1)}(x)| \\ &\leq C|a-b| \sup_{x\in(a,b)} |P'(x)| \\ &\leq C \sup_{x\in(a,b)} |P(x)|.\end{aligned}$$

This completes the proof. □

Remark 2.6. Suppose that $F \in \mathcal{P}^B_N(I)$ and F is either decreasing or increasing on $(a,b) \subset I$. Then for $1 \leq n \leq N$, $0 \leq \theta \leq 1$,

$$|b-a|^n |F^{(n)}(a)|^\theta |F^{(n)}(b)|^{1-\theta} \leq C|F(b)-F(a)| \tag{2.9}$$

with $C = C(N,B,n)$. It follows from Lemma 2.4. Indeed, $|F^{(n)}(a)|^\theta |F^{(n)}(b)|^{1-\theta} \leq \sup_{x\in(a,b)} |F^{(n)}(x)|$, and $\sup_{x\in(a,b)} |F(x)-F(a)| = |F(b)-F(a)|$ because F is decreasing or increasing on I. Then, applying Lemma 2.4 to $F(x)-F(a)$ and interval (a,b), we get (2.9).

The following lemma is an extension of Lemma 3.1 in [BOS] which covers the case $n = 1$.

Lemma 2.7. *Let I be a finite interval. Suppose that $P \in \mathcal{P}^B_N(I)$ and P' has constant sign on $(a,b) \subset I$. Then for $1 \leq n \leq N$, there is a constant $C = C(N,B,n)$, independent of P, a,b, such that for $s \in I$ and $\alpha > 0$,*

$$\int_{\{t\in(a,b):\alpha|P^{(n)}(s)P^{(n)}(t)|^{\frac{1}{n+2}} > |P(t)-P(s)|\}} |P(t)-P(s)|dt \leq C\alpha^{\frac{n+1}{n}} |P^{(n)}(s)|^{\frac{1}{n+2}}.$$

Proof of Lemma 2.4. Obviously, the set $\{t \in (a,b) : \alpha|P^{(n)}(s)P^{(n)}(t)|^{\frac{1}{n+2}} > |P(t)-P(s)|\}$ is contained in two minimal subintervals (t_0, s) and (s, t_1) of (a,b). So for $i = 0,1$,

$$\alpha|P^{(n)}(s)P^{(n)}(t_i)|^{\frac{1}{n+2}} \geq |P(s)-P(t_i)|. \tag{2.10}$$

It suffices to show that the integrals of $|P(t)-P(s)|$ over the intervals (t_0,s), (s,t_1) are bounded above by $C\alpha^{\frac{n+1}{n}}|P^{(n)}(s)|^{\frac{1}{n+2}}$. We only need to consider the integral over $[t_0,s]$ since the argument is the same in both cases.

From the fact that P' has constant sign on (a,b), it follows that

$$\int_{t_0}^s |P(t)-P(s)|dt \leq \int_{t_0}^s \int_t^s |P'(v)|dv \leq (s-t_0)|P(s)-P(t_0)|.$$

Since $P \in \mathcal{P}^B_N(I)$ and P' is of constant sign on the interval $[t_0,s]$, using (2.9) in *Remark* 2.6, we see that there is a constant $C_1 = C_1(B,N,n)$ such that for

$1 \le n \le N$,

$$|s-t_0|^n |P^{(n)}(t_0)|^{\frac{n+1}{n+2}} |P^{(n)}(s)|^{\frac{1}{n+2}} \le C_1 |P(s) - P(t_0)|.$$

Form this it follows that

$$(s-t_0)|P(s)-P(t_0)| \le C_1^{1/n} |P^{(n)}(t_0)|^{\frac{-n-1}{n(n+2)}} |P^{(n)}(s)|^{\frac{-1}{n(n+2)}} |P(s)-P(t_0)|^{\frac{n+1}{n}}.$$

Hence, $(s-t_0)|P(s)-P(t_0)| \le C\alpha^{\frac{n+1}{n}} |P^{(n)}(s)|^{\frac{1}{n+2}}$ because of (2.10). This completes the proof. □

2.2. Proof of Proposition 2.1. We may assume f is nonnegative. For measurable set $E \subset \mathbb{R}^2$, let $E(x)$ be the sets $\{y : (x,y) \in E\}$. We decompose $I_1 \times I_2$ into measurable sets $A_1, \dots, A_{N-1}$ such that

$$I_1 \times I_2 = \bigcup_1^{N-1} A_j,$$

and $A_j(x)$ is an interval, and $\partial_x \partial_y S(x,y)$ has constant sign on $A_j(x)$ for all $x \in I_1$ (discarding some harmless measure zero set). To do this, set $p_x(y) = \partial_x \partial_y S(x,y)$. Since $p_x^{(N-1)}$ is positive or negative on I_2 by (2.2), we can decompose I_2 into at most $N-1$ intervals $I_x^1, I_x^2, \dots,$ on each of which p_x has constant sign. Let A_j be the set of all (x,y) such that $(x,y) \in I_x^j$. Then, one can choose the I_x^j, using smoothness of S, so that A_j is measurable.

To simplify notation, let us set

$$D_n(x,y) = |\partial_x \partial_y^n S(x,y)|^{\frac{1}{n+2}}.$$

Define

$$T_A f(t,x) = \int f(t + S(x,y), y) \chi_A(x,y) D_n(x,y) dy.$$

For the proof of Proposition 2.1, it is sufficient to show that for $A = A_1, \dots, A_{N-1}$,

$$\|T_A f\|_{n+2} \le C \|f\|_{\frac{n+2}{2}, n+2}$$

By multilinear interpolation [C1], [J] (also see [B], [BOS]) the above follows from

$$\int \prod_1^{n+2} T_A f_j(x,t) dx dt \le C \|f_1\|_1 \prod_2^{n+2} \|f_j\|_{n+1,1}. \tag{2.11}$$

Since the adjoint of T_A is given by

$$T_A^* f(t,y) = \int f(t - S(x,y), x) \chi_A(x,y) D_n(x,y) dx,$$

it is sufficient to show that for $A = A_1, \dots, A_{N-1}$,

$$\int \prod_2^{n+2} T_A f_j(v - S(x,u), x) D_n(x,u) \chi_A(x,u) dx \le C \prod_2^{n+2} \|f_j\|_{n+1,1}$$

with C independent of $v \in \mathbb{R}$ and $u \in I_2$. From now on, we assume $A = A_j$ for some $j = 1, \dots, N-1$. Hölder's inequality reduces it to showing

$$\left(\int |T_A f(v - S(x,u), x)|^{n+1} D_n(x,u)\chi_A(x,u)dx\right)^{1/(n+1)} \le C\|f\|_{n+1,1}.$$

By duality this follows from

$$\int\int f(v - S(x,u) + S(x,y), y)\chi_A(x,y)G(x,y,u)dxdy \le C\|f\|_{n+1,1}\|gD_n^{\frac{n}{n+1}}(\cdot,u)\|_{L^{\frac{n+1}{n}}(\mathbb{R})} \tag{2.12}$$

where $G(x,y,u) = D_n(x,y)D_n(x,u)\chi_A(x,u)g(x)$.

For $(v,u) \in \mathbb{R} \times I_2$, define a map $\Phi_{u,v} : I_1 \times I_2 \to \mathbb{R}^2$ by

$$\Phi_{u,v}(x,y) = (v - S(x,u) + S(x,y), y). \tag{2.13}$$

We claim $\Phi_{u,v}$ has uniformly bounded multiplicity at most M on the set $\{(x,y) \in I_1 \times I_2 : y \ne u\}$. To see this, consider $h(x) = -S(x,u) + S(x,y)$. Observe $h^{(M)}(x) = \int_u^y \partial_x^M \partial_y S$. Since $\partial_x^M \partial_y S(x,t)$ is either positive or negative on $I_1 \times I_2$, we can see $h^{(M)}$ is also positive or negative if $u \ne y$ because so is $\partial_x^M \partial_y S$. This means the map $h(x) = -S(x,u) + S(x,y)$ has uniformly bounded multiplicity at most M if $u \ne y$. From this our claim follows.

Therefore, by the change of variables

$$(\xi,\eta) = \Phi_{u,v}(x,y),$$

the left hand side of (2.12) is bounded by

$$C \iint_{\Phi_{u,v}(A)} f(\xi,\eta)G(x,y,u)\frac{\partial(x,y)}{\partial(\xi,\eta)}d\xi d\eta$$

where x, y are functions in η, ξ and $\frac{\partial(x,y)}{\partial(\xi,\eta)}$ denotes the absolute value of the determinant of the Jacobian matrix of the map $(\xi,\eta) \to (x,y)$. By Hölder's inequality in Lorentz spaces (2.12) follows from

$$\|D_n(x,y)D_n(x,u)\chi_A(x,u)g(x)\frac{\partial(x,y)}{\partial(\xi,\eta)}\|_{L_{\xi,\eta}^{\frac{n+1}{n},\infty}(\Phi_{u,v}(A))} \le C\|gD_n^{\frac{n}{n+1}}(\cdot,u)\|_{\frac{n+1}{n}}. \tag{2.14}$$

For $\lambda > 0$, let us set

$$\Delta = \left\{(\xi,\eta) \in \Phi_{u,v}(A) : D_n(x,y)D_n(x,u)\chi_A(x,u)|g(x)|\frac{\partial(x,y)}{\partial(\xi,\eta)} > \lambda\right\}.$$

To get (2.14), it suffices to show

$$|\Delta| \le C\lambda^{-\frac{n+1}{n}} \int |g(x)|^{\frac{n+1}{n}} D_n(x,u)dx. \tag{2.15}$$

Since $|\Delta| = \int_\Delta d\xi d\eta$ and $\frac{\partial(\xi,\eta)}{\partial(x,y)} = |\partial_x S(x,u) - \partial_x S(x,y)|$, reversing the change of variables $((\xi,\eta) \to (x,y))$, we see that

$$|\Delta| \le C \int\int_{\widetilde{\Delta}} |\partial_x S(x,u) - \partial_x S(x,y)| dy dx \tag{2.16}$$

where

$$\widetilde{\Delta} = \left\{ (x,y) \in A : \frac{D_n(x,y) D_n(x,u) \chi_A(x,u) |g(x)|}{\lambda} > \frac{\partial(\xi,\eta)}{\partial(x,y)} \right\}.$$

From (2.2) we see that for all $x \in I_1$, $\partial_x S(x,\cdot) \in \mathcal{P}_N^B(I_2)$. Observe that $\chi_A(x,u)\chi_A(x,y) \equiv 0$ unless both u and y are contained in $A(x)$. Since $\partial_y \partial_x S(x,\cdot)$ has constant sign on the interval $A(x)$, we can apply Lemma 2.7 to $\partial_x S(x,\cdot)$ and interval $A(x)$ with $u, y \in A(x)$. Then, we get for all $x \in I_1$, $1 \le n \le N$,

$$\int_{\tilde{\Delta}(x)} |\partial_x S(x,u) - \partial_x S(x,y)| dy \le C \left(\frac{|g(x)|}{\lambda} \right)^{\frac{n+1}{n}} D_n(x,u) \tag{2.17}$$

since $\widetilde{\Delta}(x) = \{ y \in A(x) : \lambda^{-1} g(x) |\partial_x \partial_y^n S(x,y) \partial_x \partial_y^n S(x,u)|^{\frac{1}{n+2}} > |\partial_x S(x,u) - \partial_x S(x,y)| \}$ if $u \in A(x)$. Here C is depending only on N, B, n. We put (2.17) in the right hand side of (2.16) and integrate in x to get (2.15). This completes the proof.

3. Proof of theorem 1.1

To begin with, note that

$$\partial_x \partial_y^N S(0,0) \ne 0, \quad \partial_x^M \partial_y S(0,0) \ne 0.$$

By the result due to Bak, Oberlin and Seeger [BOS] there is nothing to prove if $\widetilde{\Gamma}(S)$ has only the two extreme points $(1, M)$ and $(N, 1)$. In fact, it directly follows from Proposition 2.1 and duality.

Let (α,β) be an extreme point of $\widetilde{\Gamma}(S)$ which is neither $(1,M)$ nor $(N,1)$. For the proof of Theorem 1.1 it suffices to show that $L^{\frac{\alpha+\beta+1}{\alpha+1}}$-$L^{\frac{\alpha+\beta+1}{\alpha}}$ estimate for R since the number of extreme points of $\widetilde{\Gamma}(S)$ is obviously finite. We may assume $\beta \ge \alpha$. In the case $\beta < \alpha$, we consider the adjoint operator R^* of R. Then (β,α) is an extreme points of the reduced Newton polygon of $\widetilde{S}(x,y) = -S(y,x)$. By duality we can derive the desired from $L^{\frac{\alpha+\beta+1}{\beta+1}}$-$L^{\frac{\alpha+\beta+1}{\beta}}$ estimate for R^*.

For $0 < \delta \ll 1$, let $I = (0,\delta)$ and

$$Q = I \times I.$$

The methods in this note are not affected by the smoothness of ψ. So we divide the neighborhood of the origin into subsets of four quadrants. Then, using reflections $x \to -x$, $y \to -y$, we may replace χ_Q for ψ in (1.2) because the (reduced) Newton polygon is not changed under reflections. From now on, our analysis will be carried out on Q only.

Let

$$(A_1, B_1), (A_2, B_2), \ldots, (A_l, B_l)$$

$y = C_i x^{\gamma_i}$ $y = Y_{i,j}(x)$ $y = c_i x^{\gamma_i}$

y-axis

δ

$y = C_{i+1} x^{\gamma_{i+1}}$

δ x-axis

FIGURE 1. Decomposition of $Q = (0, \delta) \times (0, \delta)$

be the extreme points of $\Gamma(S''_{xy})$ with $B_i > B_{i+1}$. Obviously, $(A_1, B_1) = (0, N-1)$, $(A_l, B_l) = (M-1, 0)$ and $(\alpha, \beta) = (A_n + 1, B_n + 1)$ for some $n = 2, \ldots, l-1$. For $i = 1, \ldots, l-1$, we define

$$\nu_i = B_i - B_{i+1}, \quad \gamma_i = \frac{A_{i+1} - A_i}{B_i - B_{i+1}}.$$

Note that $1/\gamma_i$ is the absolute value of the slope of the face joining two extreme points (A_i, B_i) and (A_{i+1}, B_{i+1}). For convenience we also define

$$\gamma_0 = 0, \quad \gamma_l = \infty.$$

By convexity of $\widetilde{\Gamma}(S)$, $\gamma_i < \gamma_{i+1}$, and one can easily see that for $k = 2, \ldots, l-1$,

$$A_k = A_1 + \sum_{i=1}^{k-1} \nu_i \gamma_i, \quad B_k = B_l + \sum_{k}^{l-1} \nu_i. \tag{3.1}$$

We will use the following due to Rychkov [Ry].

Lemma 3.1 (Puiseux decomposition of C^∞ functions)**.** *Let F be a real-valued smooth function and $(A_1, B_1), \ldots, (A_l, B_l)$ be the extreme points of $\Gamma(F)$, and let $\nu_i = B_i - B_{i+1}$, $\gamma_i = \frac{A_{i+1} - A_i}{B_i - B_{i+1}}$ for $l = 1, \ldots, l-1$. Then there is a neighborhood U of the origin such that F admits in the region $x, y > 0$ a factorization of the form*

$$F(x, y) = U(x, y) \prod_{i=1}^{A_1} (x - X_i(y)) \prod_{i=1}^{B_l} (y - Y_i(x)) \prod_{i=1}^{l-1} \prod_{j=1}^{\nu_i} (y - Y_{i,j}(x)) \tag{3.2}$$

where

(1) *U is a real valued smooth function with $U(0,0) \neq 0$,*
(2) *X_i, Y_i are smooth on $\mathbb{R}_+ \cap U$ and $X_i(x), Y_i(x) = O(x^N)$ for any $N > 0$,*
(3) *$Y_{i,j}$ are continuous and $Y_{i,j}(x) = C_{i,j}x^{\gamma_i} + O(x^{\gamma_i+\epsilon})$ for some small $\epsilon > 0$ as $x \to 0$ with $C_{i,j} \neq 0$.*

Assuming δ is sufficiently small and replacing F by $\partial_x\partial_y S$ in Lemma 3.1, we see that for $(x,y) \in Q$,

$$\partial_x\partial_y S(x,y) = U(x,y)\prod_{i=1}^{l-1}\prod_{j=1}^{\nu_i}(y - Y_{i,j}(x)) \tag{3.3}$$

where U is a real valued smooth function with $U(0,0) \neq 0$ and $Y_{i,j}$ is a continuous function on I with $Y_{i,j}(x) = c_{i,j}x^{\gamma_i} + O(x^{\gamma_i+\epsilon})$, $c_{i,j} \neq 0$, for some small $\epsilon > 0$. Additionally we set

$$\mu_i = A_{i+1} - A_i, \quad \delta_i = 1/\gamma_i.$$

Interchanging the roles of x, y in Lemma 3.1, we also have

$$\partial_x\partial_y S(x,y) = V(x,y)\prod_{i=1}^{l-1}\prod_{j=1}^{\mu_i}(x - X_{i,j}(y)) \tag{3.4}$$

where V is a real valued smooth function with $V(0,0) \neq 0$ and $X_{i,j}$ is a continuous function on I with $X_{i,j}(y) = d_{i,j}y^{\delta_i} + O(y^{\delta_i+\epsilon})$, $d_{i,j} \neq 0$, for some small $\epsilon > 0$. Since $Y_{i,j}(x) = c_{i,j}x^{\gamma_i} + O(x^{\gamma_i+\epsilon})$, it is possible to choose $C_i, c_i > 0$ such that for $i = 1, \dots, l-1$,

$$2c_i x^{\gamma_i} < |Y_{i,j}(x)| < \frac{C_i}{2}x^{\gamma_i}$$

if $x \in I$ with sufficiently small δ.

Now we decompose Q into regions $\mathcal{N}_i$ (near the zero branches) and $\mathcal{A}_i$ (away from the zero branches) by setting

$$\mathcal{N}_i = \{(x,y) \in Q : c_i x^{\gamma_i} < y < C_i x^{\gamma_i}\}, \quad i = 1, \dots, l-1,$$
$$\mathcal{A}_i = \{(x,y) \in Q : C_i x^{\gamma_i} < y < c_{i-1}x^{\gamma_{i-1}}\}, \quad i = 1, \dots, l$$

where we set $c_0 = C_l = 1$. (See figure 1). We define a weighted operator $T[\cdot, w]$ by

$$T[f,w](t,x) = \int f(t + S(x,y), y)w(x,y)\chi_Q(x,y)dy.$$

Proposition 3.2. *If δ is sufficiently small, then there is a constant C such that for $i = 1, \dots, l-1$,*

$$\left\| T[f, \chi_{\mathcal{N}_i}|S''_{xy}|^{-1/(A_n+B_n)}] \right\|_{\frac{A_n+B_n}{A_n}} \leq C\,\|f\|_{\frac{A_n+B_n}{A_n}} \tag{3.5}$$

and for $i = 1, \dots, n-2, n, \dots, l$,

$$\left\| T[f, \chi_{\mathcal{A}_i} |S''_{xy}|^{-1/(A_n+B_n)}] \right\|_{\frac{A_n+B_n}{A_n}} \le C \left\| f \right\|_{\frac{A_n+B_n}{A_n}} \tag{3.6}$$

where $S''_{xy} = \partial_x \partial_y S$.

Since $|\partial_x \partial_y^M S|$, $|\partial_x^N \partial_y S| > c > 0$ on Q with small δ, using Proposition 2.1 (with $n = 1$), we have

$$\left\| T[f, |S''_{xy}|^{1/3}] \right\|_3 \le C \left\| f \right\|_{\frac{3}{2}}.$$

By interpolation (complex) between this and the estimates (3.5), (3.6), we see that for $i = 1, \dots, l-1$,

$$\|T[f, \chi_{\mathcal{N}_i}]\|_{\frac{A_n+B_n+3}{A_n+1}} \le C \|f\|_{\frac{A_n+B_n+3}{A_n+2}}$$

and for $i = 1, \dots, n-2, n, \dots, l$,

$$\|T[f, \chi_{\mathcal{A}_i}]\|_{\frac{A_n+B_n+3}{A_n+1}} \le C \|f\|_{\frac{A_n+B_n+3}{A_n+2}}.$$

Since $Q = (\bigcup_{i=1}^{l} \mathcal{A}_i) \cup (\bigcup_{i=1}^{l-1} \mathcal{N}_i)$, $Rf = \sum_{i=1}^{l} T[f, \chi_{\mathcal{A}_i}] + \sum_{i=1}^{l-1} T[f, \chi_{\mathcal{N}_i}]$. To finish the proof of Theorem 1.1, we have to show (recall $A_n = \alpha - 1, B_n = \beta - 1$) that

$$\left\| T[f, \chi_{\mathcal{A}_{n-1}}] \right\|_{\frac{\alpha+\beta+1}{\alpha}} \le C \left\| f \right\|_{\frac{\alpha+\beta+1}{\alpha+1}}. \tag{3.7}$$

3.1. Proof of Proposition 3.2. The proof Proposition 3.2 is to be given by combining Lemma 3.4 (below) with the following which is a form of Schur's lemma given in Yang [Y].

Lemma 3.3. *Let* $Tf(x) = \int K(x,y) f(y) dy$ *and* $1 < p < \infty$. *Suppose*

$$\int |K(x,y)| y^{-1/p} dy \le C x^{-1/p}, \qquad \int |K(x,y)| x^{-1+1/p} dx \le C y^{-1+1/p}.$$

Then, there is a constant C *such that* $\|Tf\|_p \le C \|f\|_p$.

The following can be shown by the argument in [Y] using (3.3), (3.4) but for the convenience of readers we give a proof of this.

Lemma 3.4. *Let* $\alpha_n = 1/(A_n + B_n)$ *and* $1/p_n = A_n/(A_n + B_n)$. *Then for* $k = 1, \dots, l-1$ *and* $n = 2, \dots, l-1$,

$$I_{k,n}(x) = \int \chi_{\mathcal{N}_k}(x,y) |S''_{xy}(x,y)|^{-\alpha_n} y^{-1/p_n} dy \le C x^{-1/p_n}, \tag{3.8}$$

$$\mathcal{I}_{k,n}(y) = \int \chi_{\mathcal{N}_k}(x,y) |S''_{xy}(x,y)|^{-\alpha_n} x^{-1+1/p_n} dx \le C y^{-1+1/p_n}. \tag{3.9}$$

And for $k = 1, \dots, l$ *and* $n = 2, \dots, k, k+2, \dots, l-1$

$$J_{k,n}(x) = \int \chi_{\mathcal{A}_k}(x,y) |S''_{xy}(x,y)|^{-\alpha_n} y^{-1/p_n} dy \le C x^{-1/p_n}, \tag{3.10}$$

$$\mathcal{J}_{k,n}(y) = \int \chi_{\mathcal{A}_k}(x,y) |S''_{xy}(x,y)|^{-\alpha_n} x^{-1+1/p_n} dx \le C y^{-1+/p_n}. \tag{3.11}$$

Set $K = \chi_{\mathcal{N}_k}|S''_{xy}|^{-1/(A_n+B_n)}$. By (3.8) and (3.9), $\int K(x,y)y^{-A_n/(A_n+B_n)}dy \le Cx^{-A_n/(A_n+B_n)}$, $\int K(x,y)x^{-B_n/(A_n+B_n)}dx \le Cy^{-B_n/(A_n+B_n)}$. By Minkowski's inequality

$$\left\|T[f,\chi_{\mathcal{N}_i}|S''_{xy}|^{-1/(A_n+B_n)}]\right\|_{\frac{A_n+B_n}{A_n}} \le \left\|\int K(x,y)\|f(\cdot,y)\|_{\frac{A_n+B_n}{A_n}}dy\right\|_{L^{\frac{A_n+B_n}{A_n}}(dx)}.$$

An application of Lemma 3.3 gives (3.5). The remaining $T[f,\chi_{\mathcal{A}_i}|S''_{xy}|^{-1/(A_n+B_n)}]$ can be handled in the same way using (3.10) and (3.11). This proves Proposition 3.2.

Proof of Lemma 3.4. By symmetry it is sufficient to show (3.8) and (3.10) because (3.9) and (3.11) can be shown by interchanging the roles of x, y. First we show (3.8). Comparing the sizes of factors in (3.3)(also note (3.1)) on the set $\mathcal{N}_k = \{(x,y) \in Q : c_k x^{\gamma_k} < y < C_k x^{\gamma_k}\}$, we see that if $(x,y) \in \mathcal{N}_k$,

$$|\partial_x\partial_y S(x,y)|^{\frac{-1}{A_n+B_n}} \le Cx^{-\frac{A_k}{A_n+B_n}}y^{-\frac{B_{k+1}+A_n}{A_n+B_n}}\prod_{j=1}^{\nu_k}|(y-Y_{k,j}(x))|^{\frac{-1}{A_n+B_n}}. \tag{3.12}$$

We treat the cases $\gamma_k \ge 1$ and $\gamma_k < 1$, separately.

Note that if $\gamma_k \ge 1$ then $A_n + B_n > \nu_k$ for any n. Indeed, suppose $A_n + B_n = \nu_k$ for some k. Then we have $\nu_i = 0$ for all $i \ne k$ because $A_n + B_n = \sum_{i=1}^{n-1}\nu_i\gamma_i + \sum_{i=n}^{l-1}\nu_i$ (see (3.1)). This means that $\Gamma(\partial_x\partial_y S)$ has only two extreme points. It was excluded by our assumption on (α,β) which is an extreme point neither $(1,N)$ nor $(M,1)$. By a computation we see

$$I^j_{k,n}(x) = \int_{c_k x^{\gamma_k}}^{C_k x^{\gamma_k}} |(y-Y_{k,j}(x))|^{-\nu_k/(A_n+B_n)}dy \le Cx^{\gamma_k(1-\frac{\nu_k}{A_n+B_n})}.$$

By (3.12) and Hölder's inequality

$$I_{k,n}(x) \le Cx^{-\frac{A_k+\gamma_k B_{k+1}+\gamma_k A_n}{A_n+B_n}}\Big(\prod_{j=1}^{\nu_k} I^j_{k,n}(x)\Big)^{1/\nu_k}.$$

Since $\nu_k = B_k - B_{k+1}$, we have

$$I_{k,n}(x) \le Cx^{-\frac{A_k+\gamma_k B_k-\gamma_k B_n}{A_n+B_n}}.$$

So it is sufficient to show that $A_n \ge A_k + \gamma_k B_k - \gamma_k B_n$ because $0 < x \le \delta$. This follows from the convexity of Newton polygon. Comparing the slopes of the lines connecting (A_k, B_k) and (A_n, B_n), it is easy to see that $\gamma_k \le \frac{A_n-\mathcal{A}_k}{B_k-B_n}$ if $n > k$, and $\gamma_k > \frac{A_k-\mathcal{A}_n}{B_n-B_k}$ if $n < k$.

Now we turn to the case $\gamma_k < 1$. Using the factorization (3.4), we see that if $(x,y) \in \mathcal{N}_k$,

$$|\partial_x\partial_y S(x,y)|^{\frac{-1}{A_n+B_n}} \le Cx^{-\frac{A_k}{A_n+B_n}}y^{-\frac{B_{k+1}+A_n}{A_n+B_n}}\prod_{j=1}^{\mu_k}|(x-X_{k,j}(y))|^{\frac{-1}{A_n+B_n}}.$$

Now set

$$I_{k,n}^{j}(x)=\int_{c_k x^{\gamma_k}}^{C_k x^{\gamma_k}} |(x-X_{k,j}(y))|^{-\mu_k/(A_n+B_n)}\,dy.$$

Since $X_{k,j}(y)=d_{k,j}y^{\delta_k}+O(y^{\delta_k+\epsilon})$, it is easy to see that

$$I_{k,n}^{j}(x)\le\int_{c_k x^{\gamma_k}}^{C_k x^{\gamma_k}} \left|x-2dy^{\delta_k}\right|^{-\mu_k/(A_n+B_n)}+|x-dy^{\delta_k}/2|^{-\mu_k/(A_n+B_n)}dy$$

where $d=|d_{k,j}|$. From the fact that $\delta_k=1/\gamma_k>1$ and $A_n+B_n=\sum_{i=1}^{n-1}\delta_i\mu_i+\sum_{i=n}^{l-1}\mu_i$, it follows $\mu_k<A_n+B_n$ by the same argument in the previous case. Therefore a routine computation gives

$$I_{k,n}^{j}(x)\le Cx^{\gamma_k-\frac{\mu_k}{A_n+B_n}}=Cx^{\gamma_k(1-\frac{\nu_k}{A_n+B_n})}.$$

because $\gamma_k=\mu_k/\nu_k$. Now the remaining is done by the same lines of arguments in the previous case. This completes the proof of (3.8).

Now we prove (3.10). From (3.3) and (3.1) it is easy to see that $|\partial_x\partial_y S(x,y)|\le Cx^{-\frac{A_{k+1}}{A_n+B_n}}y^{-\frac{B_{k+1}}{A_n+B_n}}$ if $(x,y)\in\mathcal{A}_k$. So it follows that

$$J_{k,n}(x)\le C\int_{C_{k+1}x^{\gamma_{k+1}}}^{c_k x^{\gamma_k}} x^{-\frac{A_{k+1}}{A_n+B_n}}y^{-\frac{B_{k+1}+A_n}{A_n+B_n}}\,dy.$$

First we consider the case $n<k+1$. In this case $\frac{B_{k+1}+A_n}{A_n+B_n}<1$. By a computation we have

$$J_{k,n}(x)\le Cx^{-\frac{A_{k+1}-\gamma_k B_n+\gamma_k B_{k+1}}{A_n+B_n}}.$$

From the convexity of Newton polygon we see $\gamma_k\ge\frac{A_{k+1}-A_n}{B_n-B_{k+1}}$ if $n<k+1$. Hence, (3.10) follows because $x\le\delta$. Secondly when $n>k+1$, $\frac{B_{k+1}+A_n}{A_n+B_n}>1$. So we have

$$J_{k,n}(x)\le Cx^{-\frac{A_{k+1}-\gamma_{k+1}B_n+\gamma_{k+1}B_{k+1}}{A_n+B_n}}.$$

Then the right hand side of the above is bounded by $x^{-\frac{A_n}{A_n+B_n}}$ because $\gamma_{k+1}\le\frac{\mathcal{A}_n-A_{k+1}}{B_{k+1}-B_n}$ if $n>k+1$. This proves (3.10). □

(3.10) and (3.11) in Lemma 3.4 do not hold when $k=n-1$. In fact, by a simple computation one can see $J_{n-1,n}\ge Cx^{-1/p_n}|\log x|$. This is due to the fact that $\frac{B_{k+1}+A_n}{A_n+B_n}=1$.

3.2. Proof of (3.7). We may assume $f\ge 0$. By interpolation with change of measures (see [BL], p. 119) (3.7) will follow from two estimates

$$\left\|T[f,\chi_{\mathcal{A}_{n-1}}x^{\frac{\alpha-1}{\beta+2}}]\right\|_{\beta+2}\le C\,\|f\|_{\frac{\beta+2}{2}}\,, \tag{3.13}$$

$$\left\|T[f,\chi_{\mathcal{A}_{n-1}}x^{\frac{\alpha-\beta-1}{2\beta+2}}]\right\|_{2}\le C\,\|f\|_{\frac{2\beta+2}{\beta+2}}\,. \tag{3.14}$$

We deduce (3.13) from Proposition 2.1. Since $\partial_x\partial_y^N S(0,0)$, $\partial_x^M\partial_y S(0,0)\neq 0$, by Proposition 2.1 with $n=\beta$ (trivially $\beta<N$), it is sufficient to show that if $(x,y)\in\mathcal{A}_{n-1}$ and δ is sufficiently small,

$$C|\partial_x\partial_y^\beta S(x,y)|\geq x^{\alpha-1}. \tag{3.15}$$

Since $\partial_x\partial_y^N S(0,0)\neq 0$, by Malgrange's preparation theorem (see [H], p. 200)

$$\partial_x\partial_y^\beta S(x,y)=U(x,y)(y^{N-\beta}+c_{N-\beta-1}(x)y^{N-\beta-1}+\cdots+c_0(x))$$

where U and $c_{N-\beta-1},\ldots,c_0$ are smooth functions with $U(0,0)\neq 0$ and $c_{N-\beta-1}(0)=\cdots=c_0(0)=0$. For $0\leq j\leq N-\beta-1$, c_j is of type a_j, that is, $c_j(x)=dx^{a_j}+O(x^{a_j+1})$ for some a_j, $d\neq 0$. Possibly $a_j=\infty$, and obviously $a_0=\alpha-1$. Since the Newton polygon of $\partial_x\partial_y^\beta S$ at the origin is equal to that of $y^{N-\beta}+c_{N-\beta-1}(x)y^{N-\beta-1}+\cdots+c_0(x)$, $\Gamma(\partial_x\partial_y^\beta S)$ is the convex hull of the union of the sets $\mathbb{R}_+\times\mathbb{R}_+ +(0,N-\beta)$ and $\mathbb{R}_+\times\mathbb{R}_+ +(a_j,j)$, $j=0,\ldots.N-\beta-1$. And note $\Gamma(\partial_x\partial_y^\beta S)+(0,\beta-1)=\{(\mu,\nu)\in\Gamma(S''_{xy}):\nu\geq\beta-1\}$. From these and convexity of $\Gamma(\partial_x\partial_y^\beta S)$, comparing the slopes of the lines joining (a_j,j) and $(a_0,0)$, we see that for $1\leq j\leq N-\beta-1$,

$$\frac{a_0-a_j}{j}=\frac{\alpha-1-a_j}{j}<\gamma_{n-1}.$$

Hence, if $y\sim x^{\gamma_{n-1}}$ (namely, $(x,y)\in\mathcal{A}_{n-1}$), all $y^{N-\beta}$, $c_{N-\beta-1}(x)y^{N-\beta-1}$, … ,$c_2(x)y^2$, $c_1(x)y$ are $O(x^{\alpha-1}x^\epsilon)$ because $\alpha-1<a_j+j\gamma_{n-1}$ for $1\leq j\leq N-\beta-1$. Therefore, (3.15) holds provided $(x,y)\in\mathcal{A}_{n-1}$ and δ is sufficiently small.

Now we turn to the proof of (3.14). Set

$$Uf=T[f,\chi_{\mathcal{A}_{n-1}}x^{\frac{\alpha-\beta-1}{2\beta+2}}].$$

By the T^*T argument, it is sufficient to show that U^*U is bounded from $L^{(2\beta+2)/(\beta+2)}$ to $L^{(2\beta+2)/\beta}$. Making the change of variables $x\to x^{\frac{\beta+1}{\alpha}}$, we see

$$U^*Uf(t,y)=$$
$$C\int\int f(t-S(x^{\frac{\beta+1}{\alpha}},y)+S(x^{\frac{\beta+1}{\alpha}},z),z)\chi_{\mathcal{A}_{n-1}}(x^{\frac{\beta+1}{\alpha}},y)\chi_{\mathcal{A}_{n-1}}(x^{\frac{\beta+1}{\alpha}},z)dxdz$$

For fixed y,z, let us define a map $x\to v$ by

$$v=S(x^{\frac{\beta+1}{\alpha}},y)-S(x^{\frac{\beta+1}{\alpha}},z).$$

The map $x\to v$ is of uniformly bounded multiplicity at most M for all $y,z\in(0,\delta)$ with $y\neq z$ since $S(\cdot,y)-S(\cdot,z)$ has multiplicity at most M if $y\neq z$. This can be seen by the argument below (2.13). So, by the change of variables $x\to v$, $U^*Uf(t,y)$ is bounded above by a constant multiple of

$$\iint f(t-v,z)\chi_{\mathcal{A}_{n-1}}(x^{\frac{\beta+1}{\alpha}},y)\chi_{\mathcal{A}_{n-1}}(x^{\frac{\beta+1}{\alpha}},z)\,|\partial x/\partial v|\,dvdz$$

where $x = x(v)$. Set

$$k(v,y,z) = \chi_{\mathcal{A}_{n-1}}(x^{\frac{\beta+1}{\alpha}}(v),y)\chi_{\mathcal{A}_{n-1}}(x^{\frac{\beta+1}{\alpha}}(v),z)\,|\partial x/\partial v|$$

and note that $U^*Uf(t,y) \le C\int f(t-v,z)k(v,y,z)dv$. We claim that

$$\|k(\cdot,y,z)\|_{L^{\frac{\beta+1}{\beta},\infty}(dv)} \le C|y-z|^{-1+\frac{1}{\beta+1}}. \tag{3.16}$$

Assuming this for the moment, we prove (3.14). By a generalized Young's inequality (cf. [F], p.232), $U_{y,z}F(t) = \int F(t-v)k(v,y,z)dv$ is, in particular, bounded from $L^{\frac{2\beta+2}{\beta+2}}(\mathbb{R})$ to $L^{\frac{2\beta+2}{\beta}}(\mathbb{R})$ with operator norm $C|y-z|^{-1+\frac{1}{\beta+1}}$. Therefore, Minkowski's inequality gives

$$\|U^*Uf\|_{L^{\frac{2\beta+2}{\beta}}(\mathbb{R}^2)} \le C\|\int |y-z|^{-1+\frac{1}{\beta+1}}\|f(\cdot,z)\|_{L^{\frac{2\beta+2}{\beta+2}}}dz\|_{L^{\frac{2\beta+2}{\beta}}(dy)}.$$

By the fractional integration theorem we get (3.14).

We now prove the claim (3.16). It suffices to show

$$|\{v : |k(v,y,z)| > \lambda\}| \le C\lambda^{-\frac{\beta+1}{\beta}}|y-z|^{-1}. \tag{3.17}$$

Observe $|\{v : |k(v,y,z)| > \lambda\}| = \int_{\{v:|k(v,y,z)|>\lambda\}} dv$. Reversing the change of variables ($v \to x$) gives

$$|\{v : |k(v,y,z)| > \lambda\}| \le C\int_{\left\{x\in(0,\delta):\chi_{\mathcal{A}_{n-1}}(x^{\frac{\beta+1}{\alpha}},y)\chi_{\mathcal{A}_{n-1}}(x^{\frac{\beta+1}{\alpha}},z)|\frac{\partial x}{\partial v}|>\lambda\right\}} |\partial v/\partial x|dx.$$

Set $\mathcal{A}_{n-1}(z) = \{x : (x,z) \in \mathcal{A}_{n-1}\}$, and note that $\chi_{\mathcal{A}_{n-1}}(x^{\frac{\beta+1}{\alpha}},y)\chi_{\mathcal{A}_{n-1}}(x^{\frac{\beta+1}{\alpha}},z)$ $= 0$ unless $x^{\frac{\beta+1}{\alpha}} \in \mathcal{A}_{n-1}(y)\cap\mathcal{A}_{n-1}(z)$. Since $\partial v/\partial x = \partial_x(S(x^{\frac{\beta+1}{\alpha}},y) - S(x^{\frac{\beta+1}{\alpha}},z))$, we see

$$|\{v : |k(v,y,z)| > \lambda\}| \le C\int_\Delta |\partial_x(S(x^{\frac{\beta+1}{\alpha}},y) - S(x^{\frac{\beta+1}{\alpha}},z))|dx$$

where

$$\Delta = \left\{x \in (0,\delta) : \frac{1}{\lambda} \ge \left|\partial_x(S(x^{\frac{\beta+1}{\alpha}},y) - S(x^{\frac{\beta+1}{\alpha}},z))\right|, x^{\frac{\beta+1}{\alpha}} \in \mathcal{A}_{n-1}(y)\cap\mathcal{A}_{n-1}(z)\right\}.$$

Since $\{v : |k(v,y,z)| > \lambda\}| \le C|\Delta|/\lambda$, it is sufficient for (3.17) to show that there is a constant C such that

$$|\Delta| \le C\lambda^{-\frac{1}{\beta}}|y^\beta - z^\beta|^{-\frac{1}{\beta}}. \tag{3.18}$$

Using (3.3) and (3.1), it is easy to see that if $(x,y) \in \mathcal{A}_{n-1}$, $S''_{xy}(x,y)$ has constant sign and $|S''_{xy}(x,y)| \sim x^{A_n}y^{B_n} = x^{\alpha-1}y^{\beta-1}$. By these, if $(x,y),(x,z) \in \mathcal{A}_{n-1}$, then $|\partial_x S(x,y) - \partial_x S(x,z)| \sim |\int_z^y x^{\alpha-1}t^{\beta-1}dt|$. This implies that if $x^{\frac{\beta+1}{\alpha}} \in \mathcal{A}_{n-1}(y)\cap\mathcal{A}_{n-1}(z)$, then

$$|\partial_x(S(x^{\frac{\beta+1}{\alpha}},y) - S(x^{\frac{\beta+1}{\alpha}},z))| \sim x^{\frac{\beta-\alpha+1}{\alpha}}\left|\int_z^y x^{\frac{\beta+1}{\alpha}(\alpha-1)}t^{\beta-1}dt\right| \sim x^\beta|y^\beta - z^\beta|.$$

So $|\Delta| \le C|\{x \in (0,\delta) : x^\beta|y^\beta - z^\beta| \le 1/\lambda\}|$. From this (3.18) follows.

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